DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Spatial externalities, R&D spillovers, and endogenous technological change

Spyridon Tsangaris

Anastasios Xepapadeas

Athanasios N. Yannacopoulos

Working Paper Series

22-25

October 2022

Spatial externalities, R&D spillovers, and endogenous technological change

Spyridon Tsangaris[a,*], Anastasios Xepapadeas[b], Athanasios N. Yannacopoulos[c]

October 3, 2022

Abstract

We incorporate the spatial dimension into a standard expanding variety growth model based on R&D. The spatial interaction is introduced through spatial production spillovers, knowledge diffusion across space, and the capability for spatial heterogeneity. Forward-looking agents who operate in a finite continuous geographic area choose how much to innovate at each point in time and space. We study the properties of equilibrium and optimal allocations and argue that the characteristics are different from those of the non-spatial model, which alter the appropriate policy measures. We show how spatial interactions may lead regions with spatial homogeneity to differ in their growth rates and areas with spatial heterogeneity to share the same growth rates in the long run. Finally, we present numerical examples to illustrate the different dynamic outcomes and stylized facts from the US economy.

Keywords: endogenous growth, knowledge diffusion, R&D, scale effects, spatial development, spatial production externalities

[a]Department of Regional and Economic Development, Agricultural University of Athens
[b]Department of Economics, University of Bologna
[c]Department of Statistics, Athens University of Economics and Business
[*]Corresponding author
E-mail addresses: tsangarisp@aua.gr (S. Tsangaris), anastasio.xepapadeas@unibo.it (A. Xepapadeas), ayannaco@aueb.gr (A. N. Yannacopoulos).

1 Introduction

Economic activity is distributed very unequally across locations, even within the same country. Income inequality, different employment concentration, clustering, uneven growth, and different investment rates reflect the existence of agglomeration forces and spatial externalities. The understating of spatial dynamics that lead to different spatial patterns may answer several interesting questions. Why are there significant growth rate variations within some countries while other countries face economic convergence? Is this an endogenous process? Does the distance between regions with high and low growth rates matter? What are the appropriate policy measures that bridge this gap? Are these policies differentiated by location? Is it optimal for the whole economy to bridge this gap?

Several empirical papers provide evidence on externalities, spillovers, and knowledge diffusion across space. For instance, Van Oort (2002) studies data from municipalities in the Netherlands, whereas Tian et al. (2010) focus on China prefectures from 1991 to 2007, while Zhang et al. (2020) use both cross-country and China's provincial data. In addition, Paci and Usai (1999) and Mion (2004) use data from Italy, while Sedgley and Elmslie (2004), Rosenthal and Strange (2008), and Ciccone and Hall (1993) examine the USA economy. Moreover, Ertur et al. (2006) investigate 138 European regions over the period 1980-1995, whereas Dekle and Eaton (1999) analyze data from Japanese prefectures. Finally, Comin, Dmitriev, et al. (2012) review data from 161 countries over the last 140 years, while Comin and Hobijn (2010) review 166 countries over the previous two centuries. These findings, among others, highlight the need for the incorporation of spatial interactions into growth models.

However, the vast majority of economic growth models take the country as the unit of measure, assuming that economic activity spreads evenly over the country. Over the previous two decades, a few economists have contributed to filling this gap in the literature. They have attempted to combine economic growth with spatial agglomeration into models with forward-looking agents and a continuum spatial domain. Besides, such models incorporate capital mobility or spillovers. This research is surveyed by Quah (2002), Brito (2004), Brock and Xepapadeas (2008, 2009, 2020), Boucekkine, Camacho, and Zou (2009), Boucekkine, Camacho, and Fabbri (2013), Brock, Xepapadeas, and Yannacopoulos (2014), Fabbri (2016), Ballestra (2016), Boucekkine, Fabbri, et al. (2019) which study the social optimum problems in the context of Solow, Ramsey, and AK frameworks.

Moreover, Desmet and Rossi-Hansberg (2010, 2014) present a spatial growth model with two sectors where areas can accumulate technology through investment in innovation, which is the outcome of profit-maximizing agents. They assume that goods markets clear sequentially and agents solve a static problem. In other words, agents are not forward-looking and make their decisions based on static optimization. However, this structure makes the social planner problem intractable, as argued in Desmet and Rossi-Hansberg (2014).

This paper incorporates geographically ordered space into an endogenous growth model where growth results from intentional investment decisions made by forward-looking profit-maximizing agents, specifically the expanding variety growth model, developed by Romer (1990) and expanded by many economists like Aghion and Howitt (1992), Jones (1995), Aghion, Akcigit, et al. (2014), Acemoglu and Restrepo (2018), and Jones and Kim (2018). The model we present follows the basic structure of endogenous growth models. However, the economic activity takes place within a continuous finite spatial domain. The spatial components are introduced through knowledge diffusion across space, spatial production externalities, and the capability for spatial heterogeneity. These kinds of diffusion and externalities across space are modeled by an integrable kernel function, which models the interconnections between regions. Similar theoretical modeling has previously been used to model spatial production externalities and knowledge diffusion in Lucas (2001), Quah (2002), Lucas and Rossi-Hansberg (2002), Rossi-Hansberg (2004), Brock and Xepapadeas (2009), and Kyriakopoulou and Xepapadeas (2013, 2017).

The internalization of spatial interactions reveals distortions to the allocation of resources to R&D that have not been previously studied. Some of them promote under-investment, whereas others boost over-investment, and there is a case that the same distortion has different effects on different areas. For example, knowledge diffusion and production externalities may enhance the R&D process in some areas while acting as a deterrent to other regions since they decrease the value of existing patents and act as forces that reduce the number of researchers. Additionally, we demonstrate how the spatial model can confront the main empirical critique of the expanding variety growth model, namely the scale effect feature. So, this model presents different dynamic outcomes from its non-spatial counterpart, which affect appropriate policy measures. Finally, numerical examples are presented, using micro and macro data, and we use the US economy as a case study to present some stylized facts.

The rest of the article is organized as follows. Section 2 describes the model, how we incorporate spatial interaction across space, and analyzes the decentralized model, and the social planner problem, while section 3 describes the welfare properties. Section 4 includes the proposed policy measures while section 5 presents numerical examples and stylized facts. Section 6 concludes.

2 The model

We consider an economy developing on the spatial domain $\mathbb{S} = [S_0, S_1]$. Time is continuous $t \epsilon [0, \infty)$ and $s \epsilon \mathbb{S}$ denotes a specific point on the spatial domain. We assume that each location produces one final good $Y(t, s)$ and let $L(s)$ denote the total number of agents at each spatial point. That is, we assume that labor is fixed and immobile.

The economy of each site consists of three sectors: a research sector, an intermediate-goods sector, and a final-goods sector. The research sector creates new ideas. Ideas and inventions correspond to the creation of new capital goods that can be used as inputs in the final-goods sector. The intermediate-goods firm manufactures the capital goods, and the final-goods sector produces the final good. Agents at each location are used either to the final-goods sector (L_y) or to engage in the research sector and create new knowledge (L_A). We consider that the production process in the final-goods sector and the research sector depends not only on the knowledge accumulation in the location s but also on the knowledge accumulated in the neighboring regions.

2.1 Market equilibrium

The decentralized setup follows the common assumption, in this literature, that the economy consists of three sectors. Each of these sectors will be discussed in turn.

2.1.1 The final-goods sector

The final-goods sector of each region consists of a large number of firms. These firms are fully competitive in input and output markets. The final good is the numeraire good, which may be either consumed or invested. The final output is produced using labor and the capital-intermediate goods $x_j(t, s)$ according to the production function:

$$Y(t,s) = e^{b_1 z(t,s)} L_y(t,s)^{1-a} \left(\sum_{j=1}^{A(t,s)} x_j(t,s)^{\gamma a}\right)^{\frac{1}{\gamma}} \tag{1}$$

where $L_y(t,s)$ is the labor used to produce output, $A(t,s)$ is the stock of knowledge at any given point in time and space and γ is the substitution parameter which determines the degree of substitutability between intermediate goods. In other words, higher γ corresponds to more substitutable input[1].

The stock of knowledge or ideas in this model corresponds to the creation of new intermediate goods that can be used by the final-goods sector. That is, A measures the number of capital-intermediate goods that are available to be used in the final-goods sector and is taken as given by the firms. Finally, z is the production externality, which depends on how many workers are employed in the final-goods sector at all locations and represents positive knowledge spillovers, which has to be modeled. To do so, we use integral equations.

Assumption 1 *We assume that the production externality is described by the following integral equation:.*

$$z(t,s) = c_1 \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ln(L_y(t,s'))ds' \tag{2}$$

where $e^{-c_1(s-s')^2}$ is a kernel function that models the effects that position s' has on s. This kernel formation can reveal the positive effects and knowledge spillovers coming from the labor force by neighboring regions (i.e., workers at a spatial point benefit from labor in nearby areas). The parameter b_1 in (1) shows the intensity of this impact. Since equation (2) quantifies spatial knowledge spillovers, it is reasonable to assume that the influence between s and s' decays as the distance between them is increasing. A high c_1 indicates that the effects between distant locations are weak. In other words, the higher c_1 is, the more profitable it is for the final good producers of location s, if nearby areas have a high level of employment to the final-goods sector.

In the decentralized setup each agent does not realize that its actions affect productivity in other areas.

Assumption 2 *Each firm at each spatial site regards knowledge spillovers as exogenous. So, the spatial externality is not internalized.*

[1] The choice of the production function in the final-goods sector would affect the demand function for the intermediate sector. With a Cobb Douglas production function, as in Romer (1990), and capital share equal to 1/3, the monopolist intermediate firm would charge a price that is three times greater than its marginal cost. This value is far away from empirical evidence. The CES specification eliminates this problem. More detailed information about the link between the production function, the markup, the capital share, and the empirical evidence can be found in Jones and Williams (2000).

In other words, we assume that final good firms and inventors operating at site s treat the production externalities and knowledge spillovers produced nearby as exogenous parameters. This myopic view makes the conclusions derived in the case of the equilibrium studied here, differ from those of the optimal solution.

The final-goods producers maximize their profits, taking the price of their inputs, labor (w_y) and capital goods/varieties (p_{xj}) as given and regards the production externality z as exogenous-fixed parameter z^{ex}. The problem of the representative final-goods producer at each site is:

$$\max_{L_y(t,s),x_j(t,s)} e^{b_1 z^{ex}(t,s)} L_y(t,s)^{1-a} (\sum_{j=1}^{A(t,s)} x_j(t,s)^{\gamma a})^{\frac{1}{\gamma}} - \sum_{j=1}^{A(t,s)} p_{x_j} x_j(t,s) - w_y(t,s) L_y(t,s) \quad (3)$$

Simplifying the notation, the optimal rules are:

$$w_y = (1-a)\frac{Y}{L_y} \quad (4)$$

and

$$p_{x_j} = a e^{b_1 z^{ex}} L_y^{1-a} (\sum_{j=1}^{A} x_j^{\gamma a})^{\frac{1}{\gamma}-1} x_j^{\gamma a - 1} \quad (5)$$

Clearly, equation (4) shows that there is a regional wage divergence since Y and L_y are space dependent.

2.1.2 The intermediate-capital goods production

The intermediate-goods sector at each region consists of monopolists who produce the capital goods. Only one firm can manufacture each capital good (patent protection). That is, the number of capital goods $A(t,s)$ is equal to the number of intermediate firms. Moreover, firms understand that they face a downward slopping demand for their output. So, an intermediate good firm chooses the level of output $x_j(t,s)$ to maximize its revenue minus the variable cost:

$$\Pi_{int}(t,s) = \max_{x_j} \gamma p_{x_j}(t,s) x_j(t,s) - (r(t,s)+m) x_j(t,s) \quad (6)$$

where m is the depreciation rate, r the interest rate, and p_{x_j} is given by (5). This implicit assumption behind this specification is that the technology of producing intermediate goods can transform one unit of raw capital into one unit of specialized capital good. What gives to the intermediate firms the knowledge to transform goods to inputs for the output sector is the blueprints hold. A blueprint is simply the technology or the know-how for transforming goods to intermediate inputs. Dropping the subscript, and given the symmetry, we have that:

$$p_x = \frac{r+m}{\gamma a} \tag{7}$$

The intermediate-goods firm decides whether or not to enter the market. The analysis of the firm entry into capital good market resembles the arbitrage argument followed by Romer (1990). So, let us recall very briefly the basic condition which gives the price $p_{R\&D}$ of a blueprint (patent) . Since Π_{int} is the profit from the new patent and r the instantaneous real interest rate, then:

$$r(t,s)p_{R\&D}(t,s) = \Pi_{int}(t,s) + \dot{p}_{R\&D}(t,s) \tag{8}$$

where $\dot{p}_{R\&D}(t,s)$ is the partial derivative of $p_{R\&D}(t,s)$ with respect to time. This condition implies that in equilibrium at every point in time and space the interest earned from investing $p_{R\&D}$ in the capital market must be equal to the blueprint producer's revenue plus the gain or loss on the resale price of the patent.

2.1.3 The research sector - The production of blueprints

To model technological progress, we base on the below R&D equation:

$$\frac{\partial A(t,s)}{\partial t} = \delta(t,s)e^{c_2 q(t,s)}e^{c_3 h(t,s)}A(t,s)L_A(t,s)^{\lambda(t,s)} \tag{9}$$

where $\delta(t,s) > 0$ is the productivity of research activities, $A(t,s)$ is the existing stock of knowledge, and $L_A(t,s)$ is the number of researchers and engineers. The parameter $\lambda(t,s)$ lies between zero and one since, as argued by Jones (1995), the duplication and overlap of research reduce the total number of innovations produced by L_A units of labor. That is, suppose that it is not L_A but rather a smaller percentage that belongs in the R&D equation occurring because of duplication in the R&D process[2]. The term $e^{c_2 q(t,s)+c_3 h(t,s)}$ is introduced to model spatial interaction into the R&D processes, where

[2]Another interpretation for this parameter is that there are more low-skilled researchers when a larger portion of the labor force enters the research sector.

q follows:

$$q(t,s) = c_2 \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ln(L_A(t,s'))ds' \tag{10}$$

Equation (10) has similar interpretation as equation (2). In other words, equation (10) quantifies the positive effects and production externalities coming by neighboring regions from the aggregate employment in R&D and positively affects the rate at which R&D creates new ideas in the locations s.

Assumption 3 *The growth rate of the R&D process depends both on the current level of knowledge in a specific site and on a weighted average of growth rates of neighborhood areas.*

Under Assumption 3 we have that:

$$h(t,s) = c_3 \int_{S_0}^{S_1} e^{-c_3(s-s')^2} \frac{\partial A(t,s')/\partial t}{A(t,s')} ds' \tag{11}$$

So, a higher growth rate of R&D leads to higher growth rates for the neighboring regions. That is to say, location s exploits the innovations developed by other regions in order to promote the advancement of knowledge in the local economy. The parameters b_2 and b_3 indicate the intensity of spillovers while c_2 and c_3 show if the effects between distant locations are weak or strong. Since diffusion parameters c_2 and c_3 decline exponentially with distance, very high diffusion parameters mean that only the R&D process in nearby areas is affected positively.

Remark 1 *Note that we distinguish between the two sources of economic interaction, namely knowledge spillover and technical diffusion. The reason is that knowledge spillover lies in learning by doing as argued by Arrow (1962), while technical diffusion takes place through the spread of technology as argued by Eaton and Kortum (1999).*

The R&D equation implies that there is no aggregate uncertainty in the R&D process. Although there will be uncertainty for each R&D lab, this specification turns out to be consistent if it is assumed that with many different blueprint producers, the R&D equation holds deterministically. Each R&D lab is small relative to the R&D sector. Therefore, it can predict that its contribution to the aggregate level of technology would be negligible. Taking into account the duplication effect we can express the rate $\bar{\delta}$ at which R&D creates new ideas as:

$$\bar{\delta}(t,s) = \delta A(t,s) e^{b_2 q(t,s)} e^{b_3 h(t,s)} L_A(t,s)^{\lambda-1} \tag{12}$$

Each lab takes as given the rate at which R&D creates new ideas and considers $\bar{\delta}(t,s)$ as an exogenous-fixed parameter. Let $p_{R\&D}$ be the price of a new design-blueprint and w_A the wage. Then the problem for an individual R&D lab is:

$$\max_{L_A(t,s)} p_{R\&D}(t,s)\bar{\delta}(t,s)L_A(t,s) - w_A(t,s)L_A(t,s) \tag{13}$$

Simplifying the notation, the wage at the research sector is:

$$w_A = p_{R\&D}\delta A L_A^{\lambda-1} e^{b_2 q} e^{b_3 h} \tag{14}$$

As is evident from the equation above, there is a regional wage divergence in the research sector as well.

2.1.4 Steady-state properties

We focus on the properties of the balanced growth path, or steady-state, where all model variables need to grow at a constant rate.

Proposition 1 *The labor force devoted to R&D at each location for the decentralized economy is given by:*

$$L_A^{eq}(s,t) = \frac{L(s)}{1 + \frac{1}{a\gamma}(\sigma\frac{1-\gamma\alpha}{\gamma(1-\alpha)} - \frac{1-\gamma}{\gamma(1-\alpha)} + \frac{\rho}{g_A(s,t)})} \equiv \psi(\frac{\partial lnA(s,t)}{\partial t}) \tag{15}$$

where $g_A(s,t)$ is the growth rate of A.

Proof. The definitions of the decentralized equilibrium, along with the derivation of the steady-state allocation, could be found in Appendix A. ■

Not surprisingly, the solution in the non-spatial case coincides with the solution of the spatial counterpart whence $b_2 = b_3 = 0$, and $\lambda = 1$, $\gamma = 1$. That is, setting $b_2 = b_3 = 0$ and $\lambda = \gamma = 1$, equation (15) reveals that:

$$L_A^{eq} = \frac{\delta L - \frac{\rho}{a}}{\delta(1 + \frac{\sigma}{a})} \tag{16}$$

which is the equilibrium outcome in the Romer's model. As can be seen from (15) the R&D share depends on the growth rate of A. So, by substituting (15) into (9) we end up with a non-linear integral equation of the Hammerstein type, where the unknown variable is $g_A(s,t)$:

$$g_A = \delta\left(\frac{L}{1+\frac{1}{a\gamma}(\sigma\frac{1-\gamma\alpha}{\gamma(1-\alpha)}-\frac{1-\gamma}{\gamma(1-\alpha)}+\frac{\rho}{g_A})}\right)^{\lambda}.$$

$$exp\{b_2c_2\int_{S_0}^{S_1} e^{-c_2(s-s')^2}ln\left(\frac{L}{1+\frac{1}{a\gamma}(\sigma\frac{1-\gamma\alpha}{\gamma(1-\alpha)}-\frac{1-\gamma}{\gamma(1-\alpha)}+\frac{\rho}{g_A})}\right)ds' + b_3c_3\int_{S_0}^{S_1} e^{-c_3(s-s')^2}g_A ds'\} \tag{17}$$

A solution of the above equation may be constructed using an iterative interpolation algorithm[3].

2.2 The social planner problem

In the social planner formulation, we would like to share optimally at each point in time and space, the labor force between R&D and the final good production, taking into account both time and space externalities. That is, the social planner internalizes the spatial externalities in the production process and the R&D process and solves:

$$\max_{C(t,s),L_A(t,s)} \int_{S_0}^{S_1}\int_0^{\infty} e^{-\rho t}\frac{C(t,s)^{1-\sigma}-1}{1-\sigma}dtds \tag{18}$$

where $C(t,s) \equiv c(t,s)L(t,s)$, subject to[4]:

$$\frac{\partial K(t,s)}{\partial t} = e^{b_1 z(t,s)}K(t,s)^a A(t,s)^{\frac{1}{\gamma}-a}L_y(t,s)^{1-a} - C(t,s) - mK(t,s) \tag{19}$$

$$\frac{\partial A(t,s)}{\partial t} = \delta A(t,s)L_A(t,s)^{\lambda}e^{b_2 q(t,s)}e^{b_3 h(t,s)} \tag{20}$$

$$L(s) = L_y(t,s) + L_A(t,s)\ \forall s \tag{21}$$

[3] The Mathematica code is given in Appendix C.

[4] Because of the symmetry between the different intermediate goods and the fact that the sum of the intermediate goods equals the total capital stock in each site we can express x in term of A and K.

Proposition 2 *The optimal allocation of labor is:*

$$L_A^{sp}(t,s) = \frac{L(s)}{1 + \frac{1}{\lambda(t,s)d(t,s)}(\sigma - 1 + \frac{\gamma(1-\alpha)}{1-\gamma\alpha} g_A(t,s) b_3 c_3 H(t,s) + \frac{\gamma(1-\alpha)}{1-\gamma\alpha} \frac{\rho}{g_A(t,s)})} \quad (22)$$

where d reflects the interdependence between production externalities in the R&D and the final-goods sector. Stronger production externalities in the R&D lead to $d > 1$. On the contrary, stronger spillovers in the final-goods sector lead to $d < 1$. In the special case where these effects have the same impact then $d = 1$.

Proof. The analytic solution of the social planner problem is given in the Appendix B. ■

As was the market equilibrium case, the solution in Romer's model coincides with the solution of the spatial counterpart when $b_1 = b_2 = b_3 = 0$ and $\lambda = \gamma = 1$. So, setting $b_1 = b_2 = b_3 = 0$ and $\lambda = \gamma = 1$ equation (22) reveals that:

$$L_A^{sp} = \frac{\delta L - \rho}{\delta \sigma} \quad (23)$$

which is the optimal outcome in the Romer's model.

The term $e^{b_2 q} e^{b_3 h}$ models positive spatial externalities in the R&D process. Therefore, externalities in the term $\frac{\rho}{g_A} = \frac{\rho}{\delta L_A^{\lambda} e^{b_2 q} e^{b_3 h}}$ acts as a force that increases the number of research workers according to (22) since $\frac{\gamma(1-\alpha)}{1-\gamma\alpha} > 0$ for plausible parameter values. On the other hand, the term $g_A b_3 c_3 H$ reflects the fact that the diffusion of technology decreases the need for R&D since the social planner internalizes the spatial externality. In other words, the social planner understands that the high diffusion of knowledge allows a region to specialize in the final-goods production since it can exploit innovations developed by other regions. Thus, this term acts as a force that decreases the number of researchers. The interdependence between these two terms, along with d and λ, determines in which location the spatial model chooses a higher level of employment in the R&D sector than the non-spatial model. To put it another way, the presence of spatial production externalities in the production of the final output and the R&D process and the knowledge spillovers cause the difference between the spatial and the non-spatial case.

Substituting (22) into the R&D equation (20) we obtain:

$$g_A = \delta\left(\frac{L}{1 + \frac{1}{\lambda d}(\sigma - 1 + \frac{\gamma(1-\alpha)}{1-\gamma\alpha} g_A b_3 c_3 H + \frac{\gamma(1-\alpha)}{1-\gamma\alpha} \frac{\rho}{g_A})}\right)^{\lambda}.$$

$$exp\{b_2c_2\int_{S_0}^{S_1} e^{-c_2(s-s')^2} ln(\frac{L}{1+\frac{1}{\lambda d}(\sigma-1+\frac{\gamma(1-\alpha)}{1-\gamma\alpha}g_Ab_3c_3H+\frac{\gamma(1-\alpha)}{1-\gamma\alpha}\frac{\rho}{g_A})})ds'+b_3c_3\int_{S_0}^{S_1} e^{-c_3(s-s')^2}g_Ads'\} \tag{24}$$

3 Social planner vs. equilibrium outcome

As was expected, there is a divergence between the social planner and the equilibrium outcome. Seven factors give rise to this divergence. Comparison between (15) and (22) will highlight these factors. First, there is the fact that the intermediate-goods sector is monopolistic. Making comparison between (15) and (22) we can see that the equilibrium outcome includes the term $\frac{1}{a\gamma}$. This term is the monopoly markup. As a result, intermediate goods producers produce too little of each variety. Second, there is an inter-regional spillover effect which may lead to over-invest in R&D. Specifically, the term $g_Ab_3c_3H$ in (22) depicts the fact that the diffusion of technology decreases the need for R&D and acts as a force that decreases the number of researchers. So, the social planner internalizes knowledge diffusion and decreases the number of researchers when a region can exploit the R&D from nearby regions. The presence of spatial externalities in the final good and the R&D production process are two additional reasons why the social planner outcome differs from the equilibrium outcome. Namely, the ratio $\frac{1+\frac{b_2c_2}{\lambda}\int_{S_0}^{S_1} e^{-c_2(s-s')^2}ds'}{1+\frac{b_1c_1}{1-\alpha}\int_{S_0}^{S_1} e^{-c_1(s-s')^2}ds'} \equiv d$ shows the effect of these externalities in the optimal outcome and shows that the social planner takes into account the spatial externalities at all sectors while the equilibrium outcome does not take into account the production externality in the final-goods sector. In other words, d shows the interdependence between externalities in different sectors. The fifth reason is the externality due to the duplication of research, the parameter λ. The decentralized equilibrium does not take into account this parameter, which may cause to over-invest in R&D. The sixth reason is the inter-regional spillover effect which affects the decision of a firm to enter or not into capital good market through the arbitrage equation $p_{R\&D}(t,s) = \frac{\Pi_{int}(t,s)}{r(t,s)-\frac{1-\gamma}{\gamma(1-\alpha)}g_A}$. The diffusion of knowledge boosts R&D and increases intermediate goods. This allows the final-goods sector to use different bundles of inputs, decreasing the value of patent rights and discouraging intermediate firms from entering the market, reducing resources from the R&D sector. The term $\frac{1-\gamma}{\gamma(1-\alpha)}$ in (15) depicts this distortion, that the diffusion of technology decreases the value of an existing patent and acts as a force that decreases the number of researchers. The final reason is the intertemporal spillover effect since private firms do not internalize the fact that a new design increases all future researchers' productivity. To witness this, let us assume that we can implement a policy that corrects the inefficiency caused by the monopoly power in the production of intermediates and the inefficiency which is caused when the agents do not take into account the knowledge spillovers (i.e., the term $g_A b_3 c_3 H$). Moreover, for simplicity, assume that the production externalities in the first and third sectors have the same magnitude, viz. $d = 1$, that there is no duplication of research i.e., $\lambda = 1$ and that the substitution parameter γ is equal to one. These assumptions simplify the comparison between (15) and (22) and emphasize the last reason which causes the difference between the equilibrium and the regulator's optimum. Specifically, we have to compare $L_A^{eq} = \frac{L}{1+\sigma+\frac{\rho}{g_A}}$ with $L_A^{sp} = \frac{L}{\sigma+\frac{\rho}{g_A}}$. Note that in the decentralized economy, the externalities across time in the R&D process are not internalized, causing a dynamic inefficiency in the allocation of labor, i.e., $L_A^{eq} < L_A^{sp}$. The monopoly markup and the intertemporal spillover effect have already been noted by Romer (1990), while the duplication of research has been noted by Jones (1995). The other reasons are closely linked to the spatial aspect of the model. That is, the presence of spatial externalities in the production of the final output and the R&D process and the presence of knowledge spillovers.

4 Policy analysis

The divergence between the social planner and the equilibrium outcome indicates the need to implement appropriate policy measures to correct the market imperfections. Our goal is to introduce several policy measures in order to induce the private economy to attain the social optimum without canceling the profit motive for the R&D and the intermediate sectors to create new types of products. We propose the following policy measures based on the market imperfection, which we would like to correct.

4.1 Subsidies in the production of the intermediate goods

Consider that the policymaker establishes a lump-sum tax to finance a subsidy τ_x at the production of intermediate goods. Such a subsidy alters the problem for an individual intermediate firm:

$$\Pi_{int} = \underset{x_j}{max}\ \gamma p_{x_j} x_j - \frac{(r+m)}{(1+\tau_x)} x_j \tag{25}$$

Therefore, the price of intermediates would be:

$$p_x = \frac{r+m}{\gamma a(1+\tau_x)} \tag{26}$$

and the quantity:

$$x = (\frac{\gamma(1+\tau_x)}{r+m})^{\frac{1}{1-a}} a^{\frac{2}{1-a}} e^{\frac{b_1 z}{1-a}} A^{\frac{1-\gamma}{\gamma(1-\alpha)}} L_y \tag{27}$$

while the final good would be given by:

$$Y = a^{\frac{2a}{1-a}} e^{\frac{b_1 z}{1-a}} A^{\frac{1-\gamma\alpha}{\gamma(1-\alpha)}} (\frac{\gamma(1+\tau_x)}{r+m})^{\frac{a}{1-a}} L_y \tag{28}$$

and the wage at the output sector would be:

$$w_y = (1-a)\frac{Y}{L_y} = (1-a)a^{\frac{2a}{1-a}} e^{\frac{b_1 z}{1-a}} A^{\frac{1-\gamma\alpha}{\gamma(1-\alpha)}} (\frac{\gamma(1+\tau_x)}{r+m})^{\frac{a}{1-a}} \tag{29}$$

Thus, implementation of this policy would decrease the price and increase the quantity of intermediate goods as we can see from (26) and (27). Moreover, this policy would affect the final-goods sector by increasing the final output as we can see from (28). Although, since the wage at the final-goods sector would have been increased as demonstrated by (29), the allocation of the labor force between the sectors of the economy would remain the same. This can be confirmed following the same solution procedure as in Appendix A. So, the subsidy at the production of intermediate goods would not have impact on the growth rate, the interest rate, and the allocation of the labor force. Setting the same subsidy amount at all regions:

$$\tau_x = \frac{1-\gamma a}{\gamma a} \tag{30}$$

would eliminate the monopoly pricing problem.

4.2 Patent subsidy

On the one hand, knowledge diffusion boosts economic activity in nearby locations. On the other hand, it acts as a deterrent to a firm's decision to buy a patent and participate in the capital goods market. This happens because a new intermediate good may replace an existing one. So, an intermediate firm would pay less for a patent since the diffusion of knowledge will accelerate the price reduction. This means fewer funds in the R&D sector. In this way, the combination of patent protection and high knowledge diffusion from nearby sites decreases the incentives for R&D. Consider that the policymaker would like to neutralize this negative effect and establish a lump-sum funding scheme to assist intermediate good firms to enter the market. This subsidy will differ based on location and spatial knowledge diffusion intensity and will take place only once when potential firms bid for a patent. So, let $\tau_v(t,s)$ the subsidy and using the arbitrage equation we have:

$$\frac{p_{R\&D}}{1+\tau_v} = \frac{\Pi_{int}}{r(t,s) + \frac{\gamma-1}{\gamma(1-\alpha)} g_A} \tag{31}$$

Setting the spatial dependent subsidy:

$$\tau_v = \frac{(\gamma-1) g_A(t,s)}{\gamma(1-\alpha) r(t,s)} \tag{32}$$

one can see that:

$$p_{R\&D}(t,s) = \frac{\Pi_{int}}{r} \tag{33}$$

This policy would enhance the growth rate and would alter the labor allocation in order to cope with the inefficiency caused by the knowledge diffusion. Simply put, it will increase the number of researcher:

$$L_A^{eq} = \frac{L}{1 + \frac{1}{a\gamma}\left(\frac{\sigma(1-\gamma\alpha)}{\gamma(1-\alpha)} + \frac{\rho}{g_A}\right)} \tag{34}$$

4.3 Taxes/subsidies on labor in the final-good production

Let us assume that we have implemented a policy that corrects the inefficiency caused by the knowledge diffusion and the monopoly pricing problem. Then, a tax or subsidy, τ_w, on labor can correct the inefficiency caused by the misallocation of labor. The problem of the firm in the final-goods sector is:

$$\max_{L_y, x_j} e^{b_1 z^{ex}} L_y^{1-a} \left(\sum_{j=1}^{A} x_j^{\gamma a}\right)^{\frac{1}{\gamma}} - \sum_{j=1}^{A} p_{x_j} x_j - (1+\tau_w) w_y L_y \tag{35}$$

Therefore, the wage is given by:

$$w_y = \frac{(1-a)}{(1+\tau_w)} \frac{Y}{L_y} \tag{36}$$

With some algebra, one can show that in this case the labor working in the research sector is:

$$L_A^{eq} = \frac{L}{1 + \frac{1}{(1+\tau_w)a\gamma}(\sigma\frac{1-\gamma\alpha}{\gamma(1-\alpha)} - \frac{1-\gamma}{\gamma(1-\alpha)} + \frac{\rho}{g_A})} \tag{37}$$

The tax/subsidy rate will be spatial dependent, and the policymaker will adjust the rate in each location. For some parameter values, there is a case that the appropriate policy is a tax for some sites, and for other sites, the appropriate policy is a subsidy. The policymaker must take into account not only the sign and magnitude of parameter values in each location but also the sign and magnitude of parameter values in the nearby areas along with the intensity of the positive knowledge diffusion and the production spillovers. For example, ceteris paribus, the level of population in location s may affect the optimal allocation of labor in location s' and change the optimal policy measures completely. This effect highlights the fact that the inclusion of space into the model can significantly alter the results and the policy measures.

Remark 2 *Each policy alone does not allow the economy to reach the first-best. It is a policy scheme that combines uniform subsidies in the production of the intermediate goods and site-specific patent subsidies as well spatially-dependent taxes/subsidies on labor in the final-goods sector that will enable the economy to achieve the first-best Pareto optimum outcome.*

5 Numerical experiments and stylized facts

5.1 Numerical experiments

5.1.1 Spatial Homogeneity

The contribution of this section is to predict the dynamics of the distribution of economic activity in space and helps us to better understand the differences between the decentralized economy and the social planner problem. The fundamental question is whether we invest too little or too much in R&D. The answer to this question will determine the appropriate policy measure in order to bridge the difference between the social planner and the decentralized model. In attempting to answer this question, it will be necessary to assign numerical values to the parameters and notice how some key parameters alter the spatial distribution of economic activity. Although the numerical values for this exercise were based on calibration parameters used in literature, this is not an attempt to replicate a real economy's spatial evolution. We want to show how the inclusion of the space dimension into an endogenous growth model leads to different dynamic outcomes from the standard model, altering the key results and predictions and, therefore, the appropriate policy instruments.

As argued by Jones and Williams (2000), substitution parameter γ ranges between 0.5 and 2.77. We set $\gamma = 1.2$. We follow Jones (1995) regarding the capital share, $a = 0.33$. The spatial domain's length is chosen to be $S = 2\pi$, the interest rate is $\rho = 0.04$, and the risk aversion coefficient is $\sigma = 2.5$. These parameter values will not be varied in any of the following simulations. One might argue that different locations may have different sets of production and utility elements. In other words, all the aforementioned parameters have different values at various locations. The combinations are numerous, and for this reason, we will allow only the total population to vary across regions. This is a well-grounded assumption if we study areas of the same country since they have similar laws, customs, institutions, production, and consumption characteristics. We set the total number of agents at each spatial point at $L = 100$ for the first simulations, and we will change the population distribution for the following simulations. The key parameter λ is allowed to take values in the range $(0, 1]$. The rates at which knowledge diffusion and production spillovers affect the economic activity b_1, b_2, b_3 are over the range of 0.01 to 0.04[5]. The strength of knowledge diffusion and the production externalities $c_{1,}\ c_2,\ c_3$ depend on the length of the spatial domain. Change in these parameters affects the rate at which diffusion and externalities decay with distance. Lucas (2001), modeling the spatial structure of cities, let the parameter vary over the values 0.1, 1, and 5, while Lucas and Rossi-Hansberg (2002) assume higher values. Comin, Dmitriev, and Rossi-Hansberg (2012), analyzing data from 161 countries over the last 140 years, estimate the diffusion parameter at 1.5, given that the spatial domain is 1,000 kilometers. Based on these findings, Desmet and Rossi-Hansberg (2014) assume a spatial domain of 5,000 kilometers which is the distance between the East Coast and the West Coast of the United States, and set the diffusion parameter equal to 7.5. Given that our purpose is not to model the economy of a specific country but to investigate and illustrate the effects of spatial diffusion and externalities, we assume that $c_{1,}\ c_2,\ c_3$ ranging from 1 to 7.5. Finally, the parameter δ is equal to 0.01[6].

[5]We follow the literature setting low values for b_1, b_2, b_3 although we do not have to worry about the "No-Black-Hole" assumption, described in Fujita, Krugman, and Venables (1999) and Fujita and Thisse (2013), due to labor immobility.

[6]In view of the fact that we use baseline parameter values and since we have normalized the population to 100, the parameter δ is to range between 0.01 and 0.04 in order for the R&D equation

To begin with, assume that there are no spatial externalities, that is $b_1 = b_2 = b_3 = 0$ or $c_1 = c_2 = c_3 = 0$. Then, the labor force engaged in the R&D sector is the same at all sites and equal to 9.5% in both problems when $\lambda = 0.39$. On the one hand, when λ is greater than 0.39, the decentralized economy under-invests in R&D. On the other hand, when λ is less than 0.39 then the decentralized economy over-invests in R&D. However, this is not the case when spatial interactions are taken into account. Figure 1 shows the distributions of labor in the R&D sector for $\lambda = 0.35$ for the non-spatial model (left) and the spatial model (right) when production externalities in the final sector, and knowledge diffusion are low relative to production externalities in the R&D sector. Specifically when $b_1 = b_3 = 0.01, b_2 = 0.02$ and $c_1 = c_3 = 1, c_2 = 2$. The solid line corresponds to the equilibrium outcome, while the dashed line corresponds to the social planner outcome.

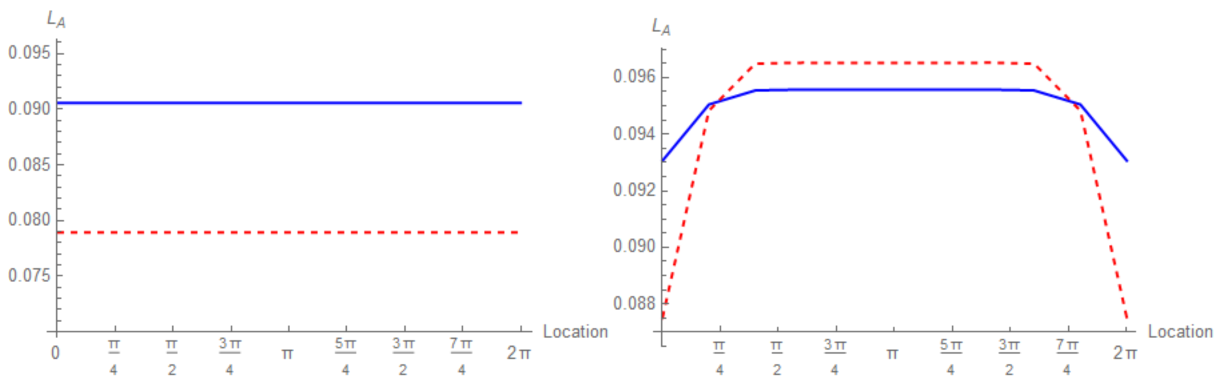

Figure 1: R&D shares: social optimum (dashed line) vs. the decentralized economy (solid line) for $\lambda = 0.35$. Non-spatial model (left) vs. spatial model (right) when production externalities in the final sector and knowledge diffusion are low relative to production externalities in the R&D sector.

As we can see from figure 1, without internalizing the spatial interactions and $\lambda = 0.35$ the non-spatial model (left) suggests that the decentralized economy over-invests in R&D. As a result, spatially-uniform subsidies on labor in the final good production would eliminate the distortion. However, taking into account spatial interactions and assuming that production externalities in the output sector and knowledge diffusion are low relative to production externalities in the R&D sector the social optimum is that center regions have to devote greater labor share into R&D[7]. This happens because the

to fit cross-country and intra-country growth rates data.

[7]The same result could be obtained with a lower duplication externality but with a higher interest rate or a higher substitution parameter or a higher elasticity of labor in the final-goods sector.

regulator chooses the allocation of labor in order to maximize the aggregate regional welfare. In other words, the regulator wants to exploit the high production externalities from the R&D sector, and for this reason, the optimal allocation of labor in the R&D sector has a bell-shaped pattern. So, taking into account the spatial interaction, we have that at the boundaries of the spatial domain, the optimal R&D labor is less than the equilibrium outcome. At the same time, in the central regions the decentralized economy undertakes too little R&D. Therefore, spatially-dependent instruments are required. Specifically, at the boundaries, the appropriate policy is a subsidy on labor in the final good production. In contrast, in the central regions, the appropriate policy is a tax on the final good production to swift labor to the R&D sector. This example shows that even with similar parameter values at the production function and R&D equation and identical initial conditions, firms and R&D labs' location is essential. Thus, spatial interaction implies the possibility of divergence even among regions with similar characteristics.

Let us now choose a value for λ where the competitive equilibrium under-invests in R&D in the non-spatial case. For this reason, the appropriate policy would have promoted R&D at the same magnitude for all the spatial domain. Figure 2 shows the distributions of labor in the R&D sector for $\lambda = 0.4$ for the non-spatial model (left) and the spatial model (right) when production externalities in the final-goods sector are high relative to production externalities in the R&D sector and the knowledge spillovers. Specifically when $b_1 = 0.025, b_2 = b_3 = 0.01$ and $c_1 = 2, c_2 = c_3 = 1$.

As we can see from figure 2, without internalizing the spatial interactions and $\lambda = 0.40$ the non-spatial model (left) suggests that the decentralized economy under-invests in R&D. As a result, spatially-uniform Pigouvian taxes on labor in the final good production in order to swift labor in the R&D, would eliminate the distortion. Again, taking into account spatial interactions, we have spatially-heterogeneous patterns. As shown in figure 2 the competitive outcome follows the same pattern as in the previous case. On the contrary, the social planner chooses higher R&D intensity at the boundaries than the center altering the outcome from the previous case. This happens because the regulator wants to exploit the high production externalities from the final-goods sector. For this reason, the optimal allocation of labor in the R&D sector has a U-shaped pattern, and for the final output sector, a bell-shaped pattern. Therefore, we need spatially-dependent policy instruments. To be more specific, the optimal policy measures are a tax on labor in the final good production at the boundaries and a subsidy in the central regions. So, the internalization of spatial externalities may alter entirely the policy measure leading to a spatially dependent taxes/subsidies scheme.

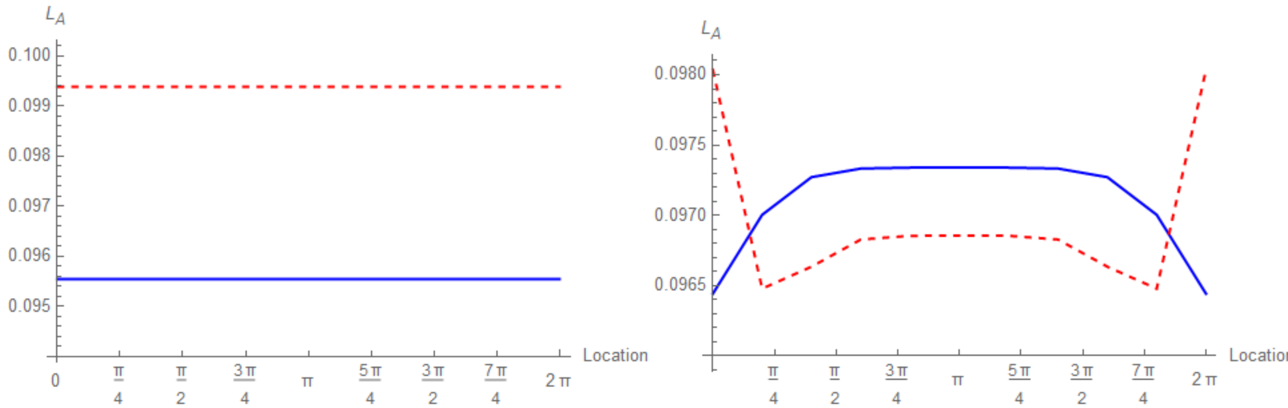

Figure 2: R&D shares: social optimum (dashed line) vs. the decentralized economy (solid line) for $\lambda = 0.40$. Non-spatial model (left) vs. spatial model (right) when production externalities in the final-goods sector are high relative to production externalities in the R&D sector and the knowledge spillovers.

An analogous increase in values of production externalities and knowledge spillovers leads to the same pattern. However, the differences between regions regarding the optimal labor allocation are higher than in the previous cases.

5.1.2 Spatial heterogeneity

Up until now, we have assumed that there are no spatial asymmetries and that all regions are identical. For the next simulations, we fix the parameter λ at 0.39 and will alter the population distribution although the total population at the whole spatial domain will remain the same. Since each site has a different population density, it is reasonable to assume that the rate at which production externalities affect labor productivity depends on the size of the labor force i.e., the population. Regions with a low level of labor force face a greater rate at which production externalities affect labor productivity. In other words, a small village is influenced more by a nearby metropolitan area than a high population density city located near the same metropolitan area. Thus, b_1 and b_2 have different values at different locations. A suitable specification is: $b_1(s) = b_2(s) = v(\frac{Log(maxL)-Log(L(s))}{L(s)})$[8]. So the lowest-density regions face the highest positive impact. The parameter v shows the intensity of this impact. Although, we

[8]This specification is compatible with the flat population distribution. Adding a constant term, $b(s) = b^c + v(\frac{Log(maxL)-Log(L(s))}{L(s)})$, we have a constant b parameter for the uniform population distribution and a space-dependent parameter for the non-uniform case.

choose v in a way that b_1 and b_2 are always and everywhere below the highest acceptable value.

An important feature of expanding variety growth models is that a larger population means a higher growth rate. This positive relationship between the size of the population and the growth rate is called scale effect, and it is the major criticism of the theory of endogenous growth. A lot of empirical papers have investigated the link between the level of population and the growth rate of the economy. Kremer (1993) studies the technological change between one million b.c. to 1990 and finds that regions with larger initial populations and no technological contact with others starting with similar technology would have more significant technological change. On the other hand, many studies using cross-country evidence and time-series data from the last century argue that there is no significant relationship between growth rates and the level of population (see, e.g., Jones (1995), Evans (1996), Dinopoulos and Thompson (1999), Barro and Sala-i-Martin (2004), and Laincz and Peretto (2006)). Moreover, they found that developed countries, at least, are converging to a similar growth rate in the long run. These results are in line with many other studies that look at the growth rates on a regional level. For example, regarding the US economy see Barro and Sala-i-Martin (1992), Tsionas (2000), and Young, Higgins, and Levy (2008). Previous studies use the neoclassical growth model as a framework and diminishing returns to capital as the crucial element for convergence. Relying on empirical findings, many economists have argued that the observable beta convergence is against endogenous growth theory and that technological progress is not an important source of growth.

We would like to show that convergence may be driven by knowledge diffusion and production externalities than differences in capital accumulation and diminishing returns. In order to do so, we have to deal with the scale effect problem. Is there any way, in this theoretical context, to eliminate the scale effect problem? The answer to the previous question depends on the richness of spatial interactions. As we can see from figure 3, when the distance decay parameters are high ($c_1 = c_2 = c_3 = 7.5$), like those found for the US economy, the scale effect problem is eliminated[9].

[9]Another way to deal with the scale effect problem is through the concavity in the returns to research activity. Otherwise stated, we have to include a congestion cost into the model. The logic behind this is simple. Suppose all the population of a country (the spatial domain), and therefore all the researchers, live in one region. In that case, this will affect the elasticity of labor at the R&D equation, namely the parameter λ. Defining the duplication parameter as $\overline{\lambda}(s) = \lambda(1 - \frac{L(s)}{\int_{S_0}^{S_1} L(s)ds})^{\psi}$ the scale effect problem is eliminated. If all the population resides in one spatial point, then the congestion cost is enormous. On the other hand, regions with a low population level face lower duplication externalities.

The first column in figure 3 shows three different population distributions. The first graph displays a U-shaped pattern, while the second displays a bell-shaped pattern. The third row refers to a non-symmetric distribution. The second column shows the growth rates of the decentralized economy for each population distribution. The solid line corresponds to the spatial model, while the dashed line corresponds to the case with no spatial interactions ($c_1 = c_2 = c_3 = 0$).

Regarding the non-spatial model, the scale effect feature is evident since the dashed lines follow the same pattern as the population distribution. On the contrary, the spatial model provides evidence that the growth rates tend to be higher in low populated regions in the balanced growth path. Since we have the growth rate of A at the balanced growth path for each site we can find the level of A given that we have the initial level of A. From (9), the R&D equation, it is evident that the most densely populated regions have a higher initial level of A. Even if we impose the same initial level $A(0, s)$ for all regions, the term $L_A(0, s)^\lambda$, ceteris paribus, will lead regions with a high level of population to produce a proportionally higher level of A. After that, knowledge spillovers rise, and ultimately this allowed poor regions to leave stagnation behind. So, the spatial model exhibits a scale effect at the aggregate level of A but does not exhibit a scale effect on the growth rate.

We note a sharp difference between the spatial and non-spatial models. In the non-spatial model, regions with a high population level and a high level of A tend to grow faster than the low populated regions. This deepens the differences between areas. On the other hand, in the spatial model, we reach the opposite conclusion, namely that spatial interactions force to convergence. This happens because growth at each site depends on three factors. Firstly, the level of population, which is correlated with the number of researchers and the level of new ideas. Secondly, the location and the proximity to more developed areas. Thirdly, the magnitude of spatial interactions. Therefore, strong spatial spillovers smooth over the differences between regions. However, the speed of convergence is very slow since the differences between growth rates are relatively small.

Regarding the second row, it is worth mentioning that the lower growth rates in the boundaries are caused by the combination of bell-shaped population distribution and one-dimensional spatial modeling. Specifically, these areas receive positive impacts only from one side of the spatial domain.

The parameter ψ shows the intensity of the congestion cost.

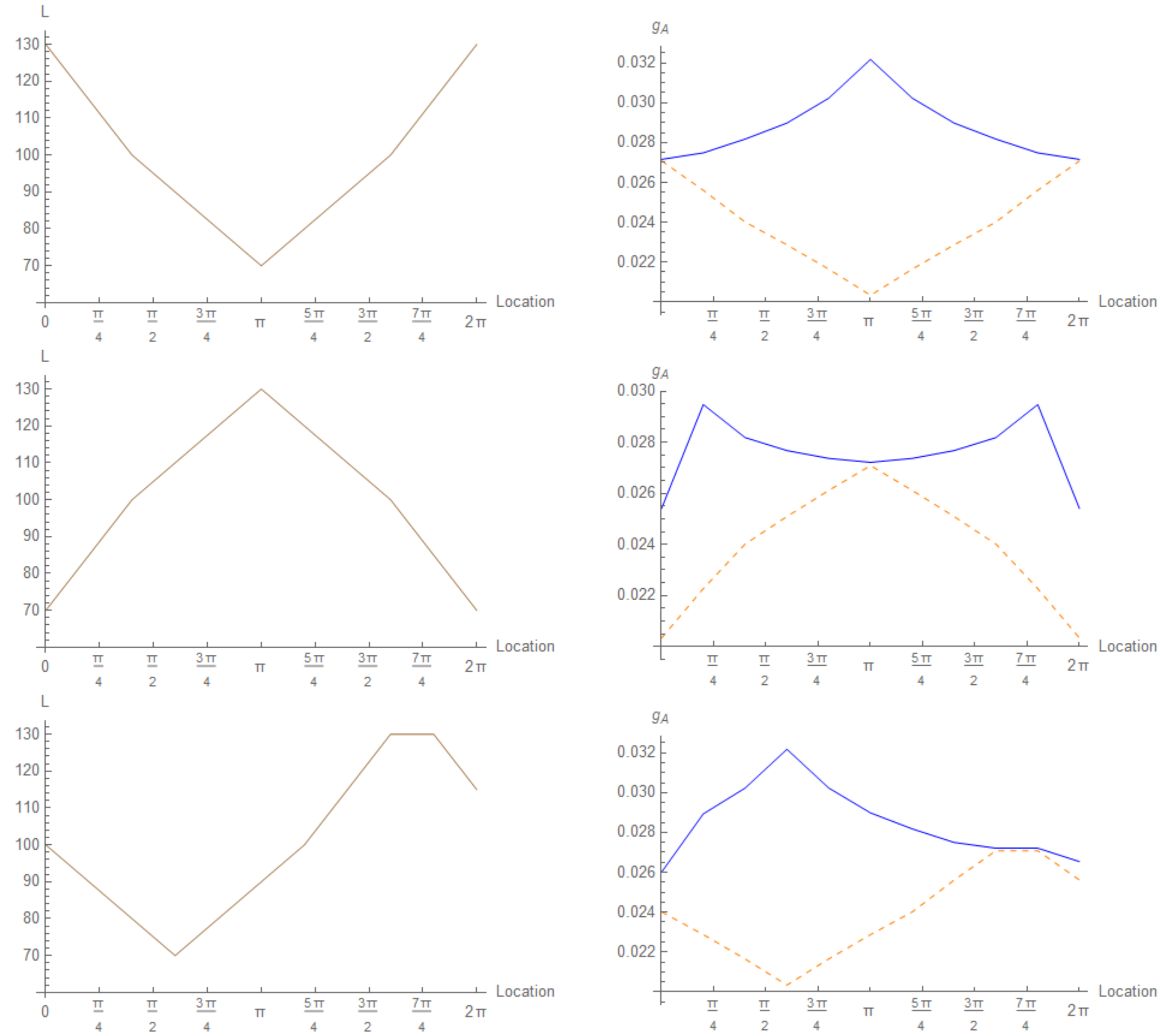

Figure 3: The first column shows the population distributions. The second shows the growth rates. The solid line corresponds to the equilibrium outcome when spatial interactions exist ($c_1, c_2, c_3 = 7.5$), while the dashed line corresponds to the equilibrium outcome when there are no spatial interactions ($c_1, c_2, c_3 = 0$).

Figure 4 shows the distribution of the labor force in the R&D sector. The first column shows the population distributions. The second presents the researchers' allocations in the spatial model, and the third column displays researchers' allocations when spatial interactions are not internalized. The solid line corresponds to the equilibrium outcome, while the dashed line corresponds to the social planner outcome. Comparing the first column with the third, it is evident that the population level and the percentage of researchers and engineers follow the same pattern. Moreover, the social planner of the non-spatial model suggests that most populous regions have to devote more labor to R&D. With this policy, the social planner wants to exploit the scale effect feature that the majority of idea-based growth models exhibit. On the other side, the spatial model features different outcomes. As we can see from the second column, the allocation of scientists and engineers engaged in R&D, in the decentralized case, exhibits a more flat distribution. Regarding the social optimum allocation, the less populous areas have to devote more resources to R&D. This is a totally different result compared to the non-spatial model. The economic interpretation is that the social planner wants to take full advantage of the externalities and knowledge diffusion coming from densely inhabited areas. This does not mean that the less populous regions have to employ more scientists and engineers in absolute terms than the most populated areas. The policy suggestion is that less dense regions have to employ more scientists and engineers in R&D as a share of the total labor force, that is L_A/L.

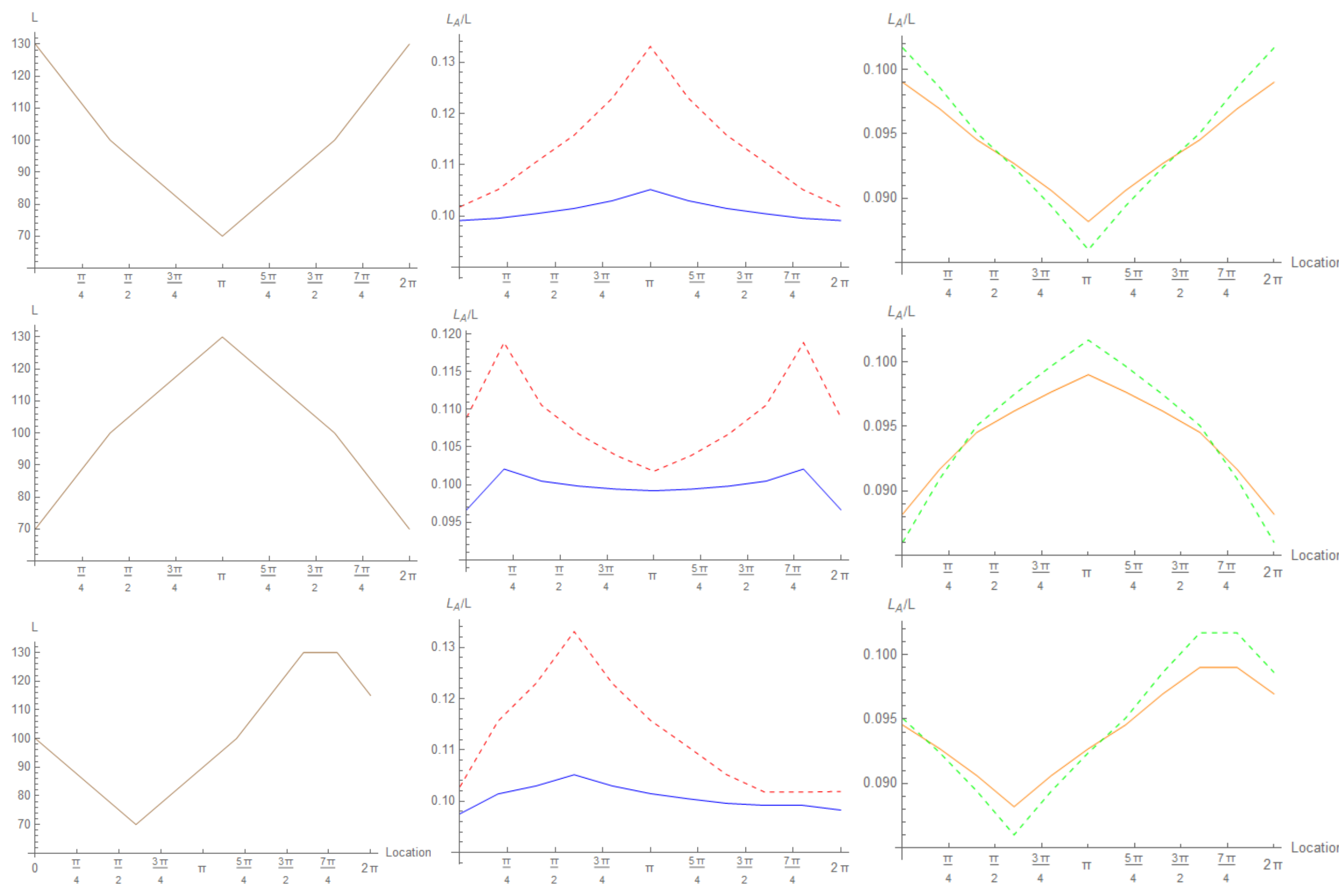

Figure 4: The population distribution (first column) and labor engaged in the R&D sector as a share of the total labor force in the spatial model (second column) and the non-spatial counterpart (third column). The solid line corresponds to the equilibrium outcome, while the dashed line corresponds to the social planner outcome.

By implementing this policy, the low densely populated areas exhibit approximately 15% higher growth rates as shown in figure 5.

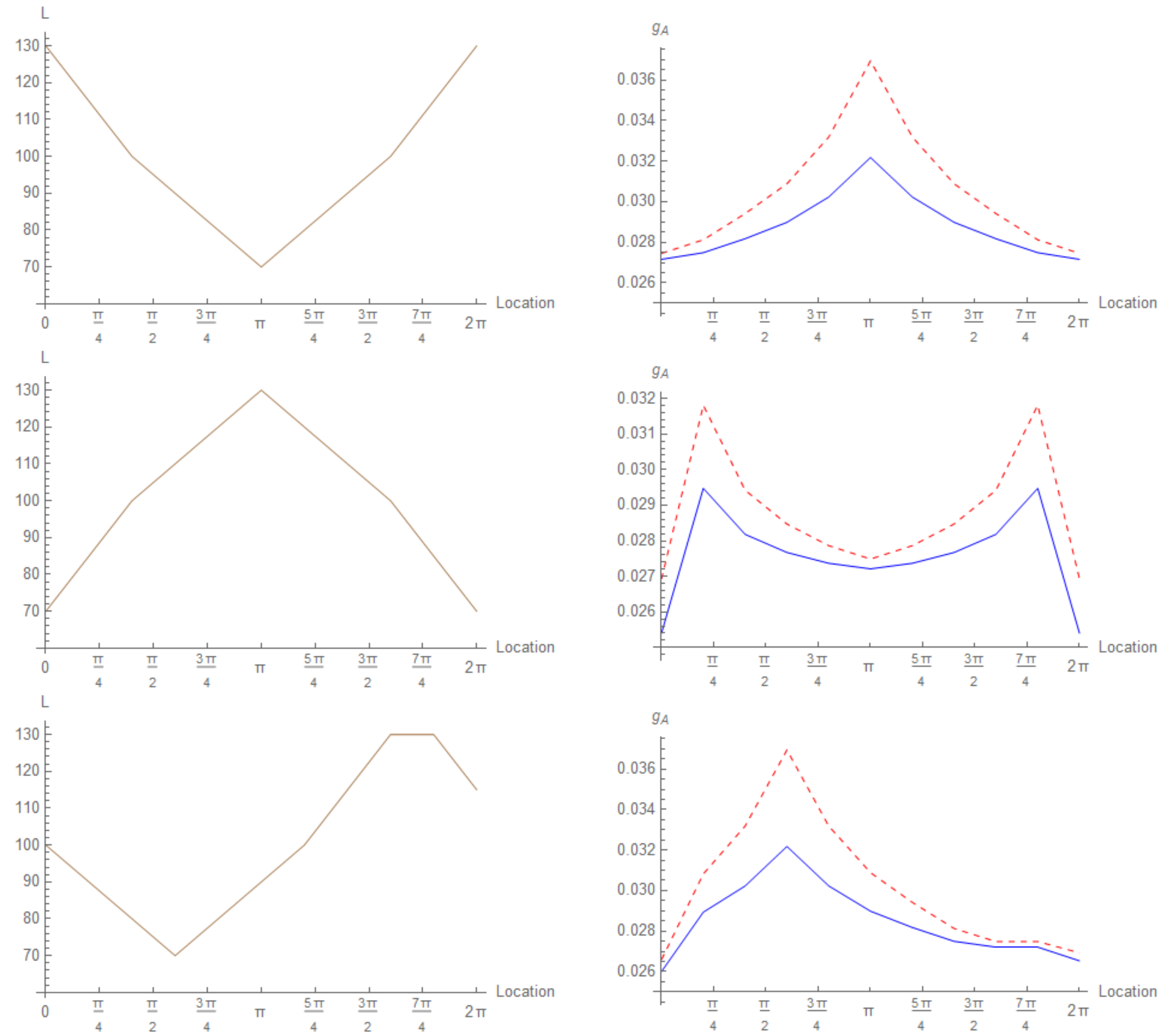

Figure 5: The first column shows the population distributions. The second column shows the growth rates in the spatial model. The solid line corresponds to the competitive equilibrium outcome, while the dashed line corresponds to the social planner outcome.

5.2 Stylized facts

An empirical evaluation of the above findings has to start with identifying the appropriate measurement to estimate the existing stock of knowledge. Unfortunately, we do not have any generally accepted measurement unit to measure the R&D output. According to Brown and Svenson (1988), R&D production could take the form of new products, patents, processes, publications, and fact/knowledge. The outcomes of these are cost reduction, product or sales improvements, and capital avoidance. So, it is impossible to create a measurement system that includes and evaluates all the aspects of the R&D process correctly[10].

A typical indicator for measuring R&D productivity is the number of patents. Although much of the R&D output is not patented and the value of each patent may vary a lot, the patents data "are easily available and they provide a tremendous amount of information about the invention, the inventor and her employer" as reported by Jaffe et al. (1993). Moreover, Griliches (1990), analyzing the sources of growth and the rate of technological change, pointed out that "In this desert of data, patent statistics loom up as a mirage of wonderful plentitude and objectivity."

Our analysis is based on 50 states of the US over the period 1963-2014. The patent data are obtained by the United States Patent and Trademark Office (USPTO), while the population data for each state is derived from the United States Census Bureau (USCB)[11].

Figure 6 shows the level of the population (left map) in the US and the level of patents in 1963 (right map)[12]. Figure 7 shows the same data as figure 6 but for 2014. As we can see, there is a direct relationship between the level of population and the level of patents.

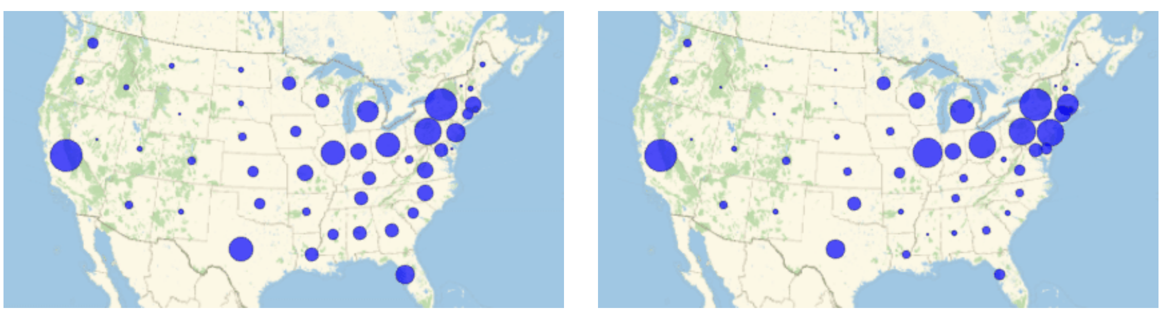

Figure 6: Population level (left) vs. number of patents (right), 1963

[10]For more thorough coverage of problems in measuring knowledge-based growth, see Aghion and Howitt (1998, ch.12).

[11]The database and a Tableau visualization are available at: https://developer.uspto.gov/visualization/utility-patents-state-over-time

[12]The states of Alaska and Hawaii are not shown in the figure, but they are included in the analysis.

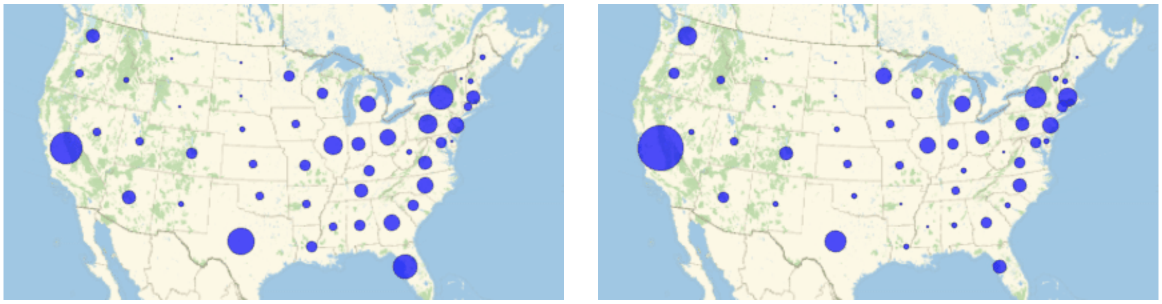

Figure 7: Population level (left) vs. number of patents (right), 2014

Figure 8 shows the average growth rate of patents between 1963 and 2014. A careful comparison with figure 6 reveals an inverse relationship between them.

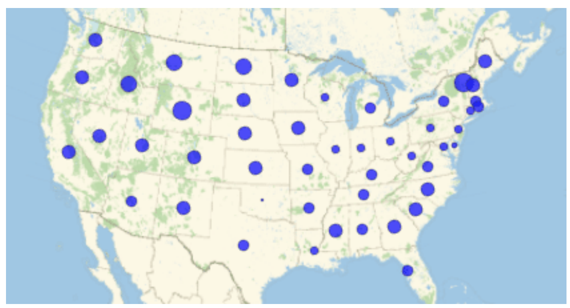

Figure 8: Average growth rate of patents, 1963-2014

Scatter plots between population and the level of patents for 1963 and 2014 confirm the tableau visualization about the connection between the level of population and the patents level (figures 9 and 10). An interesting finding is the inverse relationship between the initial level of R&D output, as measured by the patent level in 1963, and the average growth rates over 1963-2014.

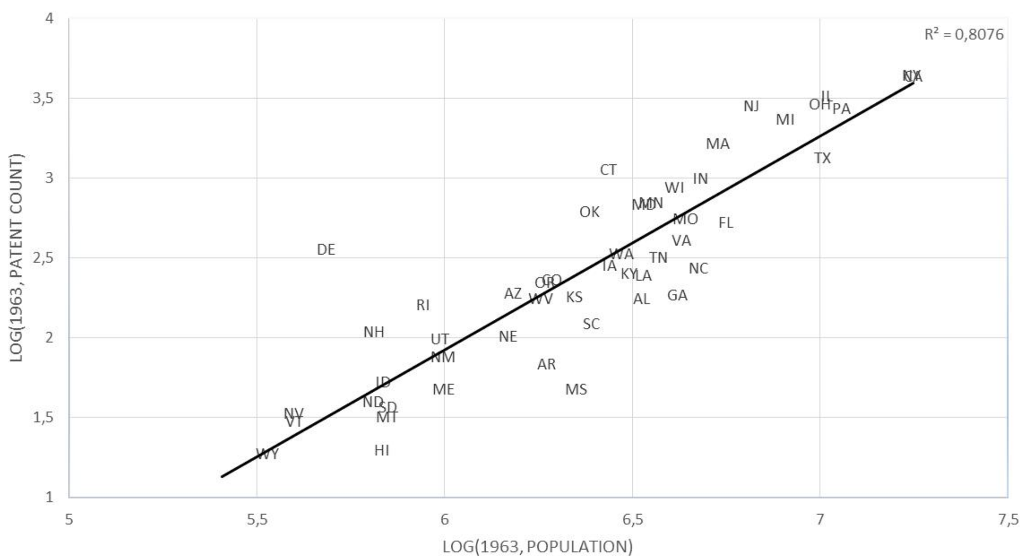

Figure 9: Number of patents vs. population level for 1963

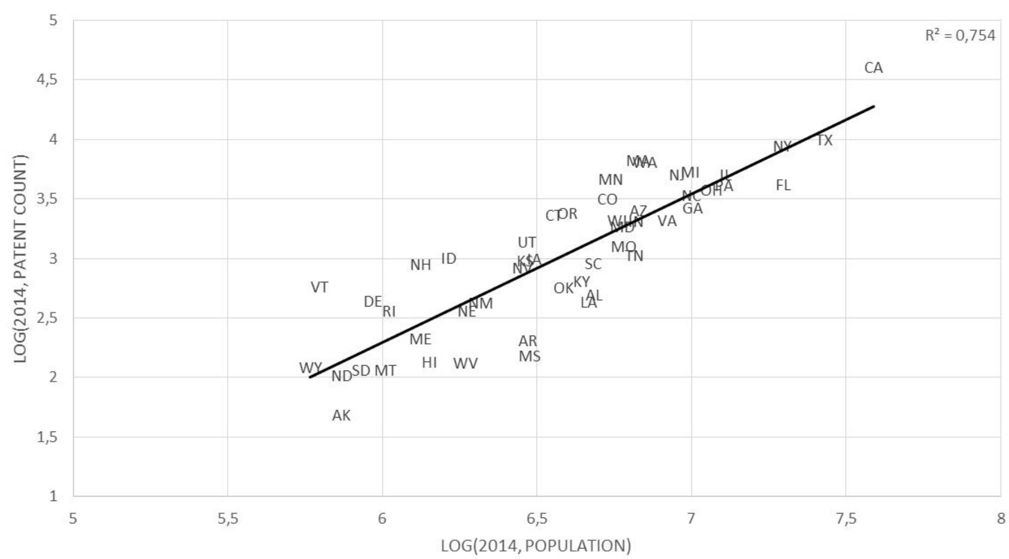

Figure 10: Number of patents vs. population level for 2014

Figure 11 displays the average growth rates from 1963 to 2014 against the level of patents in 1963, i.e., the initial level of R&D output, while figure 12 displays the average per capita growth rates from 1963 to 2014 against the level of patents in 1963. The simple correlation is 0.517 for the patent count growth rate and 0.562 for per capita terms.

The results provide evidence that states with an initial low level of R&D output tend to grow faster than states with a high level of R&D output. Therefore, we can argue that this inverse relationship provides evidence of convergence in the sense that the growth rate is higher the lower the initial level of R&D. This link is stronger for low populated states. Figure 13 shows the connection between the initial level of R&D and the per capita growth rate of patents but only for those states where the level of population is below the average population size (34 states) while figure 14 shows the same link for the 20 less densely populated states in 1963. The simple correlations are 0.611 and 0.705, respectively.

As can be seen, the stylized facts match with the numerical examples since they document a positive relationship between the population level and the existing stock of knowledge which is negatively related to the subsequent growth rates. This phenomenon indicates the existence of convergence in the sense that states with a low stock of knowledge tend to grow faster than states with high R&D stock.

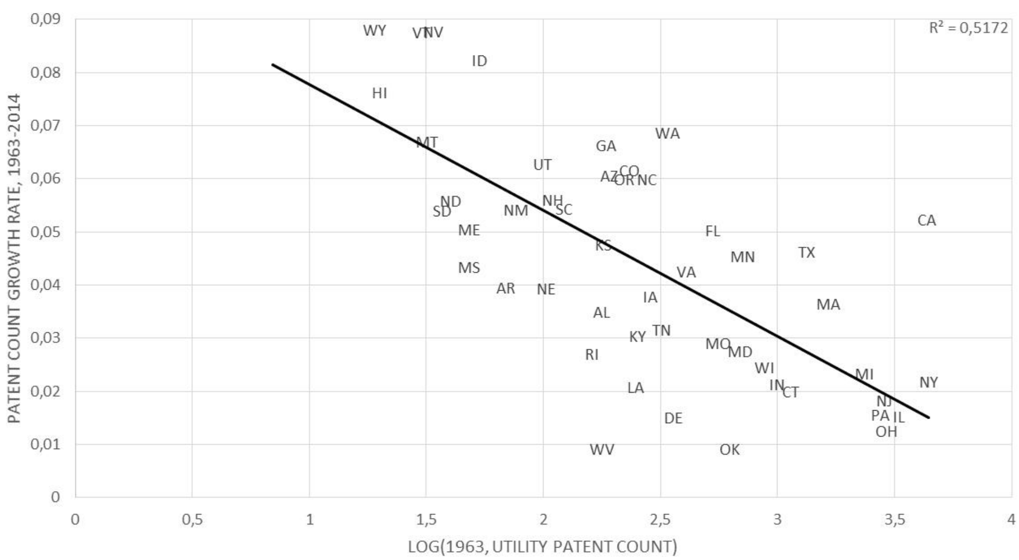

Figure 11: Growth rate from 1963 to 2014 vs. 1963 level of patents

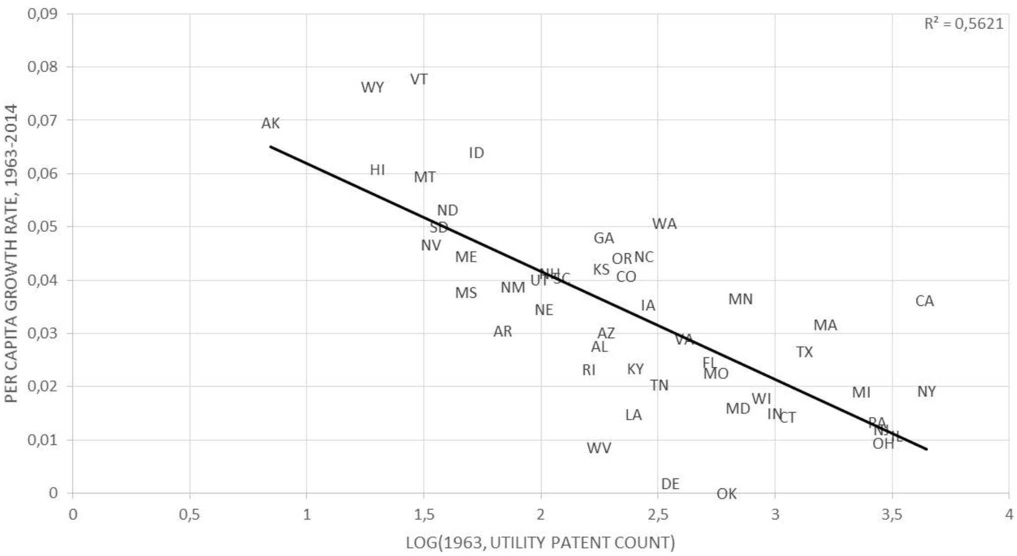

Figure 12: Per capita growth rate from 1963 to 2014 vs. 1963 level of patents

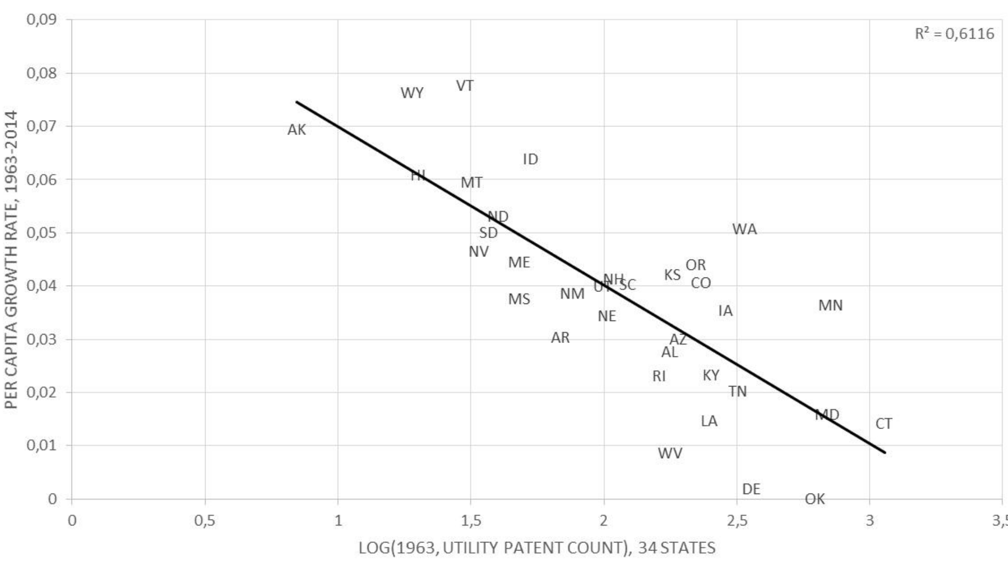

Figure 13: Per capita growth rate from 1963 to 2014 vs. 1963 level of patents, sample of 34 states

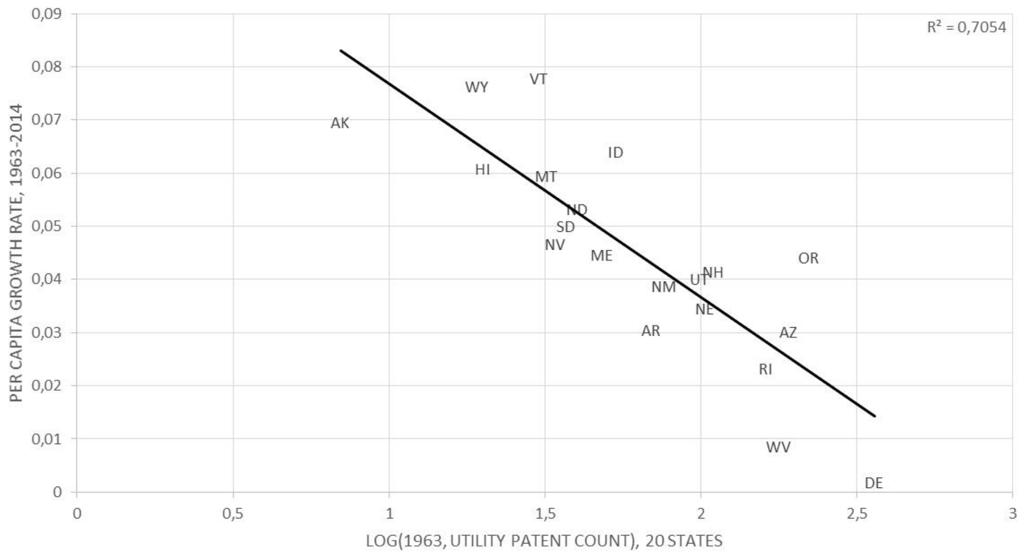

Figure 14: Per capita growth rate from 1963 to 2014 vs. 1963 level of patents, sample of 20 states

6 Conclusion

In this paper, we revisit in a spatial context, the endogenous growth theory where the technological change, which is driven by forward-looking profit-maximizing agents, expands the variety of inputs. The spatial interactions are introduced as positive externalities in the production function, the R&D equation, and through knowledge diffusion. The decentralized economy and social planner problems show the different dynamic outcomes that emerge with spatial interactions' internalization, highlighting the importance of space in the growth models. As a result, when spatial inequalities are embodied in these kinds of models it may alter the answer to the fundamental question of how much a country has to invest in R&D. As the numerical experiments illustrate, the answer has to take into account the economic activity across space and over time, since the parameter values of one region within a country may affect the economic activity at other regions and overturn the appropriate policies. Eventually, we argue how spatial interaction can eliminate the scale effect problem and explain the economic convergence in the US economy.

A critical component of growth is the level of human capital. Our analysis assumed different population distributions without investigating the mechanism behind the formation of a specific distribution. In other words, the factors that affect location decisions. The incorporation of labor mobility in this context is left for future research.

Appendix A. Deriving the steady-state allocation of labor

Solution of the Model

To solve for the competitive equilibrium outcome we define an equilibrium as a a set of functions $\{L_A,\ L_y,\ p_{R\&D},\ p_x,\ w_A,\ w_y\ ,A,\ x,\ q,\ z,\ h,\ c\}$ such that:

- Firms solve problems (3), (6) and (13) and take all prices as given.
- The total demand for capital from the intermediate-goods firms equals the total capital stock in each site. In other words, the below condition holds:

$$\sum_{j=1}^{A(t,s)} x_j(t,s) = K(t,s) \tag{38}$$

- The no-arbitrage condition (8) holds.
- The labor market clears and $L(s) = L_y(t,s) + L_A(t,s)$ for every s.
- The wages are equal in the final-goods and the R&D sector.
- z, q, and h satisfy (2), (10) and (11) respectively.
- The accumulation of capital is described by $\dot{K} = Y - C - mK$ for each location.
- Consumers solve the following problem:

$$\max_{c(t,s)} \int_0^\infty e^{-\rho t} \frac{c(t,s)^{1-\sigma} - 1}{1-\sigma} dt \tag{39}$$

subject to evolution of asset holdings denoted by $\overline{a}$:

$$\frac{\partial(\overline{a}(t,s))}{\partial t} = r(t,s)\cdot(\overline{a}(t,s)) + w(t,s) - c(t,s) \tag{40}$$

We can now solve for the decentralized model. Firstly, we can rewrite the arbitrage equation (8) as:

$$p_{R\&D}(t,s) = \frac{\Pi_{int}(t,s)}{r(t,s) - g_{p_{R\&D}}(t,s)} \tag{41}$$

where $g_{p_{R\&D}}(t,s)$ is the rate of the capital gain or loss derived from the change in the value of the resale price of the patent.

Moreover, the labor market equilibrium guarantees that the wages are equal in the final-goods and the R&D sector, that is $w_A = w_y \equiv w$.

$$(1-a)\frac{Y}{L_y} = p_{R\&D}\delta A L_A^{\lambda-1} e^{b_2 q} e^{b_3 h} \tag{42}$$

By using the above equation along with (1), (5) and (7) we get:

$$p_{R\&D} = \frac{(1-a)\alpha^{\frac{2\alpha}{1-\alpha}}}{\delta(r+m)^{\frac{a}{1-a}}} \frac{\gamma^{\frac{a}{1-a}}}{e^{b_2 q} e^{b_3 h}} \frac{e^{\frac{b_1 z}{1-a}}}{L_A^{\lambda-1}} A^{\frac{1-\gamma a}{\gamma(1-\alpha)}-1} \tag{43}$$

If we take logs and derivatives of both sides with respect to time, we can see that:

$$g_{p_{R\&D}} = \frac{1-\gamma}{\gamma(1-\alpha)} g_A \tag{44}$$

So, substitute the above equation into the arbitrage equation (41) we have:

$$p_{R\&D}(t,s) = \frac{\Pi_{int}(t,s)}{r(t,s) - \frac{1-\gamma}{\gamma(1-\alpha)} g_A} \tag{45}$$

From (14) and (41) it follows that:

$$w = \frac{\Pi_{int}}{r - g_{p_{R\&D}}} \delta A L_A^{\lambda-1} e^{b_2 q} e^{b_3 h} \tag{46}$$

Combining this with (5), (6) and (7) we find that:

$$w = \frac{(1-\alpha)}{r - g_{p_{R\&D}}} a\gamma\delta L_A^{\lambda-1} e^{b_2 q} e^{b_3 h} Y \tag{47}$$

From (4) and (47) we get:

$$L_A = \frac{L}{1 + \frac{r - g_{p_{R\&D}}}{a\gamma g_A}} \tag{48}$$

Balanced growth path

We define a balanced growth path as an equilibrium path where all model variable's need to grow at a constant rate. From the household's maximization problem defined by (39) and (40) we have that

$$\frac{\dot{c}(t,s)}{c(t,s)} = \frac{1}{\sigma}(r(t,s) - \rho) = g_c(t,s) \tag{49}$$

where $g_c(t,s)$ is the growth rate of consumption. On a balanced growth path g_c in (49) is constant. Given that ρ and σ are constants, it follows that r is also constant. Since the interest rate is constant, for each location, it follows that the saving rate is constant. So, from $C = (1 - sr)Y$ it follows that $g_c = g_y$, where sr is the saving rate. The accumulation of capital is described by $\dot{K} = Y - C - mK$ for each location. Since $Y = C + I$ and $I = srY$ we have that $\dot{K} = srY - mK$, particularly $g_y = g_k$.

Furthermore, given (38) we can rewrite (1) as:

$$Y = A^{\frac{1}{\gamma} - a} e^{b_1 z} L_y^{1-\alpha} K^a \tag{50}$$

By taking logs and derivatives of both sides with respect to time and using the fact that $g_y = g_k$ we can show that:

$$g_y = \frac{\frac{1}{\gamma} - \alpha}{1 - \alpha} g_A \tag{51}$$

So, the economy at each site s grows due to capital deepening which is driven entirely by the expansion of varieties which is caused by the R&D process and the presence of spatial externalities. All in all, we have that:

$$g_k = g_y = g_c = \frac{\frac{1}{\gamma} - \alpha}{1 - \alpha} g_A \tag{52}$$

The R&D equation (9) reveals that:

$$g_A = \frac{\dot{A}}{A} = \delta e^{b_2 q} e^{b_3 h} L_A^{\lambda} \tag{53}$$

From (49), (52), and (53) we have that:

$$r = \sigma \frac{\frac{1}{\gamma} - \alpha}{1-\alpha} \delta e^{b_2 q} e^{b_3 h} L_A^{\lambda} + \rho \tag{54}$$

Substituting (44), and (54) into (48) it follows that:

$$L_A^{eq} = \frac{L}{1 + \frac{1}{a\gamma}(\sigma \frac{1-\gamma\alpha}{\gamma(1-\alpha)} - \frac{1-\gamma}{\gamma(1-\alpha)} + \frac{\rho}{g_A})} \equiv \psi(\frac{\partial lnA}{\partial t}) \tag{55}$$

Finally, substituting (55) into (53) we end up with a non-linear integral equation of the Hammerstein type:

$$g_A = \delta(\frac{L}{1 + \frac{1}{a\gamma}(\sigma \frac{1-\gamma\alpha}{\gamma(1-\alpha)} - \frac{1-\gamma}{\gamma(1-\alpha)} + \frac{\rho}{g_A})})^{\lambda}.$$

$$exp\{b_2 c_2 \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ln(\frac{L}{1 + \frac{1}{a\gamma}(\sigma \frac{1-\gamma\alpha}{\gamma(1-\alpha)} - \frac{1-\gamma}{\gamma(1-\alpha)} + \frac{\rho}{g_A})}) ds' + b_3 c_3 \int_{S_0}^{S_1} e^{-c_3(s-s')^2} g_A ds'\} \tag{56}$$

Appendix B. Deriving the optimal allocation of labor

Simplifying the notation and taking into account the spatial dimension of the problem (18 - 21) we can set up the Hamiltonian as:

$$H^{cv}(C, L_A, K, A, t, s, \mu_\kappa, \mu_A) = \frac{C^{1-\sigma} - 1}{1-\sigma} + \mu_\kappa\{e^{b_1 z} K^{\alpha} A^{\frac{1}{\gamma} - a}(L - L_A)^{1-a} - C - mK\} + \mu_A\{\delta A L_A^{\lambda} e^{b_2 q} e^{b_3 h}\} \tag{57}$$

where μ_κ and μ_A are the co-state variables. The first-order conditions for maximization of H^{cv} can be written[13]:

[13]Notice that for different values of s, s' the function $V(s) = \eta \int_{S_0}^{S_1} e^{-\eta(s-s')^2} ln(v(s'))ds'$ can be rewritten as:

$$\frac{\partial H^{cv}}{\partial C} = 0 \Rightarrow C^{-\sigma} = \mu_\kappa \tag{58}$$

$$\frac{\partial H^{cv}}{\partial L_A} = 0 \Rightarrow \tag{59}$$

$$\mu_\kappa\{(1-a)e^{b_1 z}K^\alpha A^{\frac{1}{\gamma}-\alpha}(L-L_A)^{-a}(-1)+e^{b_1 z}K^\alpha A^{\frac{1}{\gamma}-\alpha}(L-L_A)^{1-a}b_1 c_1\frac{-1}{L-L_A}\int\limits_{S_0}^{S_1} e^{-c_1(s-s')^2}ds'\}$$

$$+\mu_A\{\lambda\delta A L_A^{\lambda-1}e^{b_2 q}e^{b_3 h} + \delta A L_A^\lambda e^{b_2 q}e^{b_3 h}b_2 c_2\frac{1}{L_A}\int\limits_{S_0}^{S_1} e^{-c_2(s-s')^2}ds'\} = 0 \tag{60}$$

$$\dot{\mu}_\kappa = -\frac{\partial H^{cv}}{\partial K} + \rho\mu_\kappa \Rightarrow \tag{61}$$

$$\dot{\mu}_\kappa = -\mu_\kappa(\alpha e^{b_1 z}K^{a-1}A^{\frac{1}{\gamma}-a}L_y^{1-a} - m) + \rho\mu_\kappa \tag{62}$$

$$\dot{\mu}_A = -\frac{\partial H^{cv}}{\partial A} + \rho\mu_A \Rightarrow \tag{63}$$

$$\dot{\mu}_A = -\{\mu_\kappa(\frac{1}{\gamma} - \alpha)e^{b_1 z}K^\alpha A^{\frac{1}{\gamma}-\alpha-1}(L-L_A)^{1-a} + \mu_A\delta L_A^\lambda e^{b_2 q}e^{b_3 h}+$$

$$\mu_A\delta A L_A^\lambda e^{b_2 q}e^{b_3 h}b_3 c_3(-\frac{\dot{A}}{A^2})\int\limits_{S_0}^{S_1} e^{-c_3(s-s')^2}ds'\} + \rho\mu_A \tag{64}$$

Take log and differentiate equation (58):

$$-\sigma g_c = \frac{\dot{\mu}_\kappa}{\mu_\kappa} \tag{65}$$

From (62) follows that:

$$-\frac{\dot{\mu}_\kappa}{\mu_\kappa} + m + \rho = a e^{b_1 z}K^{a-1}A^{\frac{1}{\gamma}-a}L_y^{1-a} \tag{66}$$

$$\eta\{ln(v(S_0)) + e^{-\eta(S_0-s)^2}ln(v(s)) + e^{-\eta(S_0-S_1)^2}ln(v(S_1))\mid_{s=S_0} + \ldots + e^{-\eta(s-S_0)^2}ln(v(S_0)) + ln(v(s))+$$

$$e^{-\eta(s-S_1)^2}ln(v(S_1))\mid_{s=s} + \ldots + e^{-\eta(S_1-S_0)^2}ln(v(S_0)) + e^{-\eta(S_1-s)^2}ln(v(s)) + ln(v(S_1))\mid_{s=S_1}\}$$

Therefore, $\frac{\partial V(s')}{\partial v(s)} = \frac{\eta}{v(s)}[e^{-\eta(S_0-s)^2} + \ldots + 1 + \ldots + e^{-\eta(S_1-s)^2}] = \frac{\eta}{v(s)}\int_{S_0}^{S_1} e^{-v(s-s')^2}ds'$

Taking logs and derivatives of the above relationship we have that:

$$g_k = \frac{\frac{1}{\gamma} - a}{1-a} g_A \tag{67}$$

From (60) we get:

$$\frac{\mu_\kappa}{\mu_A} = \frac{\delta A L_A^{\lambda-1} e^{b_2 q} e^{b_3 h} (\lambda + b_2 c_2 \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ds')}{e^{b_1 z} K^a A^{\frac{1}{\gamma}-\alpha} (L - L_A)^{-a} (1 - \alpha + b_1 c_1 \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ds')} \tag{68}$$

Take logs:

$$ln(\mu_\kappa) - ln(\mu_A) = ln(\delta) + (1 - \frac{1}{\gamma} + \alpha) ln(A) + (\lambda - 1) ln(L_A) + b_2 q + b_3 h + ln(\lambda + b_2 c_2 \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ds')$$

$$-b_1 z - alnK + aln(L_y) - ln(1 - \alpha + b_1 c_1 \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ds') \tag{69}$$

Taking into account that the quantities $ln(\delta)$, $ln(\lambda + b_2 c_2 \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ds')$, $ln(1 - \alpha + b_1 c_1 \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ds')$ are constants along the balance growth path and that $\frac{\dot{L}_y}{L_y} = \frac{\dot{L}_A}{L_A} = 0$ for each location we can differentiate the above equation and get:

$$\frac{\dot{\mu}_\kappa}{\mu_\kappa} - \frac{\dot{\mu}_A}{\mu_A} = (1 - \frac{1}{\gamma} + \alpha) g_A - \alpha g_k \tag{70}$$

Since $g_k = \frac{\frac{1}{\gamma}-a}{1-a} g_A$ we have that:

$$\frac{\dot{\mu}_\kappa}{\mu_\kappa} - \frac{\dot{\mu}_A}{\mu_A} = \frac{1 - \frac{1}{\gamma}}{1-a} g_A \tag{71}$$

Using the fact that $g_c = g_y = g_k = \frac{\frac{1}{\gamma}-a}{1-a} g_A$ and substituting (65) into (71) we get:

$$\frac{\dot{\mu}_A}{\mu_A} = \frac{\sigma\gamma a - \sigma - \gamma + 1}{\gamma(1-a)} g_A \tag{72}$$

Dividing (64) by μ_A we have that:

$$\frac{\dot{\mu}_A}{\mu_A} = -\frac{\mu_\kappa}{\mu_A} (\frac{1}{\gamma} - \alpha) e^{b_1 z} K^\alpha A^{\frac{1}{\gamma}-\alpha-1} L_y^{1-a} - \delta L_A^\lambda e^{b_2 q} e^{b_3 h} +$$

$$\delta L_A^{\lambda} e^{b_2 q} e^{b_3 h} b_3 c_3 g_A \int_{S_0}^{S_1} e^{-c_3(s-s')^2} ds' + \rho \tag{73}$$

Combining the above equation with (68), (72) and given that $g_A = \frac{\dot{A}}{A} = \delta L_A^{\lambda} e^{b_2 q} e^{b_3 h}$ we get:

$$\frac{\sigma\gamma a - \sigma - \gamma + 1}{\gamma(1-a)} = -\frac{(L - L_A)}{L_A} \frac{(1-\gamma\alpha)}{\gamma} \frac{\lambda}{(1-a)} \frac{1 + \frac{b_2 c_2}{\lambda} \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ds'}{1 + \frac{b_1 c_1}{1-\alpha} \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ds'} - 1 +$$

$$g_A b_3 c_3 \int_{S_0}^{S_1} e^{-c_3(s-s')^2} ds' - \frac{\rho}{g_A} \tag{74}$$

Setting:

$$d \equiv \frac{1 + \frac{b_2 c_2}{\lambda} \int_{S_0}^{S_1} e^{-c_2(s-s')^2} ds'}{1 + \frac{b_1 c_1}{1-\alpha} \int_{S_0}^{S_1} e^{-c_1(s-s')^2} ds'} \tag{75}$$

and:

$$H \equiv \int_{S_0}^{S_1} e^{-c_3(s-s')^2} ds' \tag{76}$$

we can simplify equation (74):

$$L_A^{sp} = \frac{L}{1 + \frac{1}{\lambda d}(\sigma - 1 + \frac{\gamma(1-\alpha)}{1-\gamma\alpha}(g_A b_3 c_3 H + \frac{\rho}{g_A}))} \tag{77}$$

Appendix C. Mathematica program

This appendix contains the Mathematica code used to calculate the market equilibrium and social planner outcome. Specifically, it calculates the first numerical experiment presented in section 5.1.1 (i.e., figure 1). All numerical experiments were performed by simply changing the parameters as described in the previous sections. To solve equation (17) we contract an iterative interpolation algorithm. It should be noted that the iterative scheme converges for any plausible initial value for g_A.

In[∘]:=
```
ClearAll["Global`*"]
```

Define constants:

In[∘]:=
```
a = 0.33;
δ = 0.01;
ρ = 0.04;
σ = 2.5;
b1 = 0.01;
c1 = 1;
b2 = 0.02;
c2 = 2;
b3 = 0.01;
c3 = 1;
S0 = 0;
S1 = 2 * Pi;
λ = 0.35;
γ = 1.2;
L = 100;
```

Market equilibrium

The non-linear Fredholm integro-differential equation for the long-run growth rate:

$$g_A = \delta * \left(\frac{L}{1+\frac{1}{a\gamma}\left(\frac{\sigma\,(1-\gamma a)}{\gamma\,(1-a)}-\frac{1-\gamma}{\gamma*(1-a)}+\frac{\rho}{g_A}\right)}\right)^{\lambda} * \exp \left\{b_2 * c_2 * \int_{S_0}^{S_1}\left(e^{-c_2\,(r-s)^2} * \mathrm{Log}\left[\frac{L}{1+\frac{1}{a\gamma}\left(\frac{\sigma\,(1-\gamma a)}{\gamma\,(1-a)}-\frac{1-\gamma}{\gamma*(1-a)}+\frac{\rho}{g_A}\right)}\right]\right)dr + b_3 * c_3 \int_{S_0}^{S_1}\left(e^{-c_3\,(r-s)^2} * g_A\right)dr\right\}$$

We obtain a numerical solution by exploiting an iterative interpolation algorithm:

In[∘]:=
```
approxsoln[r_] = 0.02;
```

In[∘]:=
```
iterstep := (values = Table[{r,
     δ * (L / (1 + 1/(a*γ) (σ*(1-γ*a)/(γ*(1-a)) - (1-γ)/(γ*(1-a)) + ρ/(approxsoln[r]))))^λ * Exp[b2 * c2 * NIntegrate[(e^(-c2 (r-s)^2) *
        Log[L / (1 + 1/(a*γ) (σ*(1-γ*a)/(γ*(1-a)) - (1-γ)/(γ*(1-a)) + ρ/(approxsoln[r])))]), {s, S0, S1}] + b3 * c3 *
      NIntegrate[(e^(-c3 (r-s)^2) * (approxsoln[s])), {s, S0, S1}]]}, {r, S0, S1, (S1 - S0)/10}];
   approxsoln[r_] = InterpolatingPolynomial[values, r])
```

In[∘]:=
```
Do[iterstep, {20}]
```

In[∘]:=
```
p1 = ListPlot[values, Joined → True, Ticks → {Range[0, 2 * Pi, Pi / 4], Automatic},
  PlotStyle → {Blue, Medium}, AxesLabel → {"Location", "g_A"}]
```

Calculate the labor force devoted to R&D:

```
In[•]:= ga = values[[All, 2]];

In[•]:= la = L / (1 + (1 / (a * γ)) ((σ * (1 - γ * a)) / (γ * (1 - a)) - (1 - γ) / (γ * (1 - a)) + ρ / ga));

In[•]:= position = values[[All, 1]];

In[•]:= p2 = ListPlot[Thread[{position, la / 100}],
   Joined → True , Ticks → {Range[0, 2 * Pi, Pi / 4], Automatic},
   PlotStyle → {Blue, Medium}, AxesLabel → {"Location", "L_A"}]
```

The Social optimum problem

The non-linear Fredholm integro-differential equation for the long-run growth rate:

$$g_A = \delta * \left(\frac{L}{1+\frac{1}{\lambda d}\left(\sigma - 1 + \frac{\gamma\,(1-a)}{1-\gamma a}\, g_A\, b_3\, c_3\, H + \frac{\gamma*(1-a)}{1-\gamma a}\,\frac{\rho}{g_A}\right)}\right)^{\lambda} * \exp$$

$$\left\{b_2 * c_2 * \int_{S_0}^{S_1}\left(e^{-c_2\,(r-s)^2} * \operatorname{Log}\left[\frac{L}{1+\frac{1}{\lambda d}\left(\sigma - 1 + \frac{\gamma\,(1-a)}{1-\gamma a}\, g_A\, b_3\, c_3\, H + \frac{\gamma*(1-a)}{1-\gamma a}\,\frac{\rho}{g_A}\right)}\right]\right) dr + \right.$$

$$\left. b_3 * c_3 \int_{S_0}^{S_1}\left(e^{-c_3\,(r-s)^2} * g_A\right) dr\right\}$$

We obtain a numerical solution by exploiting a combination of iteration and interpolation:

```
In[•]:= approxsoln[r_] = 0.02;

In[•]:= iterstep := (values = Table[
      {r, δ * (L / (1 + (1 / (λ * ((1 + ((b2 * c2) / λ) * NIntegrate[(e^(-c2*(r-s)^2)), {s, S0, S1}]) /
                  (1 + ((b1 * c1) / (1 - a)) * NIntegrate[(e^(-c1*(r-s)^2)), {s, S0, S1}])))) *
             (σ - 1 + ((γ * (1 - a)) / (1 - γ * a)) * (b3 * c3 * (approxsoln[r]) *
                   NIntegrate[(e^(-c3*(r-s)^2)), {s, S0, S1}] + ρ / (approxsoln[r])))))^λ *
         Exp[b2 * c2 * NIntegrate[(e^(-c2*(r-s)^2) * Log[(L / (1 + (1 / (λ * ((1 + ((b2 * c2) / λ) *
                           NIntegrate[(e^(-c2*(r-s)^2)), {s, S0, S1}]) / (1 + ((b1 * c1) / (1 -
                              a)) * NIntegrate[(e^(-c1*(r-s)^2)), {s, S0, S1}])))) * (σ - 1 +
                     ((γ * (1 - a)) / (1 - γ * a)) * (b3 * c3 * (approxsoln[r]) * NIntegrate[
                           (e^(-c3*(r-s)^2)), {s, S0, S1}] + ρ / (approxsoln[r]))))))]),
             {s, S0, S1}] + b3 * c3 * NIntegrate[(e^(-c3*(r-s)^2) * (approxsoln[s])),
             {s, S0, S1}]]}, {r, S0, S1, (S1 - S0) / 10}];
   approxsoln[r_] = InterpolatingPolynomial[values, r])

Do[iterstep, {20}]

In[•]:= p3 = ListPlot[values, Joined → True, PlotStyle → {Red, Dashed, Medium},
   Ticks → {Range[0, 2 * Pi, Pi / 4], Automatic}, AxesLabel → {"Location", "g_A"}]
```

Calculate the labor force devoted to R&D:

```
In[•]:= ga = values[[All, 2]];
```

```
In[•]:= d = Table[{r, (1 + ((b2 * c2) / λ) * NIntegrate[(ⅇ^(-c2 * (r - s)^2)), {s, S0, S1}]) /
            (1 + ((b1 * c1) / (1 - a)) * NIntegrate[(ⅇ^(-c1 * (r - s)^2)), {s, S0, S1}])},
        {r, S0, S1, (S1 - S0) / 10}][[All, 2]];

In[•]:= H = Table[{r, NIntegrate[(ⅇ^(-c3 (r - s)^2)), {s, S0, S1}]}, {r, S0, S1, (S1 - S0) / 10}][[All, 2]];

In[•]:= la = L / (1 + (1 / (λ * d)) * (σ - 1 + ((γ * (1 - a)) / (1 - γ * a)) * (b3 * c3 * ga * H + ρ / ga)));

In[•]:= p4 = ListPlot[Thread[{position, la / 100}],
          Joined → True, PlotStyle → {Red, Dashed, Medium},
          Ticks → {Range[0, 2 * Pi, Pi / 4], Automatic}, AxesLabel → {"Location", "L_A"}]
```

Market equilibrium vs Social optimum

```
In[•]:= Show[p2, p4, PlotRange → Automatic,
         AxesOrigin → {0, 0.087}, Ticks → {Range[0, 2 * Pi, Pi / 4], Automatic}]
```

References

Acemoglu, D., & Restrepo, P. (2018). The race between man and machine: Implications of technology for growth, factor shares, and employment. *American Economic Review*, *108*(6), 1488–1542.

Aghion, P., Akcigit, U., & Howitt, P. (2014). What do we learn from schumpeterian growth theory? *Handbook of economic growth* (pp. 515–563). Elsevier.

Aghion, P., & Howitt, P. (1992). A model of growth through creative destruction. *Econometrica*, *60*(2), 323–351.

Aghion, P., & Howitt, P. (1998). *Endogenous Growth Theory*. The MIT Press.

Arrow, K. (1962). The economic significance of learning by doing. *Review of Economic Studies*, *29*, 155–73.

Ballestra, L. V. (2016). The spatial AK model and the Pontryagin maximum principle. *Journal of Mathematical Economics*, *67*, 87–94.

Barro, R. J., & Sala-i-Martin, X. (1992). Convergence. *Journal of Political Economy*, *100*(2), 223–251.

Barro, R. J., & Sala-i-Martin, X. (2004). *Economic Growth, 2nd Edition*. The MIT Press.

Boucekkine, R., Camacho, C., & Fabbri, G. (2013). Spatial dynamics and convergence: The spatial AK model. *Journal of Economic Theory*, *148*(6), 2719–2736.

Boucekkine, R., Camacho, C., & Zou, B. (2009). Bridging the gap between growth theory and the new economic geography: The spatial Ramsey model. *Macroeconomic Dynamics*, *13*(1), 20–45.

Boucekkine, R., Fabbri, G., Federico, S., & Gozzi, F. (2019). Growth and agglomeration in the heterogeneous space: A generalized ak approach. *Journal of Economic Geography*, *19*(6), 1287–1318.

Brito, P. (2004). *The Dynamics of Growth and Distribution in a Spatially Heterogeneous World* (Working Papers Department of Economics No. 2004/14). ISEG - Lisbon School of Economics and Management, Department of Economics, Universidade de Lisboa.

Brock, W., & Xepapadeas, A. (2008). Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control. *Journal of Economic Dynamics and Control*, *32*(9), 2745–2787.

Brock, W., & Xepapadeas, A. (2009). *General Pattern Formation in Recursive Dynamical Systems Models in Economics* (Working Papers No. 2009.49). Fondazione Eni Enrico Mattei.

Brock, W., Xepapadeas, A., & Yannacopoulos, A. (2014). Optimal agglomerations in dynamic economics. *Journal of Mathematical Economics*, *53*, 1–15.

Brown, M. G., & Svenson, R. A. (1988). Measuring R&D productivity. *Research-Technology Management*, *31*(4), 11–15.

Ciccone, A., & Hall, R. E. (1993). *Productivity and the density of economic activity* (tech. rep.). National Bureau of Economic Research.

Comin, D., Dmitriev, M., & Rossi-Hansberg, E. (2012). *The spatial diffusion of technology* (tech. rep.). National Bureau of Economic Research.

Comin, D., & Hobijn, B. (2010). An exploration of technology diffusion. *American Economic Review*, *100*(5), 2031–59.

Dekle, R., & Eaton, J. (1999). Agglomeration and land rents: Evidence from the prefectures. *Journal of Urban Economics*, *46*(2), 200–214.

Desmet, K., & Rossi-Hansberg, E. (2010). On spatial dynamics. *Journal of Regional Science*, *50*(1), 43–63.

Desmet, K., & Rossi-Hansberg, E. (2014). Spatial development. *American Economic Review*, *104*(4), 1211–43.

Dinopoulos, E., & Thompson, P. (1999). Scale effects in Schumpeterian models of economic growth. *Journal of Evolutionary Economics*, *9*(2), 157–185.

Eaton, J., & Kortum, S. (1999). International technology diffusion: Theory and measurement. *International Economic Review*, *40*(3), 537–570.

Ertur, C., Le Gallo, J., & Baumont, C. (2006). The European regional convergence process, 1980-1995: Do spatial regimes and spatial dependence matter? *International Regional Science Review*, *29*(1), 3–34.

Evans, P. (1996). Using cross-country variances to evaluate growth theories. *Journal of Economic Dynamics and Control*, *20*(6-7), 1027–1049.

Fabbri, G. (2016). Geographical structure and convergence: A note on geometry in spatial growth models. *Journal of Economic Theory*, *162*, 114–136.

Fujita, M., Krugman, P., & Venables, A. (1999). *The Spatial Economy: Cities, Regions, and International Trade*. The MIT Press.

Fujita, M., & Thisse, J.-F. (2013). *Economics of Agglomeration: Cities, Industrial Location, and Globalization*. Cambridge University Press.

Griliches, Z. (1990). *Patent Statistics as Economic Indicators: A Survey* (Working Paper No. 3301). National Bureau of Economic Research.

Jaffe, A., Trajtenberg, M., & Henderson, R. (1993). Geographic localization of knowledge spillovers as evidenced by patent citations. *The Quarterly Journal of Economics*, *108*(3), 577–598.

Jones, C. (1995). R&D-based models of economic growth. *Journal of Political Economy*, *103*(4), 759–784.

Jones, C., & Kim, J. (2018). A schumpeterian model of top income inequality. *Journal of Political Economy*, *126*(5), 1785–1826.

Jones, C., & Williams, J. (2000). Too much of a good thing? The economics of investment in R&D. *Journal of Economic Growth*, *5*(1), 65–85.

Kremer, M. (1993). Population growth and technological change: One million BC to 1990. *The Quarterly Journal of Economics*, *108*(3), 681–716.

Kyriakopoulou, E., & Xepapadeas, A. (2013). Environmental policy, first nature advantage and the emergence of economic clusters. *Regional Science and Urban Economics*, *43*(1), 101–116.

Kyriakopoulou, E., & Xepapadeas, A. (2017). Atmospheric pollution in rapidly growing industrial cities: Spatial policies and land use patterns. *Journal of Economic Geography*, *17*(3), 607–634.

Laincz, C. A., & Peretto, P. F. (2006). Scale effects in endogenous growth theory: An error of aggregation not specification. *Journal of Economic Growth*, *11*(3), 263–288.

Lucas, R. (2001). Externalities and cities. *Review of Economic Dynamics*, *4*(2), 245–274.

Lucas, R., & Rossi-Hansberg, E. (2002). On the internal structure of cities. *Econometrica*, *70*(4), 1445–1476.

Mion, G. (2004). Spatial externalities and empirical analysis: The case of Italy. *Journal of Urban Economics*, *56*(1), 97–118.

Paci, R., & Usai, S. (1999). Externalities, knowledge spillovers and the spatial distribution of innovation. *GeoJournal*, *49*(4), 381–390.

Quah, D. (2002). Spatial agglomeration dynamics. *American Economic Review*, *92*(2), 247–252.

Romer, P. (1990). Endogenous technological change. *Journal of Political Economy*, *98*(5, Part 2), S71–S102.

Rosenthal, S. S., & Strange, W. C. (2008). The attenuation of human capital spillovers. *Journal of Urban Economics*, *64*(2), 373–389.

Rossi-Hansberg, E. (2004). Optimal urban land use and zoning. *Review of Economic Dynamics*, *7*(1), 69–106.

Sedgley, N., & Elmslie, B. (2004). The geographic concentration of knowledge: Scale, agglomeration, and congestion in innovation across US states. *International Regional Science Review*, *27*(2), 111–137.

Tian, L., Wang, H. H., & Chen, Y. (2010). Spatial externalities in China regional economic growth. *China Economic Review*, *21*, S20–S31.

Tsionas, E. G. (2000). Regional growth and convergence: evidence from the United States. *Regional Studies*, *34*(3), 231–238.

Van Oort, F. (2002). Innovation and agglomeration economies in the Netherlands. *Tijdschrift voor economische en sociale geografie*, *93*(3), 344–360.

Xepapadeas, A., & Yannacopoulos, A. (2020). Spatial growth theory: Optimality and spatial heterogeneity.

Young, A., Higgins, M., & Levy, D. (2008). Sigma convergence versus beta convergence: Evidence from US county-level data. *Journal of Money, Credit and Banking*, *40*(5), 1083–1093.

Zhang, X., Wan, G., Li, J., & He, Z. (2020). Global spatial economic interaction: Knowledge spillover or technical diffusion? *Spatial Economic Analysis*, *15*(1), 5–23.