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Optimal Control Approaches to Sustainability under Uncertainty

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Abstract

Optimal sustainable management of natural resources has been one of the major lines of research in environmental economics at least for the last two decades. Several attempts have been made in order to describe in a quantitative fashion the notion of sustainability and distinguish management policies between sustainable and non sustainable ones. Important aspects of this task are (a) appropriate modeling of the spatio-temporal dynamics of the state of the system, including the sources of uncertainty affecting either directly or indirectly the problem at hand (e.g. climate conditions, population growth, biological evolution, etc), and (b) the development of appropriate criteria for evaluating the welfare of the system under study that guarantee sustainability and viability. In this chapter, we present and discuss popular and established optimization approaches for investigating policy selection problems within the sustainability framework, from the perspective of viability and optimal control theory.

Keywords: Model Uncertainty; Optimal Control; Robust Optimal Control; Sustainability; Viability Theory;

1 Introduction to Sustainability Concepts

Sustainability has appeared in the literature with various definitions throughout the years. Probably, the most intuitive definition is provided by Asheim [9] through the notion of sustainable development. *Sustainable development* is defined in [76] as the development that meets the needs of the present without compromising the ability of future generations to meet their own needs. Based on this, [9] characterized as *sustainable* the management of the resource base by a particular generation at some point in time, if it constitutes the first part of a feasible sustainable development. Then, a stream of well-being develops in a sustainable manner if each generation's management of the resource base is sustainable according to this perspective.

However, during the last decades, a discrimination between two different kinds or approaches in sustainability has been made: *weak sustainability* vs *strong sustainability* (see e.g. [46, 47] and references therein). The first one is more economy-oriented in terms of industrial operations and concerns more the well-being of the human community. The second one concerns the environment in which the economic operations take place and especially the abiotic and biotic (ecological) system and the renewal processes that support the resources needed by the economy to progress.

The necessary condition to achieve weak sustainability is to preserve over time a certain amount of capital (including man-made capital and initial endowment of certain natural resources). Depending on the operation under study, different perspectives may be derived, e.g. the so called Solow sustainability [71] or in modern terms the *very weak sustainability*, in which is required to maintain intact the generalized production capacity of an economy. The later will enable a constant consumption per capita through time on the infinite horizon, to be in line with the inter-generation equity working hypothesis [71, 72]. In a broader perspective, the definition of *weak sustainability* [43, 69] refers to the preservation over time of the welfare potential of the overall capital base. In this view, the sustainable policy is not restricted to sustain a certain material/source or consumption level, but includes values that are related to nonconsumptive uses and the public good character of the environment (please see [46, 47] and references therein).

On a different pathway, the term *strong sustainability* usually refers to sustaining certain properties of the physical environment as time evolves. The environment's properties, and why it is needed to be sustained, depends on the problem and the operation setting. From an ecological perspective, minimum requirements of the strong sustainability are to choose policies that keep the total stock of the natural capital constant over time; therefore, this requires assessing the natural capital in terms of the ecosystem's viability. This approach leads to what is referred to as *environmental quality* which can be represented in terms of a function that includes the stocks of biological resources, the ecosystem's space and all those environmental assets which are essential for the integrity of the ecosystem. However, in order to translate this principle in practical terms, some ecological criteria have been defined - such as *safe, minimum sustainability standards (SMSS)* and *safe minimum standard of conservation (SMC)* or else Ciriacy-Wantrup's principle [24]. The first one (SMSS) contains a number of ecological-type criteria that have to be met, provided by the physical laws of the environment (e.g., rate of regeneration, assimilative capacity, etc) if not human-interfered with, and is more compactly referred to as the *very strong sustainability.* The second one (SMC), is more on the pathway of defining a safe minimum standard of conservation that allows a relative flexibility in consumption policies without entering the critical zone concerning to the ecosystem's viability. This in general does not coincide with SMSS approach, and is therefore defining less strictly the viable standards for the preservation of the ecosystem.

We summarize the following concepts of sustainability:

Very Weak Sustainability (VWS): The overall stock of capital assets should remain constant over time. The reduction of an asset is allowed, provided that another capital asset is increased to compensate for such a reduction.

Weak Sustainability (WS): A slight modification of very weak sustainability. This is a formulation of sustainability constraints that will impose some degree of restriction on resource-using economic activities. Such restrictions would not result from concerns for the ecosystems themselves, but rather would result from concerns for the ecosystems' ability to meet human needs.

Strong Sustainability (SS): The assumption here is that protecting the overall amount of capital is insufficient; natural capital must also be protected because some critical natural capitals cannot be replaced by other forms of capital.

Very Strong Sustainability (VSS): This concentrates on the scale of human development relative to global carrying capacity. In particular, it follows the axiom that when human development reaches the global carrying capacity, no forms of natural capital are substitutable.

In what follows, policy selection problems under sustainability conditions are discussed and several results concerning the aforementioned sustainability kinds are presented.

2 Optimization, Sustainability and Uncertainty

2.1 Towards a quantitative view of sustainability

Discussing sustainability issues in a quantitative manner requires the introduction of a model connecting the evolution of the stock of a resource(s), whether renewable or not, with its consumption or exploitation. Then, we may quantify sustainability in terms of the long run behaviour of the state of the resource or the consumption level. While this model can be as involved and realistic as desired, for the sake of illustration of the key ideas and methodologies in this chapter we will focus on a rather schematic evolution model. Of course, the concepts and methods presented here are extensible to any realistic model, at the cost of increasing mathematical and computational complexity.

These models will be largely divided into two main categories:

Deterministic, in which the evolution law governing the relationship between the economic variables (such as the stock of resources) and consumption is known.

Probabilistic, in which this evolution law is not known but depends on unpredictable factors governed by the laws of probability. Depending on whether the probability law is known and universally accepted or not, we have two subcategories

- *Stochastic models*, in which the probability law governing the evolution of the system is known (a case which corresponds to the concept of risk in the economics of uncertainty) and
- *Uncertainty models*, in which there is not a unique acceptable (by all concerned agents) probability law for the evolution, and multiple probability

models have to be considered in the analysis (a case which corresponds to the concept of Knightian uncertainty [44, 51, 56]).

Clearly, stochastic and primarily uncertain models are an important step towards introducing realistic effects into the models. Moreover, the above modeling frameworks, (either deterministic or stochastic/uncertain) can be extended to include effects of spatial heterogeneity in the distribution of the resources, introducing interesting implications.

2.2 A gallery of models

Typically, the optimal control formulation of policy selection problems in economics (including every perspective, e.g. environmental economics, ecological economics, industrial economics, etc) consists of three basic components: (a) the welfare criterion under which the policy (control/stimulus) is assessed, (b) the dynamical system describing the environment of the economy under study, and (c) a set of constraints which typically represent physical or human-made regulations, economic and sustainability/viability targets and assumptions.

The chosen welfare criterion, under which the feasible policies are assessed, retains an important and major role in the decision process, since its nature significantly affects the perspective under which the economy is examined (please see Section 2.3 for the relevant discussion) and the optimal decisions that are derived. In modern approaches, multi-criteria objectives are employed in order to take into account various perspectives of well being (both on the economy and environment) on the determination of optimal policies. However, the derivation of solutions for the latter case could be quite challenging since, depending on the nature of the criteria that are required and the set of feasible solutions, stochastic numerical approaches may be required which in the continuous-time case may lead to computational complexity issues.

The underlying dynamics of the economy, under which each policy selection problem is examined, in general are assumed to obey well known laws (of deterministic type); however, there is always room to include uncertainty in various aspects of the system which increases the degree of realism, especially in cases when natural resources are involved in the system. In order to address this issue, several authors (see e.g. [3, 15, 60, 61]) adopt stochastic formulations for the underlying system; it is evident that this dramatically alternates the status of the whole optimization problem to one of stochastic nature. This fact, does not allow for closed-form solutions in general, and even the characterization of the possible solutions for the problem under study could be a quite challenging task in most cases.

The set of constraints is usually comprised of target inequalities which refer either to critical thresholds for the levels of certain natural quantities (imposed by sustainability and viability assumptions corresponding to the minimum or maximal allowed levels to keep the environment at a certain quality level) usually set by experts or by biological mechanisms, e.g. regeneration rates, or to certain well being assumptions for the community, e.g. minimum social welfare standards, irreversibility of operations concerning resources extraction or capital production (see e.g. [28, 29, 47, 57]). A situation that may occur when including on the constrained set diverging requirements, e.g. guaranteed level of consumption and simultaneously guaranteed levels for the natural resources that are used in the production process, is that the final set of feasible solutions could be empty.

Let us now describe in mathematical formulation a quite general and flexible enough framework for the representation of policy selection problems. Assume that $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ denotes the states of an economy at time $t \in \mathbb{R}^+$, e.g. the states of labour, manufactured capital, population, renewable and/or exhaustible resources, etc. To provide a coinsize formulation, assume that the evolution of the states is described by the dynamics

$$\begin{cases} \frac{d}{dt}x(t) = F(x(t), u(t)) \\ u(t) \in \mathcal{U}(x(t)) \subset \mathbb{R}^m \end{cases} \tag{1}$$

or (if more appropriate for the application in mind) its discrete time version

$$\begin{cases} x(t+1) = x(t) + F(x(t), u(t)), \;\; t \in \mathbb{N}, \\ u(t) \in \mathcal{U}(x(t)) \subset \mathbb{R}^m \end{cases} \tag{2}$$

where F is the generator function for the states of the economy (comprising technology, production functions, consumption effects, renewal mechanisms of resources, etc), $u(\cdot)$ denotes the control variables (e.g. consumption, harvest rates, resources extraction rates, etc) and $\mathcal{U}(x(t))$ is the set of admissible controls at time t which depends on the states of the economy $x(t)$. The system formulation stated in (1) is a very general representation of the mechanism that produces the states of an economy depending on the chosen controls $u(\cdot)$ (or policies in more economical terms); for example consumption c or harvesting rules, and other parameters that affect the generating mechanism $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. All these possibilities, given an initial state $x(0) = x_0$ at the time that is considered as the starting point $t = 0$, can be described by the set

$$\mathcal{S}(x_0) = \{(x(\cdot), u(\cdot)) \mid x(0) = x_0, \;\; x(t) \text{ satisfying (1) } \forall t \in \mathbb{R}^+\} \tag{3}$$

which from now on will be referred to as the set of admissible states or admissible trajectories of the economy. Clearly, these states can also be considered as stochastic ones since uncertainty appears on certain aspects of the system (1) e.g., environmental conditions and effects by climate change, parameters affecting biological mechanisms, effects of possible diseases/pandemics, etc. In general, policies and states are constrained to satisfy certain well-being or viability/sustainability requirements either for the community or the environment, for instance, positivity of consumption, irreversibility of investments/resource extractions, availability of labour, scarcity of resources, etc. In this case, the states and controls $(x(\cdot), u(\cdot))$ should be admissible if satisfy a number of inequalities, let us say q in number, which comprise the set

$$\mathcal{K} = \{(x(\cdot), u(\cdot)) \mid g_i(x(t), u(t)) \geq \eta_i, \;\; i = 1, 2, ..., q, \;\; \forall t \in \mathbb{R}^+\} \tag{4}$$

where η_i for $i = 1, 2, ..., q$ denote the critical thresholds which have been set for viability/sustainability reasons or physical meaning. In the simplest case, the inequalities $g_i(x(t), u(t)) \geq \eta_i$ could represent critical levels of consumption, resource extraction or natural resources states which if exceeded, either the environment or the community well-being will be seriously disturbed. Of course, these constraints could also be of a more complex form if they represent critical thresholds for biological processes that are conducted in the physical environment or they can even be subject to uncertainty due to exogenous factors effects.

Example 1 (The Ramsey model)**.** Possibly the simplest model we may assume is that of an economy whose state is characterized by a single capital stock K, driven by the dynamical equation

$$\frac{d}{dt}K(t) = f(K(t)) - c(t) - \lambda K(t), \tag{5}$$

where f is a production function (often the Cobb-Douglas production function $f(K) = AK^\gamma$, with A representing the effects of technological progress and $\gamma < 1$), c is consumption which is to be determined optimally so as to maximize some intertemporal utility criterion on consumption and $\lambda > 0$ denotes the capital depreciation rate. This problem, first proposed in the 1920's by Frank Ramsey as a simple example of how the problem of "how much of its output should a nation consume" has initiated a long lasting discussion, with many variants to include effects such as resources and their effects in production, labour, spatial heterogeneity, stochasticity etc. The special case of a linear production function $f(K) = AK$ where A corresponds to the so called AK model [20] that has also been used a lot in modelling, especially due to the fact that it can lead to analytic solutions. This model falls into the general framework of (1) setting $X = K$ and $u = c$, with admissible controls being specified by the constraints $K, c \geq 0$. Note that model (5) may also be derived if we assume a production function $F(K, L) = f\left(\frac{K}{L}\right)$, where L is labour, and then work in terms of per capita capital $k = \frac{K}{L}$. Moreover, discrete time versions of the Ramsey model are also popular (where each time corresponds to a generation), for example

$$K(t+1) = K(t) + f(K(t)) - c(t) - \lambda K(t), \ \ t \in \mathbb{N}, \tag{6}$$

with appropriate reformulations of the interemporal utility of consumption.

We close this section by collecting several variants of the Ramsey problem, including stochastic and spatio-temporal versions.

Example 2 (Sustainability of an economy with an exhaustible resource)**.** We consider the model examined in [57], where a consumption-production type economy with a nonrenewable resource following the model of Dasgupta-Heal-Solow is investigated. The current economy dynamics are represented by the system

$$\begin{cases} \frac{d}{dt}K(t) = f(K(t), r(t)) - c(t) - \lambda K(t) \\ \frac{d}{dt}S(t) = -r(t) \end{cases} \tag{7}$$

where $S(t)$ is the exhaustible resource stock and $r(t)$ the related extraction rate, $K(t)$ is the manufactured capital, f the production function, $c(t)$ the consumption rate of the manufactured capital and λ the capital's depreciation rate. This model falls into the general framework of (1) setting $X = (K, S)$ and $u = (c, r)$, with admissible controls being specified by the constraints $K, S, c, r \geq 0$. Note the similarity with the standard Ramsey model.

Example 3 (Sustainability of an economy with exhaustible and renewable resources)**.** As a second example consider the model examined in [47] with an economy including both an exhaustible and renewable resources. The states of this economy are described by the dynamics

$$\begin{cases} \frac{d}{dt}K(t) = f(K(t), L(t), R(t), S(t), h(t), r(t)) - c(t) - \lambda K(t) \\ \frac{d}{dt}R(t) = g(t, R(t)) - h(t) \\ \frac{d}{dt}S(t) = -r(t) \end{cases} \tag{8}$$

where the states $x(t) = (K(t), L(t), R(t), S(t))'$ represent the manufactured capital, the labour, the renewable and exhaustible resource stocks, f denotes the production function for K which is assumed to be increasing and concave with respect to each input, g represents the biological mechanism related to the regeneration of the renewable resource R, $\lambda > 0$ stands for the capital K depreciation rate and the controls $u(t) = (c(t), h(t), r(t))'$ correspond to the consumption of the capital K and the extraction rates of the renewable and exhaustible resources, respectively.

Example 4 (Stochastic Ramsey problem and model uncertainty)**.** Various elements in the Ramsey model (see Example 1) are not fully known and as subject to randomness. Possible candidates can be the technological factors, modelled by A, or the fluctuations of the labour force due for example to demographics, is per capita capital is to be considered. Such fluctuations can be introduced into the model in terms of a stochastic factor process. For example, in the time discrete case one may replace (6) by

$$K(t+1) = K(t) + f(K(t), \omega(t)) - c(t) - \lambda(\omega(t))K(t), \;\; t \in \mathbb{N}, \tag{9}$$

where $\{\omega(t) : \; t \in \mathbb{N}\}$ are i.i.d. random variables, introducing stochastic shocks in this model economy. On the other hand, in the continuous time version, a popular choice for the stochastic factor process (mainly on account of central limit theorem considerations) is the Wiener process [1] $\{W(t) : \; t \geq 0\}$, and a possible generalization of (5) may be

$$dk(t) = [f(k(t)) - (n(t) - \sigma^2(t) + \lambda)k(t) - c(t)]dt - \sigma(t)k(t)dW(t), \tag{10}$$

where $n(t)$ is the mean rate of growth of labour supply (subject to the demographics, hence subject to random fluctuations modelled in terms of $W(t)$ where $\sigma(t)$ is a measure of the variance of these fluctuations), and $k = \frac{K}{L}$ is the per capita capital. Model (10) is a stochastic differential equation (in Itô form) and for each t generates a state $k(t)$ which is a random variable, whose distribution can be obtained by solving (10), based on the provided dynamics and the knowledge of the probability distribution of the stochastic factor $\{W(t) : \; t \geq 0\}$. In (10), the first contribution provides information concerning the mean behaviour of k, whereas the second contribution models fluctuations (of zero mean) around the mean behaviour.

[1]The Wiener process $\{W(t) : \; t \geq 0\}$ displays independent increments, continuous paths and satisfies $W(t+h) - W(t) \sim N(0, h)$ for every $t, h > 0$.

However, even that situation is too ideal in certain cases. For example, on account of incomplete information, there may only be postulates concerning the probabilistic law governing the fluctuations $W(t)$ around the mean behaviour, i.e., $W(t) \sim N(v(t), t)$ for some unknown drift $v(t)$. This essentially means that there is not a single acceptable probability law for the distribution of the stochastic factor W, but several plausible models P_v (around a central model P_0) are possible, meaning that if one decides to choose the probability model P_v for $W(t)$, then she assumes that $W(t) \sim N(v(t), t)$, with the consequent implications on the law of the state variable k through (10). In fact, tools from stochastic analysis and in particular Girsanov's theorem, can be used to show that the implication of choosing P_v as the relevant model for the stochastic factor W, is to affect the dynamics of the state in (10) by modifying it as

$$dk(t) = [f(k(t)) - (n(t) - \sigma^2(t) + \lambda)k(t) - c(t) + v(t)\sigma(t)]dt - \sigma(t)k(t)dW(t), \quad (11)$$

where now W follows the law P_0. Now, there are two choice variables (control variables) in (11), consumption c and the probability law i.e. the information drift v, and of course the state of the system depends on both. Model uncertainty consists in the fact that eventhough there is a true probability law for W, corresponding to some v_0, this is not known to the decision maker, so as far as she is concerned there are more than one plausible choices for v. Borrowing the terminology from game theory, a non benevolent player (Nature) chooses the control v and decides on the true model v_0, while the decision maker plays against nature in an attempt to make a decision on c that will cause her the minimum possible damage (appropriately defined). This leads to robust decision making schemes such as for example worst-case scenarios decisions or precautionary decisions using tools from differential game theory or from robust control.

Example 5 (Spatial models)**.** Many important problems in economics and sustainability have a natural spatial dependence (for example resources are not uniformly allocated in space, externalities such as pollution or its effects vary from one location in space to another etc). Such dependences can be modelled by appropriate modifications of (1) or (2). We only focus on a renewable resource harvesting model for brevity in this example, however, the same generalization can be obtained for any of the models provided above. Assuming that y models the spatiotemporal distribution of biomass and the u is a harvesting function we may consider transport mechanisms for biomass in space. For example, if the density of biomass is too high in some location in space z, then biomass may tend to migrate to other locations in space z' where density is lower as a strategy for better survival opportunities (availability of nutrient). On the other hand, biomass is reproduced locally in terms of a reproduction function F, so that the book balancing for biomass reduces to the partial differential equation

$$\frac{\partial y}{\partial t}(t, z) = D\frac{\partial^2 y}{\partial z^2}y(t, z) + F(y(t, z)) - u(t, z), \quad (12)$$

with the first term on the right hand side modelling the spatial transport of biomass from regions of high density to regions of lower density. If for any reason we are uneasy considering physical space as a continuum, we may consider it as a union of certain regions D_i, $i = 1, \dots, M$, and assume that biomass is more or less uniform in each one of them and equal to its average value over this region, y_i. Biomass can be transferred from one region to the other, resulting in modifications in the average, leading to a simplified spatial model in the place of (12), and of the form

$$\frac{dy_i}{dt}(t) = \sum_{j=1}^{M} L_{ij} y_j(t) + F(y_i(t)) - u_i(t), \;\; i = 1, \ldots, M, \tag{13}$$

where the coefficients L_{ij} provide information on the connectivity structure between the regions D_i and the propensity of biomass to transfer from one region to the other.

If required, model (12) or (13) can be extended to include stochastic fluctuations around a mean behaviour. To keep technicalities to the minimum, we consider only (13) and its stochastic version

$$dy_i(t) = \left(\sum_{j=1}^{M} L_{ij} y_j(t) + F(y_i(t)) - u_i(t) \right) dt + \sum_{j=1}^{N} \sigma_{ij}(t) dW_j(t), \tag{14}$$

where now the stochastic factors are modelled as a multidimensional Wiener process, that affect different spatial locations through the correlation coefficients σ_{ij}. Model uncertainty may further be introduced by allowing different models for the stochastic factors W_j, i.e. P_{v_j} according to which $W_j(t) \sim N(v_j(t), t)$, and using similar arguments as in Example 4 this leads to model

$$dy_i(t) = \left(\sum_{j=1}^{M} L_{ij} y_j(t) + F(y_i(t)) - u_i(t) + \sum_{j=1}^{N} \sigma_{ij}(t) v_j(t) \right) dt + \sum_{j=1}^{N} \sigma_{ij}(t) dW_j(t), \tag{15}$$

with the v_i (or rather $\sum_{j=1}^{N} \sigma_{ij}(t) v_j$) interpreted as spatially dependent information drifts due to model uncertainty which turn the decision making problem to a spatially dependent stochastic differential game against nature (in the same spirit as in Example 4).

The above spatial extension of the Ramsey or of biomass models is only provided here for the sake of example. Models displaying spatial effects are becoming increasingly important in sustainability science, see for example [37] for an interesting application in irrigation technology.

2.3 Social Welfare Criteria, Criticism and Uncertainty for the Future

A social welfare criterion represents the aggregate well-being instrument, in terms of the human community, and can be considered as the major factor according to which is introduced the dominant optic on a policy selection problem. In particular, the welfare function can be thought in general as an aggregate utility of all parties that constitute the human community that will be affected by the policy. As an aggregate index of the well being of the community it combines several aspects like: the population figures, utility functions that should sufficiently capture the behaviour of the community under study, time effects which are typically introduced by discounting approaches, and others. Below we discuss several important social welfare criteria highlighting important issues like the effect of discounting in the decision.

Criterion 1 (*Discounted Utilitarian Criteria)*. The time-discounting criterion is the most classical welfare function which first appeared in Ramsey's model [68] and adopted by many authors in the following years. Under this approach, given a common for each part in the community utility function U, depending on the consumption control variables (however in some works the authors [25, 47] consider utility to depend also on some states of the economy) the welfare function to be maximized is constructed as the discounted average over time of this utility and in particular

$$W_{DU}(c) = \int_0^\infty e^{-\rho t} U(c(t))dt, \;\; \rho > 0 \tag{16}$$

where ρ denotes the discounting parameter. There has been a debate the last two decades if one should use the exponential factor $\delta_{\exp}(t) = e^{-\rho t}$ as a discounting mechanism for the future generations and of course its meaning. The welfare criterion in its current form seems to cause the *dictatorship of the present* effect, since generations in the distant future are not taken properly into account which is like ignoring the very distant to future generations well-being. Therefore, the criticism in this approach is more on the case where $t \to \infty$ and not for more short-term planning and policy selection. The last decade, several attempts were made in the direction of improving (or making more fair) the effect of discounting either by ranking the generations with respect to their well-being and then performing the discounting (see e.g. the rank-discounted utilitarian approach proposed in [81]) or by changing the discounting nature (see e.g. [10] where the sustainable discounted utilitarian criterion is introduced and [62, 73] where hyperbolic discounting criteria are considered).

Criterion 2 (*Rawlsian Welfare)*. Another school of thought in welfare criteria is that Rawls' [70] where the welfare function is derived under the perspective of *inter-generation fairness* i.e., to choose the greatest possible and simultaneously constant consumption for all generations (leading to constant utility for all generations in the planning horizon). Clearly, this approach does not discount anything to present values and the Rawlsian welfare function is defined as

$$W_R(c) = \inf_{s \geq 0} U(c(s)) \tag{17}$$

Due to its "fairness" assumption regarding the equally treatment of the current and all forthcoming generations, this criterion is preferred in modern approaches by a lot of authors leading to interesting maximin problems. The main drawback of this approach is caused again in the case that $t \to \infty$, causing the *dictatorship of the future* effect where the current generation should sacrifice too much in order to act fairly with respect to the forthcoming generations. However, since the level of ignorance for the very distant future is too high (climate change, technological progress, substitutions between different natural resources, etc) the very long-term policy selection with such criteria could be quite problematic.

Criterion 3 (*Mixed Bentham-Rawls Criterion)*. An alternative criterion that mixes both types of dictatorships is presented by mixing the classical discounted utilitarian criterion and the Rawlsian [2]. In particular, mixing both welfare functions by a factor

$\theta \in (0,1)$ provides a hybrid criterion that combines characteristics from both perspectives however in a manner that is more plausible than other attempts that have tried to avoid both types of dictatorships. The welfare function is defined as the convex combination of the discounted utilities criterion and the Rawlsian perspective:

$$W_{MBR}(c) = \theta \int_0^\infty e^{-\rho t} U(c(t))dt + (1-\theta) \inf_{t \geq 0} U(c(t)), \ \ \theta \in (0,1) \tag{18}$$

Criterion 4 (*Chichilnisky's Criterion)***.** An attempt to avoid both dictatorship types was first conducted by Chichilnisky [23] by formulating a welfare function which takes into account both the discounted utilities welfare and the infinitesimal utility, the so called *C-criterion.* The resulting welfare function is

$$W_C(c) = \theta \int_0^\infty \delta(t) U(c(t))dt + (1-\theta) \lim_{t \to \infty} U(c(t)) \tag{19}$$

where $\theta > 0$ is the mixing factor and $\delta(t)$ represents the discounting factor that is employed (e.g. exponential, hyperbolic, etc).

The possibility of choosing discount functions $t \mapsto \delta(t)$ different from the exponential discount function $t \mapsto \delta(t) = e^{-\rho t}$, stems from the fact that, as also indicated by empirical evidence (see e.g. [36]), there is no reason why the discount factor should be constant across generations; in fact there is strong evidence that individuals use higher discount rates for the near future than for the long term. Moreover, such effects may also appear in the aggregation of time preferences for example individuals may use constant but different discount rates but the collective discount rate may be non-constant or as an effect of stochastic fluctuations (see e.g. [30], [40], [41] for an explanation of term structure of preferences driven by stochastic fluctuations, see also [65] and references therein for relevant phenomena in aggregation of time preferences in the presence of model uncertainty). The effects of discounting the future are crucial in economic policy and sustainability considerations (see for example [38], [39] or [48] and references therein) as well as for ethical considerations concerning intergenerational justice (see e.g. [67]), hence the appropriate modelling of the discount factor is of outmost importance both in theoretical or applied studies. The above considerations concerning varying discount rates led to the introduction of the concept of hyperbolic discounting (see e.g. [26] or [55] or the review [36] containing the state of the art in the field up to 2002 or [42] and references there in for more recent advances). Such notions can be covered by choosing $\delta(t) \geq 0$ to be a general function satisfying certain generic properties, for example δ being decreasing in t. However, such considerations, although they may bring more realism into the model, at the same time introduce complications, as for example the problem of time inconsistency. This is an important issue which loosely speaking (see e.g. [11] or [31] for more concrete statements) implies that if $\delta(t) \neq e^{-\rho t}$ then the time when the agent decide to assess a consumption stream may affect her decisions, for example if two consumption streams $c_1(\cdot), c_2(\cdot)$ are to be compared and assessed it may be that $c_1(\cdot) \succeq c_2(\cdot)$ if the assessment is done by discounting everything to time t_1 but $c_2(\cdot) \succeq c_1(\cdot)$ is the assessment is done by discounting everything to time t_2. That creates rationality problems that need to be seriously addressed. An interesting viewpoint on such problems is to consider them from a game theoretic point of view, looking for a subgame perfect equilibrium of the leader-follower game played by successive generations (see e.g. [30], [31] or [49]).

2.4 The viability approach to sustainability

2.4.1 Fundamental concepts in the deterministic case

Viability theory is the standard approach in economics for studying the set of admissible controls and ensure that the problem at hand has a non-empty set of solutions. It was introduced by Aubin [6], however many authors have contributed to this direction and provided several extensions to the stochastic framework and the multi-criteria setting (see e.g. [28, 29, 57, 58]). This approach relies on the notion of the so called *viability kernel* which is the set that contains all the initial states for the economy/system under study for which there exist controls that lead to state trajectories that satisfy all viability conditions that are taken into account into $\mathcal{K}$ for the selected time horizon (typically the horizon is considered infinite). Since the constraints refer to sustainability thresholds and requirements, the viability kernel contains all the initial states of the economy from which start viable economic trajectories in the sense that these trajectories that satisfy all the sustainability criteria under consideration at all times. A more formal mathematical definition of the viability kernel follows.

Definition 2.1 (Viability Kernel)**.** The viability kernel of $\mathcal{K}$ for the economy which dynamics are described in (1) is defined as the set

$$Viab(F, \mathcal{K}) = \{x_0 \mid \exists (x(\cdot), u(\cdot)) \in \mathcal{S}(x_0) \cap \mathcal{K},\ \forall t \in \mathbb{R}^+\}$$

where F refers to the special characteristics of the economy, and $\mathcal{K}$ to the conditions that have to be satisfied both by states and controls in order to consider the resulting trajectories as viable.

We present the role of the viability kernel by providing some results by examining an economy of the type discussed in Example 2 which investigated in [57]. Under the sustainability and economy viability framework, typical conditions that may be required are illustrated in Table 1.

Sustainability/Viability Constraint	Interpretation
(a) $r(t) \geq 0$	resource extraction irreversibility
(b) $S(t) \geq 0$	resource scarcity
(c) $S(t) \geq S_b \geq 0$	exhaustible resource conservation threshold
(d) $f(K(t), r(t)) - c(t) \geq 0$	production $\geq$ consumption
(e) $K(t) \geq 0$	production irreversibility
(f) $c(t) \geq c_b > 0$	minimum guaranteed consumption level

Table 1: Sustainability constraints for the economy (7)

In this case, the viability kernel is defined as

$$Viab(f, c_b, S_b) = \{(S_0, K_0) \mid \exists (c(\cdot), r(\cdot)) \text{ s. t. } (S(t), K(t)) \text{ satisfy (7) and (a)-(f)}\}$$

According to the standard approach of a Cobb-Douglas type technology of the form $f(K,r) = K^{\alpha}r^{\beta}$, $\beta < \alpha < 1$ the viability kernel with respect to the aforementioned conditions (Proposition 3, [57]) is determined as

$$Viab(f, c_b, S_b) = \begin{cases} \emptyset, & \alpha \leq \beta \\ \{(S,K) \ : \ S \geq V(K, c_b, S_b)\}, & \alpha > \beta \end{cases}$$

where $V(K, c_b, S_b) = \frac{1}{\alpha-\beta}\left(\frac{c_b}{1-\beta}\right)^{(1-\beta)/\beta} K^{(\beta-\alpha)/\beta} + S_b$. It is interesting to remark some effects related to the technological parameters and in particular the elasticities α (capital) and β (resource). The above result, states that the examined economy would be sustainable only in the case when resource elasticity is smaller than capital elasticity ($\alpha > \beta$), otherwise the viability kernel is empty and no feasible paths exist (see [71]). This relation indicates also the situation when a crisis is unavoidable leading to non sustainability of the economy since, even if in the initial times the trajectories lie in the viability kernel, it is certain that in a finite time instant the trajectories will exit this set leading to no sustainability. However, if a trajectory lies in the interior of the viability kernel and sometimes reaches the boundary, then specific paths must be followed to remain the economy to sustainable states, and in particular for the case under study the viable decisions (controls) coincide to

$$r^*K = \left(\frac{c_b}{1-\beta}\right)^{1/\beta} K^{-\alpha/\beta}, \ \ c^*(K) = c_b. \tag{20}$$

It is also very interesting to examine the evolution of the viability kernel with respect to the parameters of interest. In particular, the following comparisons can be obtained:

- Effect of production elasticity: $\alpha_1 < \alpha_2 \ \Rightarrow \ Viab(f_1, c_b, S_b) \subset Viab(f_2, c_b, S_b)$
- Effect of minimum consumption required level: $c_b^1 > c_b^2 \ \Rightarrow \ Viab(f, c_b^1, S_b) \subset Viab(f, c_b^2, S_b)$
- Effect of the minimum guaranteed resource stock: $S_b^1 > S_b^2 \ \Rightarrow \ Viab(f, c_b, S_b^1) \subset Viab(f, c_b, S_b^2)$

An important role is also played by the depreciation rate λ concerning the existence of viable solutions in this case, since based to Solow [71] if $\alpha < 1$ and $\lambda > 0$ the viability kernel is empty.

2.4.2 Sustainability, viability and maximin approaches

For a general abstract economic model of the form (1), and given a utility function $U(x(t), u(t))$ which may depend both on the states $x(t)$ and the controls of the economy $u(t)$, a maximin approach under the viability theory framework is considered in [28]. The maximin approach is in line with the Rawlsian perspective (please see Criterion 2) in policy selection which refers to the maximization of the minimal level of utility for all generations over time. In this case, the respective value function $V : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$V(x_0) = \sup_{(x(\cdot),u(\cdot))\in\mathcal{S}(x_0)} \left(\inf_{t\in\mathbb{R}^+} U(x(t), u(t))\right). \tag{21}$$

An interesting result related to the above maximin problem, is that when a regular maximin path exists then utility remains constant over time, i.e. $U(x^*(t), u^*(t)) = V(x_0)$ for all $t \in \mathbb{R}^+$, which can be interpreted as inter-generational equity from a sustainability point of view. Blending the above problem with sustainability constraints on states and controls as the ones comprising the set $\mathcal{K}$, then under the framework of viability theory, one needs to properly define the viability kernel. Given that a maximin approach is discussed, an extra condition that will be required is that

$$U(x(t), u(t)) \geq U_b, \ \ \forall t \in \mathbb{R}^+ \tag{22}$$

which refers to the minimum guaranteed utility level for all generations (corresponding to certain policies in order to guarantee that by choosing appropriately the consumption rates). The resulting viability kernel should depend on U_b, i.e.

$$Viab(U_b) = \left\{x_0 \mid \exists (x(\cdot), u(\cdot)) \in \mathcal{S}(x_0) \cap \mathcal{K}, \ U(x(t), u(t)) \geq U_b, \ \forall t \in \mathbb{R}^+\right\}. \tag{23}$$

From the above definition it can be proven that if a maximin optimal solution $(x^*(t), u^*(t))$ exists starting from x_0, then $x^*(t) \in Viab(V(x_0))$ for all $t \in \mathbb{R}^+$ (Proposition 3, [28]). This result states that the maximin trajectory remains within the viability kernel as defined in (23) and can be further characterized using the properties of the viable trajectories. What is of importance in the study of maximin problems form the viability theory perspective (or reverse) is that conditions for sustainability can be provided. In particular, a global-type condition for an economy to be sustainable is to satisfy that $x_0 \in Viab(U(x_0, u_0))$ while a more local condition (equiv. to the weak Hartwick's rule) is that $\mathcal{H}(x_0, u_0, V_x(x_0)) \geq 0$ where $\mathcal{H}$ denotes the associated Hamiltonian to the optimization problem. Then, a distinction among the various states of sustainability can be done according to the following principle:

- $U(x_0, u_0) > V(x_0)$: this is a strong indication of non sustainability of the economy. It holds that $x_0 \notin Viab(U(x_0, u_0))$, therefore the global condition is not satisfied. In terms of economy that means that the current economy states do not make possible to maintain the current utility.

- $U(x_0, u_0) = V(x_0)$: this translates to $x_0 \in Viab(V(x_0)) = Viab(U(x_0, u_0))$ i.e. the global condition is satisfied. However, the local condition may not hold and depends on the decisions u_0, whether this is a part of an optimal maximin feedback $u^*(x_0)$ or not. So, if the local condition is satisfied, sustainability holds since the trajectories are at most on the boundary of the Viability kernel. In the case where the local condition is not satisfied, the trajectories will leave the viability kernel so the sustainability characterization does not hold.

- $U(x_0, u_0) < V(x_0)$: the global condition is satisfied since $x_0 \in Viab(V(x_0)) \subset Viab(U(x_0, u_0))$ and the current utility can be sustained for sure since it is lower than the maximin value. However, depending on the controls u_0 (if there are a part of a maximin feedback) which should be checked by the local condition, the trajectories may remain (sustainability) or may leave the kernel (unsustainability).

2.4.3 Viability Theory in a Stochastic Framework

A stochastic extension of the notion of viability kernel was first introduced in [59] in a discrete setting and for given terminal horizon $T < \infty$. The framework of system (1) is expanded by substituting the generator mechanism F with G and allowing the latter to depend on some $\omega(\cdot)$ which represents contingencies that affects the economical system dynamics. This is a very plausible assumption since there is a variety of parameters that cannot be precisely modeled, like climate and environmental conditions, random events that affect the system, natural hazards, etc. In this perspective, system (1) is now extended to

$$\begin{cases} \frac{d}{dt}x(t) = G(x(t), u(t), \omega(t)), \;\; \omega(t) \in \Omega \\ u(t) \in \mathcal{U}(x(t)) \subset \mathbb{R}^m \end{cases} \tag{24}$$

where $\omega(\cdot)$ will be hereafter referred to as scenario and Ω denotes the set with all possible scenarios. Consider a number of sustainability criteria that are measured instantaneously by a set of certain indicators (measuring economical or ecological quantities):

$$\mathcal{I}_k(t, x(t), u(t)) \geq \eta_k, \;\; k = 1, 2, ..., q \tag{25}$$

whose outputs depend on the scenario that is realized through the states $x(t)$. These random constraints can be conceived as a generalization of the set of constraints $\mathcal{K}$ discussed in the deterministic setting. In this framework, it is meaningless to define a viable trajectory but rather we must introduce the notion of a *viable scenario.* In particular, for a given control $\hat{u}$ (feedback control), and initial states x_0 at time $t = 0$, the set of viable scenarios is defined as

$$\Omega_{\hat{u},x_0} = \left\{ \omega(\cdot) \in \Omega \;\middle|\; \begin{array}{l} \frac{d}{dt}x(t) = G(x(t), u(t), \omega(t)), \;\; x(0) = x_0 \\ u(t) = \hat{u}(t, x(t)) \\ \mathcal{I}_k(t, x(t), u(t)) \geq \eta_k, \;\; k = 1, 2, ..., q \\ t \in \mathbb{R}^+ \end{array} \right\} \tag{26}$$

In this stochastic framework, an appropriate metric tool to assess the *viability* of the control $\hat{u}$ is to measure the likelihood that this particular strategy will meet its sustainability objectives as determined in (25) by estimating the probability of the set with the associated *successful* scenarios. This measure will be called *viability probability*, it depends on the strategy $\hat{u}$ that needs to be assessed and of course to the thresholds $\boldsymbol{\eta} = (\eta_1, ..., \eta_q)'$ that have been set in (25). Introducing this quantity in mathematical formulation is defined as

$$\Pi(\hat{u}, \boldsymbol{\eta}) = \mathbb{P}(\{\omega(\cdot) \in \Omega_{\hat{u},x_0}\}) \tag{27}$$

The optimal strategy (control) given the thresholds η, should be obtained through the solution of the maximization problem $\Pi^*(\boldsymbol{\eta}) := \max_{\hat{u}} \Pi(\hat{u}, \boldsymbol{\eta})$ where $\Pi^*(\boldsymbol{\eta})$ denotes the maximal viability probability characterizing the optimal strategy (the strategy that is more likely to meet the viability targets comparing to the other strategies in the set of admissible controls). Clearly, the described optimization problem's complexity depends on several aspects like the number of the states, the number of controls, the time discretization size (in case that the problem's solution is approximated through a proper discretization scheme), etc. For the successful treatment of such problems, advanced and effective numerical approaches (e.g. stochastic optimization approaches [66], robust optimization approaches [14]) should be employed.

2.4.4 Characterization of Sustainability through Indicators

A very interesting approach in characterizing sustainability through some appropriately selected threshold values related to certain sustainability (or viability) issues is introduced in [58]. This work shares the perspective of the policy-makers, introducing criteria characterizing sustainability through indicators. For instance, the set of sustainability constraints defined in (25), and in particular the choice of the thresholds (minimal levels) $\eta \in \mathbb{R}^q$ upon which depends the fulfillment of each requirement, are derived through the solution of a maximization problem. The objective of the associated problem is a function $\Phi(\eta_1, \eta_2, ..., \eta_q)$ called *preferences* function, which has similar properties with a utility function and through this function can be introduced possible dependencies between the various sustainability requirements represented by the vector η. The value of function Φ at a certain point $\eta \in \mathbb{R}^q$ can be considered as the aggregate level of satisfaction concerning all the sustainability issues that are taken into account, if the minimal requirements represented by η are met. However, we should make clear that the output of this function is not time-dependent, e.g. it cannot be interpreted as a type of instantaneous utility, but rather as the total satisfaction if the requirements η are met in the whole planning horizon (finite or infinite). An important advantage of this approach is that it can be implemented in both sustainability frameworks (weak or strong) since the preference functions allow for this kind of flexibility.

Let us define the set of *achievable thresholds* as

$$\mathcal{A}(x_0) = \left\{ \eta = (\eta_1, \eta_2, ..., \eta_q)' \;\middle|\; \begin{array}{l} \exists (x(\cdot), u(\cdot)) \text{ s.t. } \frac{d}{dt}x(t) = F(x(t), u(t)),\ x(0) = x_0 \\ \mathcal{I}_k(x(t), u(t)) \geq \eta_k,\ k = 1, 2, ..., q,\ \forall t \in \mathbb{R}^+ \end{array} \right\} \tag{28}$$

which is the set of thresholds η for which there exist state trajectories and policies satisfying the dynamics of the economy. The resulting optimization problem, which can also be realised as a generalized maximin criterion, can be written as

$$\max_{\eta \in \mathcal{A}(x_0)} \Phi(\eta). \tag{29}$$

Notice that the formulation above contains as a special case if $q = 1$ and $\mathcal{I}_1(x, u) = U(x, u) \geq \eta_1$ the standard maximin utilitarian problem (Rawlsian criterion) since the preferences function is reduced to $\Phi(\eta) = \eta_1$ coinciding with the problem examined in [28]. Problem (29) is a static optimization problem which can be treated by the typical Karush-Kuhn-Tucker conditions given that the preferences function Φ is concave and the set $\mathcal{A}(x_0)$ convex.

As a particular example, consider problem 7 under the assumption of zero depreciation rate for the manufactured capital ($\lambda = 0$). Then, for a given conservation level S_b of the exhaustible resource, the maximal sustainable consumption is derived as the solution to the optimization problem

$$c^+(K_0, S_0, S_b) = \max \left\{ c_b \;\middle|\; \begin{array}{l} (K_0, S_0):\ \exists (c(\cdot), r(\cdot)) \text{ such that} \\ \forall t \in \mathbb{R}^+,\ c(t) \geq c_b,\ S(t) \geq S_b \end{array} \right\} \tag{30}$$

which for the Cobb-Douglas technology case is derived as

$$c^+(K_0, S_0, S_b) = (1-\beta)(S_0 - S_b)(\alpha - \beta)^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}} \tag{31}$$

being the upper bound for consumption. As a result, any chosen pair (c_b, S_b) that satisfies $c_b \leq c^+(K_0, S_0, S_b)$ guarantees the sustainability of the economy.

The discussed framework is also extended to the stochastic case, as discussed in 2.4.3, where the states $x(t)$ are provided by a generating mechanism G which is subject to uncertainty. For given thresholds η, we are interested in calculating the probability of the set of scenarios $\omega(\cdot)$ that the underlying economy can be characterized as sustainable in the sense that all the requirements of the form (25) are met, i.e.

$$\Pi(x_0; \eta) = \mathbb{P}\left(\left\{ \omega(\cdot) \in \Omega \;\middle|\; \begin{array}{l} \frac{d}{dt}x(t) = G(x(t), u(t), \omega(t)),\ \ x(0) = x_0 \\ u(t) = \hat{u}(t, x(t)) \\ \mathcal{I}_k(t, x(t), u(t)) \geq \eta_k,\ \ k = 1, 2, ..., q \\ t \in [0, T] \end{array} \right\} \right). \tag{32}$$

Clearly, in the deterministic case the investigated thresholds are either achievable (i.e. $\Pi(x_0; \eta) = 1$) or not (i.e. $\Pi(x_0; \eta) = 0$). Moreover, in this weaker viability framework, it is possible one to determine the set of achievable thresholds by requiring a certain probability level of occurence for the sustainability scenario. For instance, if $\tau \in (0, 1)$ denotes the confidence level, then the set which contains the thresholds η for which the sustainability scenario is possible with probability at least τ is determined as

$$\mathcal{A}(x_0, \tau) = \{\eta \mid \Pi(x_0; \eta) \geq \tau\}. \tag{33}$$

2.5 Optimal control approaches

An alternative (and possibly complementary approach) to viability is the optimal control approach to sustainability problems. The philosophy behind the optimal control approach is to define a criterion to be optimized (in principle an intertemporal utility criterion) and then choose the optimal policy in such a way so as to achieve the optimal value of this criterion. While the viability approach focuses on the initial conditions of the system such that a final target (which is sustainability compatible) is reached, the optimal control approach focuses on deriving the optimal policy that allows the decision maker to reach this target in the best possible way. Optimal control theory is a mature mathematical field, and there are established methods for the treatment of such problems ([27, 77]). In this section, we focus on rephrasing certain of the problems presented in section 2.2 into optimal control problems and the methodological framework for their treatment.

2.5.1 Deterministic optimal control

The deterministic control framework consists in choosing a general evolution model for the state of the system (either on the continuous time setting or on the discrete one) for example in the general framework of Examples 1, 2, 3 etc. and then select the optimal policy in such a way as to maximize a welfare criterion that accounts for sustainability considerations, for example one of the welfare criteria considered in Section 2.3. While any of the above models in the general formulation (2) supplemented with any of the sustainability criteria in Section 2.3 can be considered, here for the sake of brevity we contain ourselves to the simple Ramsey model (see Example 1) in order to fix ideas.

Let us therefore consider the problem of maximizing a consumption welfare criterion of the types considered in Section (2.3) for the state equation

$$\frac{dk}{dt} = \bar{f}(k) - c, \;\; k(0) = k_0 \geq 0, \;\; c \geq 0. \tag{34}$$

As shown in [11] the behaviour of the system largely depends on the choice of performance function to be maximized. For example if the standard welfare criterion

$$\sup_{c \geq 0} \rho \int_0^\infty e^{-\rho t} U(c(t)) dt \tag{35}$$

is chosen (see Proposition 1 in [11]), then the associated optimal control of maximizing (35) under the state equation (34) has a unique solution for every initial stock $k_0 > 0$ with the optimal path in both k and c being monotonic in t, and such that

$$\lim_{t \to \infty} k^*(t) = \bar{K}, \;\; \lim_{t \to \infty} c^*(t) = \bar{f}(\bar{k}).$$

The optimal path can be characterized in terms of the Pontryagin maximum principle, with the use of a shadow price for k (adjoint variable) leading to the Hamiltonian formulation

$$\begin{aligned}
&\frac{dk}{dt} = \frac{\partial H}{\partial p}(t, k, p), \\
&\frac{dp}{dt} = -\frac{\partial H}{\partial k}(t, k, p), \\
&H(k, p) = \max_{c \geq 0} e^{-\rho t} U(c) + p(f(k) - c),
\end{aligned}$$

which (upon choosing $\hat{p}(t) = p(t) e^{-\rho t}$) leads to the optimality system

$$\begin{aligned}
&\frac{dk}{dt} = f(k) - c, \\
&\frac{d\hat{p}}{dt} = (\rho - \bar{f}'(k))p, \\
&\hat{p} = U'(c),
\end{aligned}$$

which when endowed with an appropriate limiting condition at infinity (often called the transversality condition) characterizes the optimal path. The choice of the transversality condition has been a subject of heated discussion, however, a common choice is $\lim_{T \to \infty} \hat{p}(T) k(T) = 0$. This characterizes the optimal path in a nice geometric way, in the phase plane of the Hamiltonian system $(k, \hat{p})$, as the unique path that drives the system to its equilibrium state, satisfying $f(k) = c$ and $\rho - f'(k) = 0$, which in the terminology of dynamical systems corresponds to the stable manifold of the steady state (which is a saddle point). Moreover, if $\bar{f}(k) = Af(k) - \lambda k$, with A modelling the effect of technological progress in the production and λ modelling the deterioration of capital, the optimality system yields

$$\begin{aligned} Af'(k) &= \rho + \lambda, \\ c &= Af(k) - \lambda k, \end{aligned}$$

with the first equation, known as the Ramsey golden rule characterizing the optimal capital level at equilibrium and the second characterizing the optimal consumption at equilibrium in terms of k. Moreover, the asymptotic level is a strictly decreasing function of ρ. Generalizations of this approach to more involved models, as for example in the case of multiple assets and multiple policy instrument, are also possible, leading to interesting generalization of the golden rule, such as for example the Hartwick rule (see e.g. [8], [12], [22], [74], [75]).

However, this nice geometrical intuition which derives in this simple model, is difficult to turn into a strict mathematical formulation in the infinite horizon, especially because of the need of an appropriate transversality condition. Moreover, complications arise for more general discounting functions, where for example more than one candidates for optimal paths may appear (see e.g., the case of a Chichlinisky like criterion where a continuum of equilibrium strategies may be obtained [30], [31]).

An interesting way out, which in fact can turn into an alternative and very versatile methodology is the methodology of dynamic programming, which uses instead the value function for the problem, defined as

$$V(k_0) = \sup_{c \geq 0} \int_0^\infty e^{-\rho t} U(c(t), k(t)) dt, \text{ subject to } (34),$$

(where now we may consider a more general utility function depending on the state of the system as well as on consumption) considered as a function of the initial condition of the problem. Based on the principle of dynamic programming it can be shown (see e.g. [31]) that the value function V (if it is C^1) satisfies the so called Hamilton-Jacobi equation

$$\rho V(k) = \max_{c \geq 0} \{U(c, k) + (f(k) - c)V'(k)\}, \tag{36}$$

and that (conversely) if $V \in C^2$ is a solution of (36), then the optimal strategy $c = \sigma(k)$ can be obtained in terms of V by

$$\frac{\partial U}{\partial c}(\sigma(k), k) = V'(k).$$

The optimal policy $c = \sigma(k)$ is called either a feedback control or a Markov policy and it characterizes the optimal consumption in terms of the state of the system. Based on this approach Ekeland et al [31] recover the solution of the standard Ramsey problem with exponential time discounting, but also manage to go further in treating the same problem under alternative time preferences, including the preferences displaying hyperbolic discounting or the Chichilnisky criterion [23].

Importantly, this methodology allows one to tackle the issue of time inconsistency (see Section 2.3) or the fact that the transversality condition at infinity required by the Pontryagin principle may not be very straightforward to be determined for more involved models. For example, in [31] a generalized Hamilton-Jacobi equation has been derived for the corresponding leader-follower game relevant for the time inconsistent problem (see Section 2.3) of maximizing

$$J(c,k) = \int_0^\infty [\delta(t)u(c(t),k(t)) + \Delta(t)U(c(t),k(t))]dt,$$

where δ, Δ are two different discount factors and u, U are two different utility functions. Note that the above model may also have important implications where considering the problem of decision making for agents with inhomogeneous preferences. Moreover, in the same paper [31] connections with an alternative formulation of the time inconsistent case in terms of another Hamilton-Jacobi equation (proposed in [49]) for the value function, of the form

$$\begin{aligned} -\int_0^\infty \delta'(t)u(\sigma(k(t)),k(t))dt - \int_0^\infty \Delta'(t)U(\sigma(k(t)),k(t))dt \\ = \max_{c\geq 0}\{u(c,k_0) + U(c,k_0) + V'(k_0)(f(k_0)-c)\}, \end{aligned}$$

where $\sigma(\cdot)$ characterizes the equilibium strategy and its equivalence to the approach proposed in [31] were established. Through the qualitative study of the relevant Hamilton-Jacobi equation, interesting results concerning equilibria are obtained for a number of alternative time discounting models such as for example the bi-exponential case or the Chichilnisky case [23], and the analysis was extended focusing on economic intuition of these results in [11].

2.5.2 Stochastic optimal control

The above considerations can be in the case where there are unknown factors in the models, as a result of incomplete knowledge. Such unknown factors can be attributed to a number of factors, for example changing environmental conditions, population growth, unpredictable events related to e.g. technological progress or natural disasters etc. In this case the above models can be extended to stochastic models for example in the same spirit as in Example 4. Under the effects of stochasticity, the appropriate utility functional to be maximized must be replaced by an expected value; for example one may consider the maximization of an expected intertemporal utility function of the general form

$$J(c,k) = \mathbb{E}_P\left[\int_0^T U(t,k(t),c(t))dt + \Phi(k(T))\right], \tag{37}$$

where here we switch to a finite horizon problem, which perhaps simplifies sustainability concerns further, and assume that the state equation is of the general form (in the spirit of e.g. (10))

$$dk(t) = b(t, k(t), c(t))dt + \sigma(t, k(t), c(t))dW(t), \tag{38}$$

for suitable functions b, σ. Solving the optimization problem under a stochastic evolution law of the above form requires tools from the theory of stochastic optimal control. While the extension of the Pontryagin principle is feasible, it is technically involved as it requires the development of a concept called forward-backward stochastic differential equations (see e.g. [50], [79], [80]), and therefore the dynamic programming approach, in terms of the so called Hamilton-Jacobi-Bellman equation (which is an extension of the Hamilton-Jacobi equation mentioned in the previous section), is more straightforward. According to this approach, under certain technical conditions (see e.g. [80]) the value function (considered as a function of the initial time t_0 and initial state k_0) for the above problem satisfies the equation

$$\begin{aligned} &\tfrac{\partial V}{\partial t_0} + \sup_c G(t_0, k_0, V, \tfrac{\partial V}{\partial k_0}, \tfrac{\partial^2 V}{\partial k_0^2}) = 0, \\ &V(T, k_0) = \Phi(k_0), \end{aligned} \tag{39}$$

where

$$G(t_0, k_0, V, p, P) = \frac{1}{2}\sigma^2(t_0, k_0, c)P^2 + b(t_0, k_0, c)p + U(t, k_0, c),$$

with the optimal policy being determined in terms of the derivatives of the value function if a suitably smooth solution of this equation exists. If not, then useful information for the value function can be obtained in terms of the notion of viscosity solutions (see e.g. [80]). Note that the effect of stochasticity is to introduce second order terms in the Hamilton-Jacobi equation introduced in the previous section.

2.5.3 Uncertainty and Space

The effects of model uncertainty can be introduced into the framework of Section 2.5.2 in the same spirit as mentioned in Example 4. Introducing uncertainty in terms of an information drift v which allows for alternative probability models for the factor process W, we may enhance model (38) to the generalized form

$$\begin{aligned} &dk(t) = [b(t, k(t), c(t)) + \sigma(t, k(t), c(t))v(t)]\, dt + \sigma(t, k(t), c(t))dW(t), \\ &k(t_0) = k_0. \end{aligned} \tag{40}$$

Then, the performance criterion (37) is no longer applicable as it corresponds to the case where a single probability model for the stochastic factor P applies, whereas model uncertainty (as mentioned in Example 4) corresponds to the case where more than one possible models for W apply. In this case, we may impose a penalization on certain models. Following the suggestion of Hansen and Sargent (see e.g. [45]) we may consider the Kuhlback-Leibler divergence (entropy) between possible models Q_v (parameterized in terms of the drift v) and the reference model P and consider the alternative performance criterion

$$\hat{J}(k,c,v) = J(k,c) - \mathbb{E}_P\left[\frac{\theta}{2}\int_0^T v(t)^2 dt\right],$$

with the extra term corresponding to the penalty term (see [45] for details) and the parameter $\theta > 0$ corresponding to the agent's uncertainty aversion. Then, the optimal control problem takes the form of a differential game with the decision maker selecting the optimal policy c and a non benevolent agent (nature) selecting the model in terms of v, thus leading to a Nash equilibrium of the form

$$\min_v \max_c \hat{J}(k,c,v), \text{ subject to (40).}$$

The dynamic programming framework and the Hamilton-Jacobi-Bellman equation can be extended to treat the above problem and derive robust policies c under model uncertainty, in terms of the solution of the so called Hamilton-Jacobi-Bellman-Isaacs equation. Uncertainty introduces new interesting effects, such as for example the breakdown of control policies in the limit of deep uncertainty ($\theta \to \infty$) or the excessive cost of optimal control for large uncertainty (see e.g. [5]).

Moreover, the effects of spatial heterogeneities can be taken into account in the spirit introduced in Example 5. In the absence of stochasticity one may consider generalizations of the Pontryagin maximum principle in terms of forward-backward coupled partial differential equations, or if an expansion in terms of eigenfunctions is applicable in term of countable systems of ordinary differential equations (see e.g. [17] and references therein). The introduction of spatial effects introduces interesting effects such as for instance the generation of spatial patterns for optimal consumption (see e.g. [16] or the generation of depletion patterns for the resource see e.g. [78]). The Hamilton-Jacobi-Bellman framework still applies (see e.g. [20]), however, now it it is a partial differential equation on a function space, a fact that creates considerable technical difficulties (see e.g. [35]). However, explicit solutions are possible for certain special cases, a fact that sheds important light on the spatiotemporal dynamics of the controlled system. These special cases are essentially centered around the AK spatial system (see e.g. [20] or [21]) with the results providing interesting insight as to the conditions connecting the discount factor and the spatial variability of the system (in terms of the spatial transport processes involved) for spatial patterns to emerge and persist. Similar results can be obtained for more general systems, in terms of a linearized analysis around selected spatially homogeneous optimal states (see e.g. [16] or [17]). Interesting effects arise when model uncertainty is combined with spatial effects. A preliminary analysis of the corresponding differential game can be found in [13], providing existence of solutions for the corresponding Hamilton-Jacobi-Bellman-Isaacs equation in the case of moderate uncertainty and insights towards possible pathologies in the case of deep uncertainty. More progress concerning exact solutions can be obtained for a linear quadratic stochastic differential game modelling deviations from a desired optimal target, in terms of a stochastic target following optimal control problem under uncertainty. In such cases exact solutions can be obtained in terms of a Riccati equation with the solution revealing interesting behaviour concerning the spatial structure of the deviations from the target, pattern formation or breakdown of solutions for the deep uncertainty regime [18]. Finally, an interesting recent contribution is the study of uncertainty in the initial conditions of a spatially dependent AK model, using an alternative penalization scheme for plausible models in terms of the Wasserstein metric (rather than the Kullback-Leibler entropy, see [63], [65]) was introduced by Papayiannis in [64]. There the effect of spatial inhomogeneity is taken into account in terms the graph Laplacian and an explicit solution of the relevant Hamilton-Jacobi equation is provided for a power utility function for a finite horizon problem which allows for quantification and allocation of risk and uncertainty into the various spatial domains [64].

The introduction of uncertainty in sustainability modeling is expected to bring interesting phenomena into the picture and to provide interesting policy implications. For a number of interesting applications in water management we refer to [32], [52], [53] and [54]. Moreover, an interesting analysis of second best environmental policies under uncertainty can be found in [4].

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