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Estimation of Asymmetric Stochastic Volatility in Mean Models

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Here we investigate the estimation of asymmetric Autoregressive Stochastic Volatility models with possibly time varying risk premia. We employ the Indirect Inference estimation developed in Gallant and Tauchen (1996), with a first step estimator either the Generalized Quadratic ARCH or the Exponential GARCH. We employ Monte-Carlo simulations to compare the two first step models in terms of bias and root Mean Squared Error. We apply the developed methods for the estimation of an asymmetric autoregressive SV-M model to international stock markets excess returns.

1 Introduction

In economic and financial data there are some well documented statistical facts, the so-called stylized facts. The most important of these is perhaps the volatility clustering, i.e. that, on average, periods of high (low) volatility are followed period with high (low) volatility. However, volatility clustering is also observed in data from physics and specifically from turbulence, called intermittency in turbulence terminology (see Barndorff-Nielsen 1997 [8])[1]. This has led to modifications and extensions of the original ARCH (Autoregressive Conditional Heteroskedasticity) model of Engle (1982) [27] and its generalization by Bollerslev (1986) [12], so that now there is a plethora of dynamic heteroskedasticity models (see e.g. Bollerslev, Chou, and Kroner,

[1] For a comparison between the stylized facts of the temporal behavior of asset returns, and differences in velocity of the mean wind direction of a large Reynolds number wind field see Barndorff-Nielsen and Shephard (2001) [9], and Mantegna and Stanley (1996) [48].

1992 [13], Bera and Higgins, 1993 [10], Bollerslev, Engle and Nelson 1994 [14], and Francq and Zakoian, 2010 [30] for a book).

Furthermore, economic theory, and specifically financial theory, often postulates specific relationships between first and second conditional moments. For instance, in the stock market context, the first conditional moment of stock market excess returns, say μ_t, is a function of volatility, say σ_t^2, (see e.g. Merton, 1980 [50], and Glosten, Jagannathan and Runkle, 1993 [35]). In fact, rational risk averse investor require higher expected returns during high volatility periods, implying a positive relationship between expected returns and volatility, something which is supported by e.g. French et al. (1987) [31] and Campbell and Hentschel (1992) [19] and Poon and Taylor (1992) [57]. Consequently, Engle, Lilien and Robins (1987) [28] introduced the so called ARCH in Mean model (ARCH-M), which was a first attempt to capture this relationship.

Glosten, Jagannathan and Runkle (1993) [35] and Nelson (1991) [54], among others, give support to a negative relationship between unexpected part of returns volatility. French et al. (1987) [31] interpret it as indirect evidence of a positive correlation between the expected risk premium and ex ante volatility. They suggest that unanticipated large shocks to the return process induce higher expected volatility. If expected volatility and returns are positively related, the current stock price should fall. This is known as the volatility feedback theory (see e.g. Campbell and Hentschel (1992) [19]).

Finally, it has been observed that volatility is higher after the stock market has a fall than after a rise of the same size, meaning that stock returns are negatively correlated with future volatility. This phenomenon was first discussed by Black (1976) [11], who suggested that it could be due to the increase in leverage that occurs when the market of a firm falls. However, it seems that the leverage effect is too small to completely explain this asymmetric response of volatility (see e.g. Christie (1982) [21], Figlewski and Wang (2000) [29], and Schwert (1989) [59], Hasanhodzic and Lo [42], and Bollerslev, Sizova, and Tauchen (2012) [15]). This effect can be accommodated within asymmetric GARCH setup such as the Exponential GARCH of Nelson (1991) [54], the Quadratic GARCH of Sentana (1995) [60] or the model of Glosten, Jagannathan and Runkle (1993) [35]. Hence, and especially in the area of empirical finance, a literature field emerged, where researchers tried to quantify and estimate these relationships using mainly the GARCH-M specification, either symmetric or asymmetric, especially due to its inference tractability (see among others Gonzales-Rivera 1996 [36], Choudhry 1996

[20], Dunne 1999 [26], Tai 2000 [62] and 2001 [63] , Ortiz and Arjona 2001 [56], Arvanitis and Demos 2004 [6] and 2004a [7]).

All the above mentioned conditional heteroskedastic models are characterized by the fact that the mean error "moves" the next period conditional variance. However, there is another class of conditionally heteroskedastic processes where a second error processes, possibly correlated with the mean error, "drives" the conditional variance, the so-called stochastic variance processes (see e.g. Andersen 1996 [1]). The most popular of the stochastic variance models defines volatility as a logarithmic first-order autoregressive, known as the first-order autoregressive Stochastic Volatility (SV(1)) model. Even thought SV models are considered as competitive alternatives to GARCH ones their application has been limited.

One of the reasons is that in the SV setting volatility is not measurable with respect to observable past information. Hence, volatility estimation involves not only filtering but smoothing techniques, as well, making the estimation of the parameters cumbersome (see e.g. Andersen and Benzoni 2009 [3], and Broto and Ruiz 2004). Furthermore, classical parameter estimation for this model is extremely difficult, because of the non-analytic form of the likelihood function. In other hand, the conditional variance in GARCH is observable given past information, which makes (quasi-) maximum likelihood estimation quite straightforward. The estimation methods that have been proposed for SV models can be divided into two main groups; those that try to construct the full likelihood function and those that approximate it (see e.g. Taylor 1986 [64], and Harvey, Ruiz and Shephard 1994 [40]). The estimation method based on evaluating the full likelihood function can be found in, for example, Jacquier et al. (1994) [43], Kim, Shephard and Chib (1998) [45], Sandmann and Koopman (1998) [58], Fridman and Harris (1998) [32], and Koopman and Uspensky (2002) [47]. Several method of moment approaches have also been employed to estimate the SV model parameters such as the, so called, efficient method of moments (Gallant and Tauchen 1996 [34]), the Indirect Inference (Smith (1993) [61] and Gourieroux, Monfort and Renault (1993) [38]), the spectral method of moments (Singleton 2001; Chacko and Viceira 2003; Knight, Satchell, and Yu 2002), the simulated method of moments (Duffle and Singleton 1993) and the generalized method of moments (Melino and Turnbull 1990, Andersen and Sorensen 1996) .

Here we investigate the estimation of asymmetric SV models with possibly time varying risk premia, i.e. the standard deviation could appear as an explanatory variable in the mean equation (SV-M).[2] In such a way we model simultaneously the first two moments of the observed process, with errors that can be correlated. We employ the Indirect Inference estimation fro the parameters of the models (see e.g. Gourieroux et al. (1993), Andersen and Sorensen (1996), Gourieroux and Monfort (1996)). In fact, we employ the method developed in Gallant and Tauchen (1996) [34], with a first step estimator either the Generalized Quadratic ARCH (GQARCH) model of Sentana (1995) [60] or the Exponential GARCH (EGARCH) of Nelson (1991) [54]. We employ Monte-Carlo simulations to compare the two first step models in terms of bias and Mean Squared Error (MSE), contributing in this way to the question of the first step estimator. Finally, we apply the developed methods for the estimation of an SVM model to international stock markets excess returns.

In the nest section we present the model. The estimation method is presented in the following, Section 3. In Section 4 we present the simulation results and compare the two first step estimators in terms of bias and root MSE. In Section 4 we estimate the SVM employing real data. We conclude in section 5.

2 The SVM Model

The normal Autoregressive Stochastic Volatility in Mean models are given by:

$$y_t = c_t + \varepsilon_t^* = c + \lambda\sigma_t + \varepsilon_t\sigma_t \quad where, \tag{2.1}$$

$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_{t-1} \; and \tag{2.2}$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{pmatrix}\right).$$

The above model, with $c = \lambda = 0$, has been estimated by quasi maximum likelihood in Harvey and Shephard (1996) [41] and by MCMC in Meyer and Yu (2000) [51]. A similar model, with $c = \rho = 0$, has been estimated in Koopman and Uspensky (2002) [47] by simulated maximum likelihood, and, with $c = \lambda = 0$ but with non-normal error distribution, in Jacquier, Polson and Ross (2004) [44] by MCMC. However, there is an important difference

[2]For a linear, in standard deviation, in-mean model but with EGARCH errors see Hafner and Kyriakopoulou (2021) [39].

between the models considered in Jacquier et al. (1994)[43], (2004)[44] (JPR) and Koopman and Uspensky (2002) [47] and the one considered here. Specifically, instead of the above conditional variance specification 2.2 they employ the following one:

$$\ln \sigma_t^2 = \omega + \psi \ln \sigma_{t-1}^2 + \eta_t. \tag{2.3}$$

However, Yu (2002) [67] proved that the partial derivative of future volatility with respect to the error is not necessarily negative when $\rho < 0$, i.e. it could be the case that even if $\rho < 0$ future volatility could decrease with a negative error, claiming the the variance specification in 2.2 is a more "natural" one (see details in Yu 2002 [67]). Demos (2023) [23] presents the statistical properties of the two models.

It is worth noticing that the SV-M model has a relative advantage as compared to GARCH-M type of models. Let us denote by $\sigma_{t|t-1}$ the conditional expectation of the standard deviation on the $\sigma - field$ generated by the observed variables y_t up to time $t-1$, i.e.

$$\sigma_{t|t-1} = E\left(\sigma_t | \sigma\left\{y_{t-1}, y_{t-2}, \ldots, y_1\right\}\right).$$

Then adding and subtracting $\lambda\sigma_{t|t-1}$ in equation 2.1 we get:

$$y_t = c + \lambda\sigma_{t|t-1} + \lambda\left(\sigma_t - \sigma_{t|t-1}\right) + \varepsilon_t\sigma_t,$$

i.e. the first two terms on the right-hand side of the equation represents the risk premium, implying a positive λ, whereas the third terms represents the volatility feed-back term implying a negative λ (see Koopman and Uspensky (2002) [47], as well). On the other hand, for any GARCH-type specification $\sigma_t = \sigma_{t|t-1}$ and the two effects can not be separated. Of course, one could add to a GARCH-M model an extra term representing the volatility surprise, as in Campbell and Hentschel (1992) [19], i.e. add $\left(\varepsilon_t^2 - 1\right)\sigma_t^2$ a martingale sequence for most GARCH-type models (see Wu 2001 [66], as well). However, in our case $\sigma_{t|t-1}$ is far more complicated.

Let us concentrate now to the estimation procedure.

3 Estimation

The Gallant and Tauchen (1996) [34] estimator is defined as, for the vector of parameters $\xi' = (c, \lambda, \omega, \psi, \rho, \sigma_\eta)$,

$$\widehat{\xi} = arg\min_{\xi} \left(\sum_{s=1}^{S} \sum_{t=1}^{T} \frac{\partial l_t \left(y_t^s(\xi), \widehat{\beta} \right)}{\partial \beta} \right)' \Sigma \left(\sum_{s=1}^{S} \sum_{t=1}^{T} \frac{\partial l_t \left(y_t^s(\xi), \widehat{\beta} \right)}{\partial \beta} \right),$$

where $\widehat{\zeta} = \left(\widehat{\mu}, \widehat{\varphi}, \widehat{\alpha}, \widehat{\theta}, \widehat{\gamma}, \widehat{\beta} \right)'$ is the first step estimator, i.e. the maximiser of the approximate conditional Gaussian quasi log-likelihood function

$$l_T(\zeta) = -\frac{T}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^{T} \left(\ln h_t + \frac{\left(y_t - \mu - \varphi\sqrt{h_t} \right)^2}{h_t} \right) = \sum_{t=1}^{T} l_t(\zeta) \quad (3.1)$$

where

$$l_t(\zeta) = -\frac{1}{2} \ln 2\pi - \frac{1}{2} \left(\ln h_t + \frac{\left(y_t - \mu - \varphi\sqrt{h_t} \right)^2}{h_t} \right),$$

and

$$\ln h_t = \alpha + \theta z_{t-1} + \gamma \left| z_{t-1} \right| + \beta \ln h_{t-1}$$

for the EGARCH-M auxiliary or

$$h_t = \alpha + \gamma z_{t-1}^2 h_{t-1} + \theta z_{t-1} \sqrt{h_{t-1}} + \beta h_{t-1}$$

for GQARCH-M one.

Notice that as the number of auxiliary parameters is the same as the number of parameters, six, Σ is irrelevant, at least asymptotically, and consequently it is set to the Identity matrix (see e.g. Gourieroux and Monfort (1996) [37]).

3.1 $EGARCH(1,1) - M$ Auxiliary

The $EGARCH(1,1) - M$ class of models of Nelson (1991)) [54] is given by:

$$y_t = \mu + \varphi\sqrt{h_t} + z_t\sqrt{h_t}, \; t = 1, \ldots, n, \quad where \quad z_t \overset{iid}{\sim} (0,1) \quad and$$
$$\ln h_t = \alpha + \theta z_{t-1} + \gamma \left| z_{t-1} \right| + \beta \ln h_{t-1}$$

with

$$\ln h_0 = E\left(\ln h_t\right) = \frac{\alpha + \gamma E\left|z\right|}{1-\beta},$$

for $|\beta| < 1$, and

$$\ln h_1 = \alpha + \theta z_0 + \gamma\left|z_0\right| + \beta \ln h_0 = \alpha + \beta \ln h_0 = \frac{\alpha + \beta\gamma E\left|z\right|}{1-\beta}$$

assuming that $z_0 = 0$.

The Quasi normal log-likelihood is given by:

$$l\left(\alpha,\theta,\gamma,\beta\right) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\ln h_t + \frac{\left(y_t - \mu - \varphi\sqrt{h_t}\right)^2}{h_t}\right) = -\frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\ln h_t + z_t^2\right).$$

Now for $\circ = \{\mu,\varphi,\alpha,\theta,\gamma,\beta\}$ we have that (see Demos and Kyriakopoulou (2013) [24] for the EGARCH model) :

$$l_\circ = \frac{\partial l}{\partial \circ} = -\frac{1}{2}\sum_{t=1}^{T}\frac{\partial \ln h_t}{\partial \circ} - \frac{1}{2}\sum_{t=1}^{T}\frac{\partial z_t^2}{\partial \circ} = -\frac{1}{2}\sum_{t=1}^{T} h_{t;\circ} - \sum_{t=1}^{T} z_t \frac{\partial z_t}{\partial \circ},$$

where

$$\frac{\partial z_t}{\partial \mu} = \frac{\partial\left(y_t - \mu - \varphi e^{\frac{1}{2}\ln h_t}\right)e^{-\frac{1}{2}\ln h_t}}{\partial \mu} = -\frac{1}{2} z_t h_{t;\mu} - \frac{1}{2}\varphi h_{t;\mu} - e^{-\frac{1}{2}\ln h_t} = -\frac{1}{2}\left(z_t + \varphi\right)h_{t;\mu} - \frac{1}{\sqrt{h_t}},$$

$$\frac{\partial z_t}{\partial \varphi} = \frac{\partial\left(y_t - \mu - \varphi e^{\frac{1}{2}\ln h_t}\right)e^{-\frac{1}{2}\ln h_t}}{\partial \varphi} = -\frac{1}{2}\left(y_t - \mu - \varphi e^{\frac{1}{2}\ln h_t}\right)e^{-\frac{1}{2}\ln h_t} h_{t;\varphi} - \frac{1}{2}\varphi h_{t;\varphi} - 1$$
$$= -\frac{1}{2}\left(z_t + \varphi\right)h_{t;\varphi} - 1,$$

and for $@ = \{\alpha,\theta,\gamma,\beta\}$, the conditional variance parameters

$$\frac{\partial z_t}{\partial @} = \frac{\partial\left(y_t - \mu - \varphi e^{\frac{1}{2}\ln h_t}\right)e^{-\frac{1}{2}\ln h_t}}{\partial @} = \frac{\partial\left(y_t - \mu\right)e^{-\frac{1}{2}\ln h_t} - \varphi}{\partial @}$$
$$= -\frac{1}{2}\left(y_t - \mu\right)e^{-\frac{1}{2}\ln h_t} h_{t;@} = -\frac{1}{2}\left(z_t + \varphi\right)h_{t;@}.$$

Now the derivative of the conditional variance with respect to the parameters are given:

$$\begin{aligned}
h_{t;\mu} &= \frac{\partial\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln h_{t-1}\right)}{\partial\mu} \\
&= \frac{\partial\left(\theta z_{t-1}+\gamma\left(I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right)z_{t-1}+\beta\ln h_{t-1}\right)}{\partial\mu} \\
&= \left(\theta+\gamma\left(I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right)\right)\frac{\partial\left(z_{t-1}\right)}{\partial\mu}+\beta h_{t-1;\mu}= \\
&= \left(\theta+\gamma\left(I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right)\right)\left(-\frac{1}{2}\left(z_{t-1}+\varphi\right)h_{t-1;\mu}-\frac{1}{\sqrt{h_{t-1}}}\right)+\beta h_{t-1;\mu} \\
&= -\left\{\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right\}\frac{1}{\sqrt{h_{t-1}}} \\
&\quad +\left[\beta-\frac{1}{2}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)-\frac{1}{2}\varphi\left[\theta+\gamma\left(I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right)\right]\right]h_{t-1;\mu}
\end{aligned}$$

with

$$h_{1;\mu}=0,$$

as

$$\ln h_1=\frac{\alpha+\beta\gamma E\left|z\right|}{1-\beta}.$$

$$\begin{aligned}
h_{t;\varphi} &= \frac{\partial\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln h_{t-1}\right)}{\partial\varphi} \\
&= \theta\frac{\partial z_{t-1}}{\partial\varphi}+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\frac{\partial z_{t-1}}{\partial\varphi}+\beta h_{t-1;\varphi} \\
&= \left\{\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right\}\left(-\frac{1}{2}\left(z_{t-1}+\varphi\right)h_{t-1;\varphi}-1\right)+\beta h_{t-1;\varphi} \\
&= -\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right] \\
&\quad +\left[\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|-\frac{1}{2}\varphi\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\right]h_{t-1;\varphi}
\end{aligned}$$

with

$$h_{1;\varphi}=0.$$

$$
\begin{aligned}
h_{t;\alpha} &= 1+\theta\frac{\partial z_{t-1}}{\partial\alpha}+\gamma\frac{\partial z_{t-1}}{\partial\alpha}\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]+\beta h_{t-1;\alpha} \\
&= 1+\theta\left(-\frac{1}{2}\left(z_{t-1}+\varphi\right)h_{t-1;\alpha}\right) \\
&+\left[\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\left(-\frac{1}{2}\left(z_{t-1}+\varphi\right)h_{t-1;\alpha}\right)+\beta h_{t-1;\alpha} \\
&= 1+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|-\frac{1}{2}\varphi\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\right)h_{t-1;\alpha}
\end{aligned}
$$

$$
h_{1;\alpha}=\frac{1}{1-\beta}.
$$

as

$$
\ln h_1=\frac{\alpha+\beta\gamma E\left|z\right|}{1-\beta}.
$$

Now for $\circ=\{\gamma\}$ the derivatives are:

$$
\begin{aligned}
h_{t;\gamma} &= \frac{\partial\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln h_{t-1}\right)}{\partial\gamma} \\
&= \left|z_{t-1}\right|+\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\frac{\partial\left(z_{t-1}\right)}{\partial\gamma}+\beta h_{t-1;\gamma} \\
&= \left|z_{t-1}\right|+\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\left(-\frac{1}{2}\left(z_{t-1}+\varphi\right)h_{t-1;\gamma}\right)+\beta h_{t-1;\gamma} \\
&= \left|z_{t-1}\right|+\left\{\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|-\frac{1}{2}\varphi\left[\theta+\gamma I\left(z_{t-1}\geq 0\right)-\gamma I\left(z_{t-1}<0\right)\right]\right\}h_{t-1;\gamma}
\end{aligned}
$$

with

$$
h_{1;\gamma}=\frac{\beta E\left|z\right|}{1-\beta}.
$$

Now for $\circ=\{\theta\}$ the derivatives are:

$$h_{t;\theta} = \frac{\partial\left(\alpha + \theta z_{t-1} + \gamma\left|z_{t-1}\right| + \beta \ln h_{t-1}\right)}{\partial\theta}$$

$$= z_{t-1} + \left[\theta + \gamma\left(I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right)\right]\frac{\partial\left(z_{t-1}\right)}{\partial\theta} + \beta h_{t-1;\theta}$$

$$= z_{t-1} - \frac{1}{2}\left(z_{t-1} + \varphi\right)h_{t-1;\theta}\left[\theta + \gamma\left(I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right)\right] + \beta h_{t-1;\theta}$$

$$= z_{t-1} + \left[\beta - \frac{1}{2}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) - \frac{1}{2}\varphi\left[\theta + \gamma I\left(z_{t-1} \geq 0\right) - \gamma I\left(z_{t-1} < 0\right)\right]\right]h_{t-1;\theta}$$

with

$$h_{1;\theta} = 0.$$

Now for $\circ = \{\beta\}$ the derivatives are:

$$h_{t;\beta} = \frac{\partial\left(\alpha + \theta z_{t-1} + \gamma\left|z_{t-1}\right| + \beta \ln h_{t-1}\right)}{\partial\beta}$$

$$= \theta\frac{\partial\left(z_{t-1}\right)}{\partial\beta} + \left(\gamma I\left(z_{t-1} \geq 0\right) - \gamma I\left(z_{t-1} < 0\right)\right)\frac{\partial\left(z_{t-1}\right)}{\partial\beta} + \ln h_{t-1} + \beta h_{t-1;\beta}$$

$$= \ln h_{t-1} + \theta\left(-\frac{1}{2}\left(z_{t-1} + \varphi\right)h_{t-1;\beta}\right)$$

$$+ \left(\gamma I\left(z_{t-1} \geq 0\right) - \gamma I\left(z_{t-1} < 0\right)\right)\left(-\frac{1}{2}\left(z_{t-1} + \varphi\right)h_{t-1;\beta}\right) + \beta h_{t-1;\beta}$$

$$= \ln h_{t-1} + \left[\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right| - \frac{1}{2}\varphi\left(\theta + \gamma I\left(z_{t-1} \geq 0\right) - \gamma I\left(z_{t-1} < 0\right)\right)\right]h_{t-1;\beta}$$

with

$$h_{1;\beta} = \frac{\alpha + \gamma E\left|z\right|}{\left(1 - \beta\right)^2}.$$

3.2 The $GQARCH(1,1) - M$ Auxiliary

The $GQARCH(1,1)$ process of Sentana is given by:

$$y_t = \mu + \varphi\sqrt{h_t} + \sqrt{h_t}z_t, \quad z_t \overset{iid}{=\sim} N(0,1)$$
$$h_t = \alpha + \gamma z_{t-1}^2 h_{t-1} + \theta z_{t-1}\sqrt{h_{t-1}} + \beta h_{t-1}$$

with

$$h_0 = \frac{\alpha}{1-(\gamma+\beta)} \;\; and \;\; h_1 = \frac{\alpha(1-\gamma)}{1-(\gamma+\beta)} \quad OK.$$

Then for $\circ = \{\mu, \varphi, \alpha, \theta, \gamma, \beta\}$ we have that :

$$l_\circ = \frac{\partial l}{\partial \circ} = -\frac{1}{2}\sum_{t=1}^{T} \frac{\partial \ln h_t}{\partial \circ} - \frac{1}{2}\sum_{t=1}^{T} \frac{\partial z_t^2}{\partial \circ} = -\frac{1}{2}\sum_{t=1}^{T} h_{t;\circ} - \sum_{t=1}^{T} z_t \frac{\partial z_t}{\partial \circ},$$

as with the EGARCH-M auxiliary, where

$$\frac{\partial z_t}{\partial \mu} = \frac{\partial\left(y_t - \mu - \varphi\sqrt{h_t}\right)\left(\sqrt{h_t}\right)^{-1}}{\partial \mu} = y_t \frac{\partial\left(e^{-\frac{1}{2}\ln h_t}\right)}{\partial \mu} - \frac{\partial\left(\mu e^{-\frac{1}{2}\ln h_t}\right)}{\partial \mu}$$
$$= -\frac{1}{2} y_t e^{-\frac{1}{2}\ln h_t} h_{t;\mu} - e^{-\frac{1}{2}\ln h_t} + \frac{1}{2}\mu e^{-\frac{1}{2}\ln h_t} h_{t;\mu} = -\frac{1}{2}(y_t - \mu)\frac{1}{\sqrt{h_t}} h_{t;\mu} - \frac{1}{\sqrt{h_t}}$$
$$= -\frac{1}{2}(z_t + \varphi) h_{t;\mu} - \frac{1}{\sqrt{h_t}},$$

$$\frac{\partial z_t}{\partial \varphi} = \frac{\partial\left(y_t - \mu - \varphi\sqrt{h_t}\right)\left(\sqrt{h_t}\right)^{-1}}{\partial \varphi} = \frac{\partial (y_t - \mu) e^{-\frac{1}{2}\ln h_t} - \varphi}{\partial \varphi}$$
$$= -\frac{1}{2}(y_t - \mu)\frac{1}{\sqrt{h_t}} h_{t;\varphi} - 1 = -\frac{1}{2}(z_t + \varphi) h_{t;\varphi} - 1,$$

and for $\circ = \{\alpha, \theta, \gamma, \beta\}$ we have that :

$$\frac{\partial z_t}{\partial \circ} = \frac{\partial\left(y_t - \mu - \varphi\sqrt{h_t}\right)\left(\sqrt{h_t}\right)^{-1}}{\partial \circ} = \frac{\partial (y_t - \mu)\left(\sqrt{h_t}\right)^{-1}}{\partial \circ}$$
$$= -\frac{1}{2}(y_t - \mu)\frac{1}{\sqrt{h_t}} h_{t;\circ} = -\frac{1}{2}(z_t + \varphi) h_{t;\circ}. OK$$

The conditional variance derivatives, for $\circ = \{\mu, \varphi, \alpha, \theta, \gamma, \beta\}$, are:

$$h_{t;\circ} = \frac{\partial \ln h_t}{\partial \circ} = \frac{1}{h_t}\frac{\partial h_t}{\partial \circ} = \frac{1}{h_t}\frac{\partial \left(\alpha + \gamma z_{t-1}^2 h_{t-1}^2 + \theta z_{t-1}\sqrt{h_{t-1}} + \beta h_{t-1}^2\right)}{\partial \circ}$$
$$= \frac{1}{h_t}\left[\frac{\partial \alpha}{\partial \circ} + \frac{\partial \gamma}{\partial \circ} z_{t-1}^2 h_{t-1}^2 + 2\gamma z_{t-1}\frac{\partial\left(z_{t-1}\right)}{\partial \circ} h_{t-1} + \gamma z_{t-1}^2 \frac{\partial h_{t-1}}{\partial \circ}\right]$$
$$+\frac{1}{h_t}\left[\theta \frac{\partial\left(z_{t-1}\right)}{\partial \circ}\sqrt{h_{t-1}} + \theta z_{t-1}\frac{\partial \sqrt{h_{t-1}}}{\partial \circ} + \frac{\partial \theta}{\partial \circ} z_{t-1}\sqrt{h_{t-1}} + \beta \frac{\partial h_{t-1}}{\partial \circ} + \frac{\partial \beta}{\partial \circ} h_{t-1}^2\right]$$
$$= \frac{1}{h_t}\left[\frac{\partial \alpha}{\partial \circ} + \frac{\partial \gamma}{\partial \circ} z_{t-1}^2 h_{t-1}^2 + 2\gamma z_{t-1}\frac{\partial\left(z_{t-1}\right)}{\partial \circ} h_{t-1} + \gamma z_{t-1}^2 h_{t-1} h_{t-1;\circ}\right]$$
$$+\frac{1}{h_t}\left[\theta \frac{\partial\left(z_{t-1}\right)}{\partial \circ}\sqrt{h_{t-1}} + \frac{1}{2}\theta z_{t-1}\sqrt{h_{t-1}} h_{t-1;\circ} + \frac{\partial \theta}{\partial \circ} z_{t-1}\sqrt{h_{t-1}} + \beta h_{t-1} h_{t-1;\circ} + \frac{\partial \beta}{\partial \circ} h_{t-1}^2\right]$$

It follows that for $\circ = \mu$ we get

$$h_{t;\mu} = \frac{1}{h_t}\left[\beta h_{t-1} h_{t-1;\mu} - \left(2\gamma z_{t-1}\sqrt{h_{t-1}} + \theta\right)\left(\frac{1}{2}\varphi \sqrt{h_{t-1}} h_{t-1;\mu} + 1\right)\right],$$

with

$$h_{1;\mu} = 0.$$

For $\circ = \varphi$

$$h_{t;\varphi} = \frac{1}{h_t}\left[\beta h_{t-1} h_{t-1;\varphi} - \left(2\gamma z_{t-1} h_{t-1} + \theta\sqrt{h_{t-1}}\right)\left(\frac{1}{2}\varphi h_{t-1;\varphi} + 1\right)\right]$$

with

$$h_{1;\varphi} = 0.$$

Now for $\circ = \alpha$ the derivatives are:

$$h_{t;\alpha} = \frac{1}{h_t}\left\{1 + \left[\beta h_{t-1} - \frac{1}{2}\varphi\left(2\gamma z_{t-1} h_{t-1} + \theta\sqrt{h_{t-1}}\right)\right] h_{t-1;\alpha}\right\}$$

with

$$h_{1;\alpha} = \frac{1}{\alpha}.$$

For $\circ = \gamma$ the derivative is:

$$h_{t;\gamma} = \frac{1}{h_t}\left[z_{t-1}^2 h_{t-1}^2 + \left(\beta h_{t-1} - \gamma z_{t-1}\varphi h_{t-1} - \frac{1}{2}\theta\varphi\sqrt{h_{t-1}}\right) h_{t-1;\gamma}\right]$$

with

$$h_{1;\gamma} = \frac{\beta}{(1-(\gamma+\beta))(1-\gamma)}.$$

For $\circ = \theta$ the derivatives are:

$$h_{t;\circ} = \frac{1}{h_t}\left[z_{t-1}\sqrt{h_{t-1}} + \left(\beta h_{t-1} - \gamma z_{t-1}\varphi h_{t-1} - \frac{1}{2}\theta\varphi\sqrt{h_{t-1}}\right) h_{t-1;\theta}\right]$$

with

$$h_{1;\theta} = 0,$$

and for $\circ = \beta$ the derivatives are:

$$h_{t;\beta} = \frac{1}{h_t}\left[h_{t-1}^2 + \left(\beta h_{t-1} - \gamma z_{t-1}\varphi h_{t-1} - \frac{1}{2}\theta\varphi\sqrt{h_{t-1}}\right) h_{t-1;\beta}\right]$$

with

$$h_{1;\beta} == \frac{1}{1-(\gamma+\beta)}.$$

It follows that

$$l_\mu = \frac{1}{2}\sum_{t=1}^{T}\left(\left(z_t^2 - 1 + z_t\varphi\right) h_{t;\mu} + 2z_t e^{-\frac{1}{2}\ln h_t}\right) = \frac{1}{2}\sum_{t=1}^{T}\left(\left(z_t^2 - 1 + z_t\varphi\right) h_{t;\mu} + 2\frac{z_t}{\sqrt{h_t}}\right)$$

with

$$h_{1;\mu} = 0.$$

4 Monte Carlo Simulations

4.1 EGARCH and GQARCH

To compare the properties in terms of bias and Mean Squared Error for the two estimators we perform a Monte Carlo exercise. The results in Jacquier, Polson and Rossi (1994) (JPR94) and in Calzolari, Fiorentini and Sentana (2004) imply that the most most important determinant of the performance of the different estimators is the unconditional coefficient of variation of the unobserved volatility level σ_t^2, say CV, where

$$CV^2 = \frac{Var\left(\sigma_t^2\right)}{E^2\left(\sigma_t^2\right)} = \exp\left(\frac{\sigma_\eta^2}{1-\psi^2}\right) - 1.$$

Notice that when CV^2 is low, the observed process is close to Gaussian white noise, and consequently the estimation of the stochastic volatility parameters is difficult. Furthermore, CV is independent of ρ.

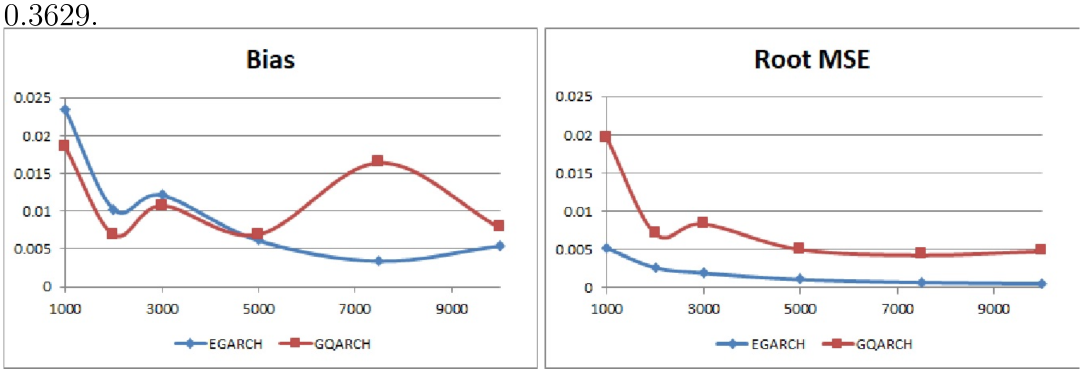
The simulated data were generated employing 3 sets of parameter values. For the the first one we set $\omega_0 = -0.1, \psi_0 = 0.9, \rho_0 = -0.8, and\, \sigma_{\eta 0} = 0.3629$ getting $CV^2 = 1.0$, for the second one we chose $\omega_0 = 0.0$, $\psi_0 = 0.9$, $\rho_0 = -0.95$ and $\sigma_{\eta 0} = 0.31623$ with $CV^2 = 0.693$ as in Monfardini (1998) [53], and for the third one we chose $\omega_0 = -0.736$, $\psi_0 = 0.9$, $\rho_0 = -0.95$ and $\sigma_{\eta 0} = 0.363$ with $CV^2 = 1.0$ as in JPR94. Notice that the third set of parameters has been employed by Andersen, Chung and Sorensen (1999) [4] and Andersen and Sorensen (1996) [5] , as well. However, the previous articles are dealing with symmetric SV models, i.e. $\rho = 0$.

In all simulations we choose $S = 200$ for $T = 1000$, 2000 and 3000, and $S = 150$ for $T = 5000$, 7500 and 10000, and perform 500 Monte Carlo simulations for each score generator. The choice of S is based mainly in time considerations, as higher value of S results in smaller asymptotic variance of the estimators and consequently increases the stability of the estimation (see below on this) but increases the time needed for the program to converge. These values of S are far smaller than the ones employed in the application with real data section.

In Figure 4.1 we present the norm of the estimated biases of the four parameters for the first parameter set for T=1000, 2000, 3000, 5000 7500 and 10000, and a measure of root MSE, as well. In fact, For T=1000, 2000 and 3000 the estimated bias for the GQARCH score generator is less the the EGARCH one, whereas the opposite is true for T=5000, 7500 and 10000. As a measure of root MSE we employ the Frobenius norm of the MSE matrix. It is obvious that the root MSE of the EGARCH generator is by far smaller the the GQARCH one for all Ts under consideration.

It is worth mentioning that in some cases the estimation routine did not converge, because either the program broke due to the fact that the values

Figure 4.1: Parameter Set 1, $\omega_0 = -0.1, \psi_0 = 0.9, \rho_0 = -0.8$, and $\sigma_{\eta 0} = 0.3629$.



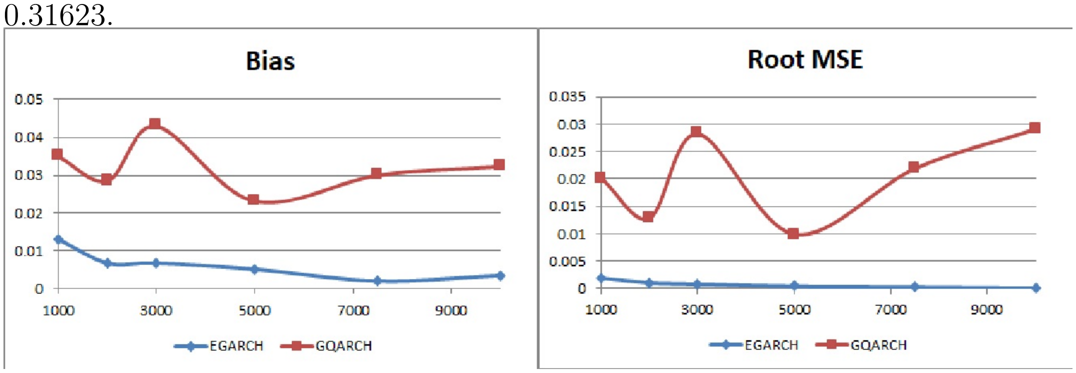
of the estimated parameters where far out of the plausible parameter range, or the program stuck for ever in a specific set of estimates (see also Andersen, Chung and Sorensen 1999). There are three possible ways to remade this phenomenon. First, we could increase S, the number of replications to estimate the expected value of the score generator, or to initialize the estimation procedure from a different point of starting values, or to discard the specific set of random numbers and replace it by another. We have chosen the third alternative. For the first set of parameters the EGARCH score generator routine failed in 31 cases for T=1000, in 12 for T=2000, and in 7 for T=3000. The GQARCH score generator routine did not failed in any of the considered sample sizes.

In Figure 4.2 presents the norm of the estimated biases and the root MSEs for the second set of parameters. In apparent that the EGARCH generator outperforms the GQARCH one, for all Ts considered. For this parameter set it is obvious that the EGARCH score generator is uniformly superior in terms of bias and root MSE.

In terms of routine failures, the GQARCH score generator failed in 22 cases for $T = 1000$, in 36 for $T = 2000$, in 25 for $T = 3000$, in 31 for $T = 5000$, in 28 for $T = 7500$, and in 21 for $T = 10000$. On the other hand, the EGARCH score generator failed in only 8 cases for $T = 3000$.

Monfardini (1998) [53] employed an Indirect Inference estimator using as first step estimators AR and ARMA models, capitalizing the autocorrelation function of the squared residuals of a symmetric SV(1) model. In Tables 1 and 2 we present the biases and the root MSE's of the two estimators of Monfardini (1998) together with ours. Of course in our case we estimate,

Figure 4.2: Parameter Set 2,$\omega_0 = 0.0$, $\psi_0 = 0.9$, $\rho_0 = -0.95$ and $\sigma_{\eta 0} = 0.31623$.



apart from the presented parameters, the dynamic asymmetry parameter ρ, as well. For the two sample sizes considered in that Monfardini (1998), $T = 1000$ and $T = 2000$ it is obvious that the EGARCH score generator is less biased and has smaller root MSE.

Table 1: Biases and Root MSE's (in parenthesis) of the 2 II Estimators in Monfardini (1998), and EGARCH and GQARCH score generators T=1000

$Method/param.$	$\omega_0 = 0.0$	$\psi_0 = 0.9$	$\sigma_{\eta 0} = 0.31623$
$Ind.Inf.1 - AR$	0.0014 (0.0197)	-0.0314 (0.1036)	0.0170 (0.1557)
$Ind.Inf.2 - ARMA$	-0.0055 (0.0239)	-0.0363 (0.1013)	0.0496 (0.160)
QML	–	-0.0327 (0.1047)	0.0319 (0.1577)
$BAYES$	–	-0.0213 (0.0540)	0.0194 (0.0941)
SEM	–	-0.0010 (0.0400)	-0.0129 (0.0570)
$GQARCH$	0.0351 (0.12098)	0.0185 (0.0518)	-0.0289 (0.1116)
$EGARCH$	**0.0002 (0.0063)**	**0.0010 (0.0143)**	**-0.0004 (0.0357)**

Monfardini (1998)

Let us turn our attention to the third parameter set. From Figure 4.3 it is obvious that the EGARCH score generator is uniformly, over all examined sample sizes, superior to the GQARCH one in terms of bias and root MSE.

In terms of routine failures, it seems that, for this parameter set, the number of failures is quite higher than the other two, which is an indication that S=200 is not enough for the programs to converge. In the that EGARCH auxiliary we have 219 failures $T = 1000$, 151 $T = 2000$, 139 $T = 3000$, 87 $T = 1000$, 56 $T = 7500$, and 47 $T = 10000$. For the GQARCH auxiliary

Table 2: Biases and Root MSE's (in parenthesis) of the 2 II Estimators in Monfardini (1998), and EGARCH and GQARCH score generators T=2000

$Method/param.$	$\omega_0 = 0.0$	$\psi_0 = 0.9$	$\sigma_{\eta 0} = 0.31623$
$Ind.Inf.1 - AR$	0.0006 (0.0108)	−0.0124 (0.0598)	0.0029 (0.1090)
$Ind.Inf.2 - ARMA$	0.0021 (0.0112)	−0.0133 (0.0600)	0.0194 (0.1104)
SEM	−	−0.0009 (0.02407)	−0.0168 (0.0438)
$GQARCH$	0.0133 (0.0712)	0.0027 (0.0415)	−0.0241 (0.1059)
$EGARCH$	**0.0003 (0.0042)**	**0.0002 (0.0103)**	**0.0012 (0.0243)**

Figure 4.3: $\omega_0 = -0.736$, $\psi_0 = 0.9$, $\rho_0 = -0.95$ and $\sigma_{\eta 0} = 0.363$.

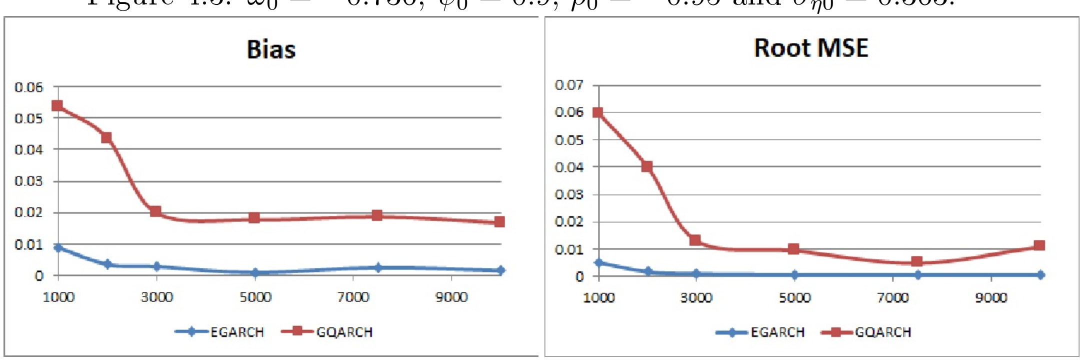


Table 3: Bias and Root MSE (in parenthesis) of MM, QML, Bayes, GQARCH and EGARCH score Generators, T=2000

$Method/param.$	$\omega_0 = -0.736$	$\psi_0 = 0.9$	$\sigma_{\eta 0} = 0.363$
$MM^{1)}$	0.124 (0.420)	0.020 (0.060)	0.053 (0.100)
$QML^{1)}$	0.117 (0.460)	0.020 (0.060)	−0.020 (0.110)
$Bayes^{1)}$	−0.026 (0.150)	−0.004 (0.020)	−0.004 (0.034)
$QML^{2)}$	**0.000 (0.010)**	−0.012 (0.050)	0.018 (0.100)
$MCL^{2)}$	−0.009 **(0.010)**	0.013 (0.020)	−0.046 (0.030)
$GMM^{3)}$	0.151 (0.311)	0.020 (0.043)	−0.086 (0.117)
$EMM^{4)}$	−0.057 (0.224)	−0.007 (0.030)	−0.004 (0.049)
$GQARCH$	0.003 (0.110)	−0.033 (0.057)	−0.026 (0.191)
$EGARCH$	−0.001 (0.038)	**0.000 (0.005)**	**−0.003 (0.022)**

1) Jacquier, Polson and Rossi (1994) Table 9, 2) Sandmann and Koopman (1998) Table 3, 3) Andersen and Sorensen (1996) Table 5, 4) Andersen, Chung and Sorensen (1999) Table 5

things are a bit worse, i.e. there are 186 failures $T = 1000$, 327 $T = 2000$, 320 $T = 3000$, 306 $T = 5000$, 350 $T = 7500$, and 337 $T = 10000$. We would like to repeat that for the Monte Carlo experiments that the routines failed can be eliminated if we increase S or choose different starting values. Increasing S is not time feasible in a simulations setup, where as changes the starting values would make inaccurate the comparisons between simulations.

For this parameter set it is fruitful to compare our results with the ones in articles where symmetric SV(1) models have been estimated. In Table 3 we compare, in terms of bias and root MSE, various estimators for only the three parameters, i.e. ω, ψ and σ_η.

It seems that the EGARCH auxiliary II estimation is performing quite well, at least for the σ_η and σ_η parameters. Notice that in our case the estimated biases and root MSEs are the ones when at the same time we estimated the ρ parameter, i.e. our third parameter set.

To further check our routines we repeated the Monte Carlo experiment of Harvey and Shephard (1996), and Yu (2005). For T=1000 and T=3000 the two Indirect estimators we consider are performing quite well, whereas for T=6000 the EGARCH auxiliary outperforms the other two estimators.

Finally, we repeat the simulations in Jacquier, Polson and Rossi (2004) (JPR04). However, notice that in JPR04 a fat tailed distribution is chosen, and the tail thickness is estimated, as opposed to our normal one. Again the

Table 4: Bias and Root MSE (in parenthesis) of QML, GQARCH and EGARCH score Generators

$Method/param.$	$\psi_0 = 0.975$	$\rho_0 = -0.9$	$ln\left(\sigma^2_{\eta 0}\right) = -4.605$
		$T = 1000$	
QML^*	-0.007 (0.034)	$\mathbf{-0.009}$ (0.132)	0.045 (0.708)
$GQARCH$	-0.006 (0.000)	0.024 (0.022)	0.135 (0.783)
$EGARCH$	$\mathbf{-0.002}$ **(0.009)**	-0.029 **(0.075)**	$\mathbf{-0.025}$ **(0.390)**
		$T = 3000$	
QML^*	-0.001 (0.007)	-0.011 (0.079)	-0.012 (0.353)
$MCMC^{**}$	0.002 (0.005)	0.019 **(0.045)**	$\mathbf{-0.010}$ **(0.209)**
$GQARCH$	0.000 (0.005)	-0.010 (0.480)	-0.048 (0.280)
$EGARCH$	**0.000 (0.004)**	$\mathbf{-0.009}$ (0.046)	-0.010 (0.223)
		$T = 6000$	
QML^*	0.000 (0.005)	-0.007 (0.058)	-0.007 (0.249)
$GQARCH$	-0.001 (0.004)	0.010 (0.033)	0.051 (0.233)
$EGARCH$	**0.000 (0.003)**	$\mathbf{-0.004}$ **(0.032)**	$\mathbf{-0.002}$ **(0.153)**

* Harvey and Shephard (1996) Table 1, ** Yu (2005) Table 5

two considered II estimators are performing quite well.

4.2 EGARCH-M and GQARCH-M

We choose again 3 set of parameters, where the conditional parameter values are the same as in the previous section. For the constant and the price of risk we set $(c_0, \lambda_0) = (0.0, 0.111)$ for the first set, $(c_0, \lambda_0) = (0, 0.111)$ for the second one, and $(c_0, \lambda_0) = (0.07, 0.08)$ for the third one.

For this set of parameters notice that $E(y_t) = 0.074$ giving an Annualized

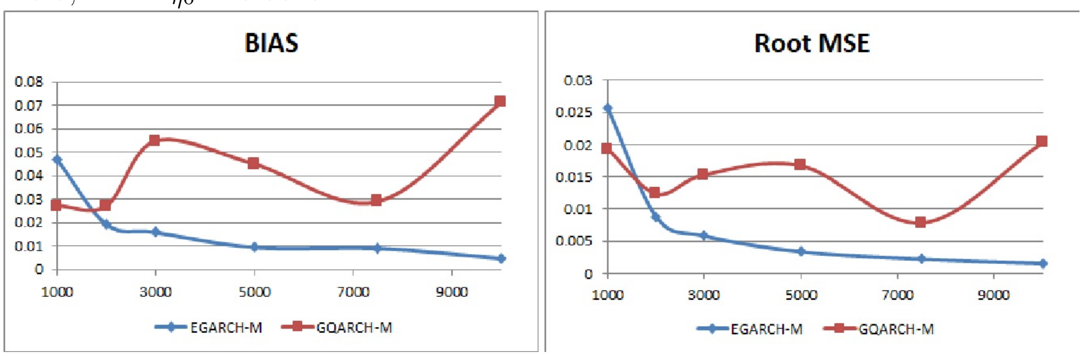
Table 5: Bias and Root MSE (in parenthesis) of Bayes, GQARCH and EGARCH score Generators

$Method/param.$	$\psi_0 = 0.95$	$\rho_0 = -0.6$	$\sigma_{\eta 0} = 0.26$
		$T = 1000$	
$Bayes^*$	0.010 (0.025)	-0.180 (0.190)	-0.010 **(0.039)**
$GQARCH$	**0.000 (0.002)**	-0.034 (0.122)	-0.040 (0.092)
$EGARCH$	0.001 (0.012)	**0.024 (0.107)**	**0.002** (0.048)

* JPR04 Table 1

Rate of Return of 3.89%, and $Var(y_t) = 0.520$. In Figure 5.4 the Bias and Root MSE are presented for the first set of parameters. Notice that only for T=1000 the GQARCH-M score generator has smaller Bias and Root MSE.

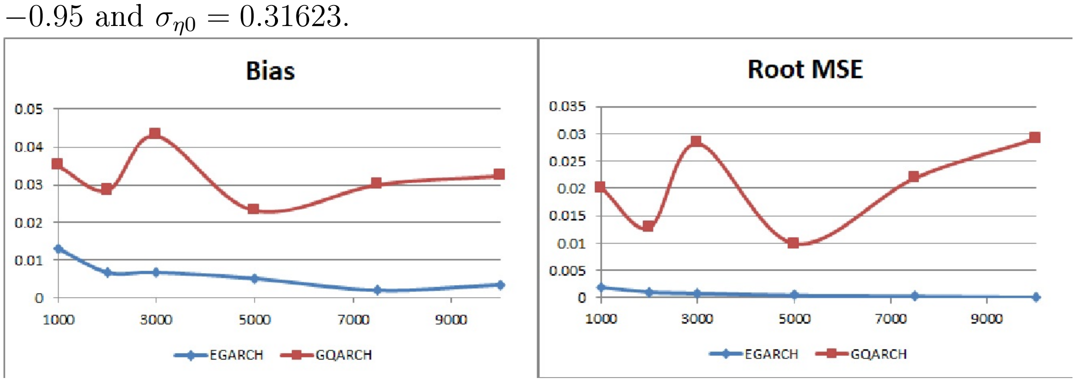
Figure 4.4: Parameter Set 1, $c_0 = 0$, $\lambda_0 = 0.111$, $\omega_0 = -0.1, \psi_0 = 0.9, \rho_0 = -0.8$, and $\sigma_{\eta 0} = 0.3629$.



For the second set of parameters we have that $E(y_t) = 0.043$, with Annualized Rate of Return 2.45%, and $Var(y_t) = 1.301$. For this set it is obvious that the Bias and Root MSE of the EGARCH-M score generator are much smaller the the respective of the GQARCH-M ones (see Figure 5.5).

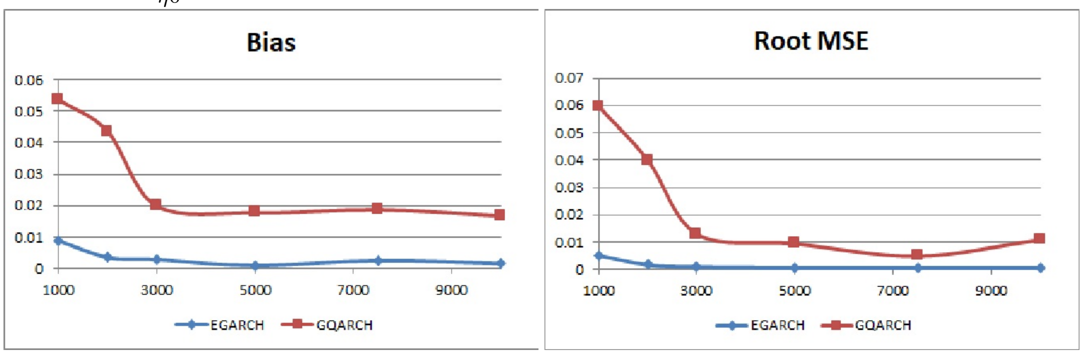
The same is true for the third set of parameters (Figure 5.6). For this set of parameters we get $E(y_t) = 0.123$, Annualized Rate of Return 6.59%, and $Var(y_t) = 0.521$.

Figure 4.5: Parameter Set 2,$c_0 = 0$, $\lambda_0 = 0.04$, $\omega_0 = 0.0$, $\psi_0 = 0.9$, $\rho_0 = -0.95$ and $\sigma_{\eta 0} = 0.31623$.



In Appendix we present all biases and root MSEs for all three parameter sets. In terms of estimated biases, in almost all cases, the auxiliary EGARCH

Figure 4.6: Parameter Set 3, $c_0 = 0.07$, $\lambda_0 = 0.08$, $\omega_0 = -0.1$, $\psi_0 = 0.9$, $\rho_0 = -0.9$ and $\sigma_{\eta 0} = 0.3629$.



estimates are closer to the true ones. With two exemption, at T=5000 for ω and ρ for the second set of parameters, the estimated root MSE of the auxiliary EGARCH estimation procedure is by far smaller than the equivalent of GQARCH estimation procedure.

In terms of program failures, the EGATRCH-M auxiliary routine failed only in 3 cases for T=1000 and 8 cases for T=10000, for the 1st set of parameters, and in 27cases for T=1000 and 5 cases for T=2000, for the 2nd set of parameters, and in 19 case for T=10000, for the 3rd set of parameters. The picture is completely different for the GQARCH-M auxiliary. In theis case the failures are as high as 322 cases for T=1000 and as low as 93 cases for T=10000, for the 1st set of parameters. Similar number of failures we get for the other two set of parameters.

It seems that the EGARCH-M auxiliary is a better choice not only in terms of bias and root MSE but in terms of easiness of routine convergence. Let us turn our attention in the estimation of the model with real data.

5 Application to International Markets.

We apply the developed methods of estimation to weekly excess returns of four indecies of international markets, i.e. the S&P, the FTSE, the DAX and the Nikkey. In this way we estimate two aglosaxon markets, a European and an Asian one. In the following Table we present some descriptive statistics for the 4 indecies, along with the period of estimation and the number of observations. It is obvious that the standard deviation of returns is almost

22 times higher than the average return. Further, in all markets the skewness and kurtosis coefficients are far from the corresponding of the normal distribution ones. The asymptotic confidence interval for the autocorrelations is $(-0.041, 0.041)$ for the 3 markets and $(-0.049, 0.049)$ for FTSE, indicating that, apart from Nikkey, either the 1st or the second order autocorrelations are significant. However, it is known that in the presents of GARCH-type effects the asymptotic distribution of the correlation coefficients are affected (see e.g. Diebold (1986) [25], Weiss (1984) [65], and Milhoj (1985) [52]). $Q(4)$ is the 4th order Ljng-Box statistic, distributed as χ_4^2 under the null of no-autocorrelation up to order 4.

Table 6: Statistics Weekly Excess Returns

Index	*S&P*	*FTSE*	*DAX*	*Nikkey*
Period	$1973-2017$	$1987-2017$	$1973-2017$	$1973-2017$
No. of Obs.	2299	1621	2300	2299
Average	0.103	0.104	0.134	0.037
Stand. Dev.	2.299	2.299	2.767	2.485
Skewness	-0.542	-0.541	-0.592	-0.614
Kurtosis	8.309	8.304	8.003	7.578
Jarque − Bera	2811.4	2807.5	2533.9	2151.9
$\widehat{\rho}(y_t, y_{t-1})$	-0.063	-0.064	-0.005	0.000
$\widehat{\rho}(y_t, y_{t-2})$	0.038	0.037	0.058	0.038
$Q(4)$	15.359	15.408	13.635	4.220
$\widehat{\rho}\left(y_t^2, y_{t-1}^2\right)$	0.267	0.267	0.203	0.217
$\widehat{\rho}\left(y_t^2, y_{t-2}^2\right)$	0.168	0.168	0.252	0.143
$Q^{(2}(4)$	363.53	363.78	425.78	201.19
Dyn.Asym.(1)	-0.198	-0.198	-0.176	-0.137

The 1st and 2nd order autocorrelation of the squared returns is significant indicating strong volatility clustering effects. This is justified by the 4th Ljung-Box statistic for the squared returns $Q^{(2}(4)$. The estimated Dynamic Asymmetry, $\widehat{\rho}(y_t^2, y_{t-1})$, is negative and significant in all cases. Notice that the theoretical dynamic asymmetry depends on the leverage effect parameter ρ as well as the parameter λ (see Demos 2023 [23] and Bollerslev and Zhou (2006) [16]).

Let us turn our attention to the estimation of the model. First, the asymptotic variance-covariance matrix is evaluated employing the formulae in

Gourieroux, Monfort, and Renault (1993) [38]. The asymptotic distribution of the II estimator of ξ, $\widehat{\xi}$ is given

$$\sqrt{T}\left(\widehat{\xi}-\xi\right) \underset{T\to\infty}{\overset{d}{\longrightarrow}} N\left(0,W\right),$$

where

$$W=\left(1+\frac{1}{S}\right)\left(\frac{\partial^2 l_\infty}{\partial\xi\partial\zeta'}I_0^{-1}\frac{\partial^2 l_\infty}{\partial\zeta\partial\xi'}\right)^{-1}.$$

As the objective functions of the auxiliary estimator, either for the EGARCH-M or GQARCH-M, is the sum of individual observations of the quasi-normal log-likelihood function (see equation 3.1) and there are not exogenous variables we have that

$$I_0=\lim_{T\to\infty}V_0\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\partial l_t\left(\zeta\right)}{\partial\zeta}\right]$$

where $V_0\left[\bullet\right]$ is the variance under the assumed true model. Employing Newey and West (1987) [55] I_0 can be consistently estimated by

$$\widehat{\Gamma}=\widehat{\Gamma}_0+\sum_{k=1}^{K}\left(1-\frac{k}{K+1}\right)\left(\widehat{\Gamma}_k+\widehat{\Gamma}_k'\right)$$

where

$$\widehat{\Gamma}_k=\frac{1}{T}\sum_{t=k=1}^{T}\frac{\partial l_{t-k}}{\partial\zeta}\left(\widehat{\zeta}\right)\frac{\partial l_t}{\partial\zeta'}\left(\widehat{\zeta}\right).$$

Further, $\frac{\partial^2 l_\infty}{\partial\xi\partial\zeta'}$ can be evaluated numerically at $\widehat{\xi}$, i.e. by $\frac{\partial^2 l_T\left(\widehat{\xi}\right)}{\partial\xi\partial\zeta'}$. Additionally, as $\dim\left(\xi\right)=\dim\left(\zeta\right)=6$, $\frac{\partial^2 l_T\left(\widehat{\xi}\right)}{\partial\xi\partial\zeta'}$ is a square non-singular matrix and it follows that the estimated asymptotic variance matrix is given by:

$$W=\left(1+\frac{1}{S}\right)\left(\frac{\partial^2 l_T\left(\widehat{\xi}\right)}{\partial\zeta\partial\xi'}\right)^{-1}\widehat{\Gamma}\left(\frac{\partial^2 l_T\left(\widehat{\xi}\right)}{\partial\xi\partial\zeta'}\right)^{-1}.$$

In Table 7 we present the estimated values of the model in equations 2.1 and 2.2 together with the asymptotic z-statistics (in parentheses). To avoid inflating the estimator variances we have chosen $S=99000$.

Table 7: Estimation

	$S\&P$		$FTSE$	
	$EGARCH-M$	$GQARCH-M$	$EGARCH-M$	$GQARCH-M$
$Parameter$				
c	-0.125	-0.145	-0.176	-0.205
	(-0.678)	(-0.601)	(-0.424)	(-0.807)
λ	0.119	0.128	0.110	0.134
	(1.249)	(0.954)	(0.584)	(1.058)
ω	0.084	0.115	0.095	0.100
	(1.156)	(1.222)	(0.863)	(1.077)
ψ	0.934	0.911	0.937	0.935
	(6.000)	(4.872)	(4.756)	(5.353)
ρ	-0.606	-0.558	-0.695	-0.600
	(-2.484)	(-2.513)	(0.279)	(-2.340)
σ_η	0.259	0.315	0.249	0.276
	(2.700)	(2.688)	(2.052)	(2.346)

DAX		$NIKKEY$	
$EGARCH-M$	$GQARCH-M$	$EGARCH-M$	$GQARCH-M$
-0.021	0.010	0.056	0.101
(-0.087)	(0.047)	(0.239)	(0.773)
0.079	0.068	-0.014	-0.019
(0.804)	(0.651)	(-0.142)	(0.279)
0.077	0.102	0.066	0.038
(0.844)	(1.152)	(0.830)	(0.893)
0.951	0.937	0.958	0.980
(5.087)	(5.593)	(5.692)	(8.007)
-0.438	-0.367	-0.348	-0.251
(-1.395)	(-1.465)	(-1.112)	(-0.726)
0.235	0.284	0.231	0.215
(2.103)	(2.577)	(2.175)	(2.917)

It is obvious that the mean constant c is highly insignificant in all cases. Consequently, we estimated the SV-M model with EGARCH-M as an auxiliary imposing the constraint that $c = 0$, but we have chosen $S = 90000$, to conserve time. The results are presented in following table. Now all estimated prices of risk are positive and significant for the $S\&P$ and DAX.

It seems that the positive effect between volatility and expected returns is stronger than the volatility feed-back effect.

Table 8: Constrained Estimation EGARCH-M Auxiliary

	$S\&P$	$FTSE$	DAX	$NIKKEY$
$Parameter$				
λ	0.053	0.023	0.070	0.015
	(2.630)	(0.986)	(2.751)	(0.713)
ω	0.079	0.086	0.077	0.067
	(1.103)	(0.0891)	(0.891)	(0.725)
ψ	0.939	0.945	0.952	0.957
	(5.953)	(5.243)	(5.308)	(4.886)
ρ	−0.591	−0.672	−0.435	−0.358
	(−2.178)	(−2.129)	(−1.369)	(−1.008)
σ_η	0.254	0.243	0.235	0.233
	(2.654)	(1.998)	(2.141)	(2.058)

Employing the formulae in Demos (2023) we can evaluate the moments of the four market returns, treating the estimates of Table 8 as the true ones. Comparing the moments in Table 9 with the sample ones presented in Table 6 it is obvious that the skewness and kurtosis coefficients are overestimated and the same applies for the 1st and 2nd order autocorrelations and the dynamic asymmetry. On the other hand the 1st and 2nd order autocorrelations of squared returns are underestimated.

Table 9: Estimated Statistics Weekly Excess Returns

Index	*S&P*	*FTSE*	*DAX*	*Nikkey*
Average	0.108 (0.103)	0.054(0.104)	0.168(0.134)	0.035(0.037)
Stand. Dev.	2.190(2.299)	2.509(2.299)	2.585(2.767)	2.561(2.485)
Skewness	0.053(−0.542)	0.022(−0.541)	0.078(−0.592)	0.016(−0.614)
Kurtosis	5.178(8.309)	5.2101(8.304)	5.409(8.003)	5.719(7.578)
$\widehat{\rho}\left(y_t, y_{t-1}\right)$	−0.068(−0.063)	−0.079(−0.064)	−0.041(−0.005)	−0.039(0.000)
$\widehat{\rho}\left(y_t, y_{t-2}\right)$	−0.064(0.038)	−0.074(0.037)	−0.039(0.058)	−0.037(0.038)
$\widehat{\rho}\left(y_t^2, y_{t-1}^2\right)$	0.163(0.267)	0.170(0.267)	0.169(0.203)	0.183(0.217)
$\widehat{\rho}\left(y_t^2, y_{t-2}^2\right)$	0.152(0.168)	0.158(0.168)	0.162(0.252)	0.173(0.143)
Dyn.Asym.(1)	0.007(−0.198)	0.003(−0.198)	0.010(−0.176)	0.004(−0.137)
Leverage	−0.173	−0.187	−0.112	−0.086
$\widehat{\rho}\left(\sigma_t^2, \sigma_{t-1}^2\right)$	0.922	0.930	0.937	0.942

6 Conclusions

We investigated the estimation of an asymmetric SV models with possibly time varying risk premia, by employing the Indirect Inference estimation procedure. As a first step estimator we employed either the GQARCH-M model or the EGARCH-M process. In the Monte-Carlo simulations the comparison the two first step models in terms of bias and root MSE, it seems, that although the GQARCH-M model performs relatively well the EGARCH-M auxiliary is almost always superior.

In the empirical application section we employed the weekly excess returns of four indices from the New York, London, Frankfurt and Tokyo. It seems that the relation between future returns and volatility is stronger than the volatility feedback, as in all cases the price of risk, $\widehat{\lambda}$, is positive and significant, apart from the one for Tokyo. Further, the estimation of the autoregressive coefficient in the variance equation indicates strong volatility clustering and leverage effects. Although the estimated coefficients are, in most cases, are relatively accurately estimated the 3rd and fourth moment they imply are quite different from the sample counterparts. The same applies for the dynamic asymmetry and the 1st order autocorrelation of the squared returns. Nevertheless, the returns 1st order autocorrelation, apart from Tokyo are successfully matched.

A possible extension of the model could be in terms of relaxing the constraint that the risk premium is the equal to the volatility feedback one. We leave this for future research.

APPENDIX

Figure 6.1: Biases Parameter c.

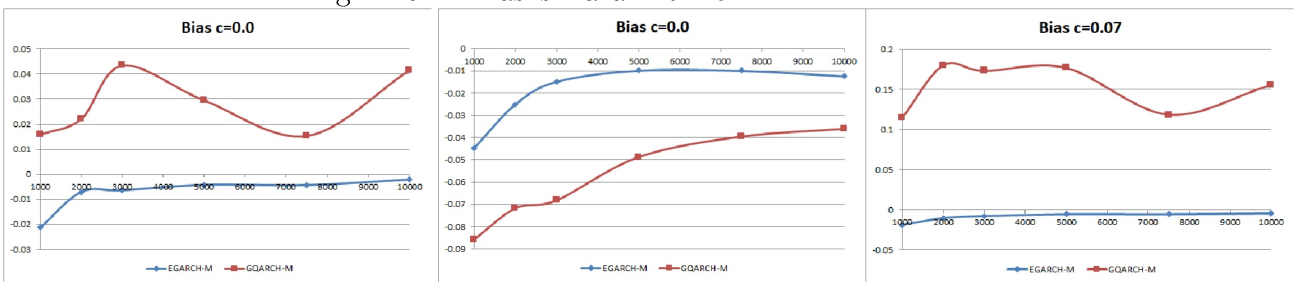


Figure 6.2: Root MSE Parameter c.

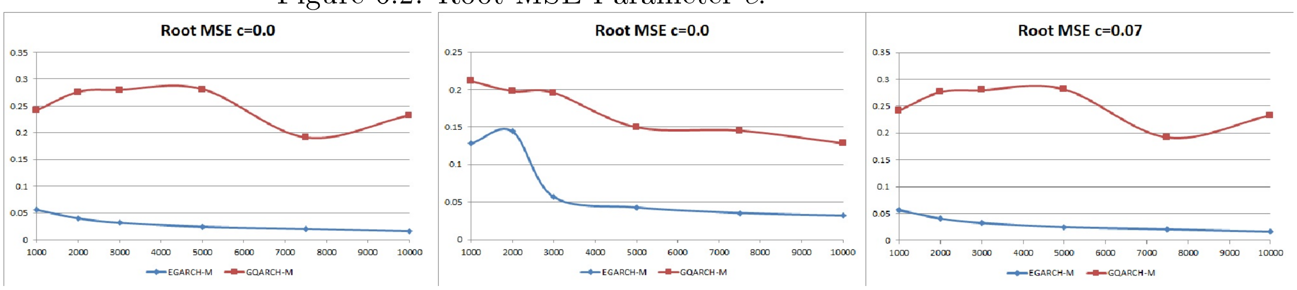


Figure 6.3: Biases Parameter λ.

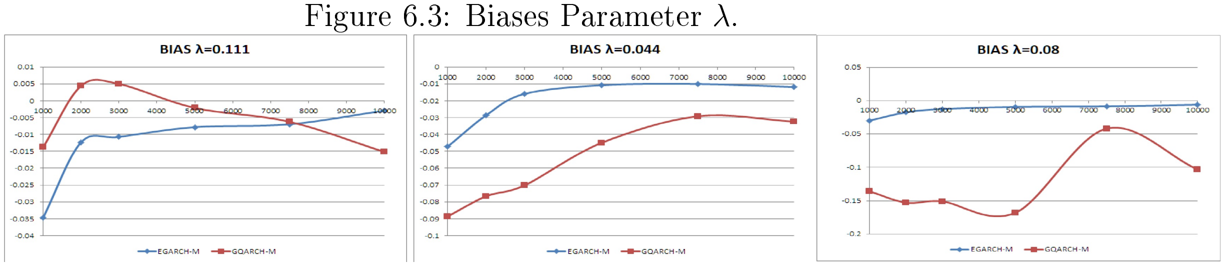


Figure 6.4: Root MSE Parameter λ.

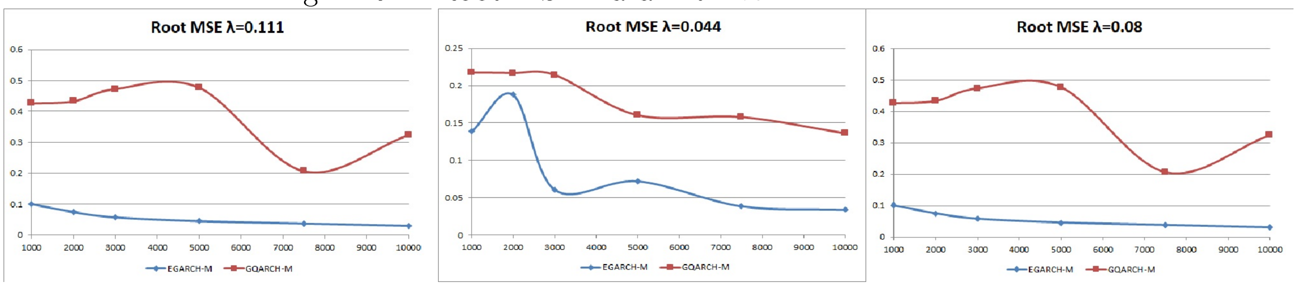


Figure 6.5: Biases Parameter ω.

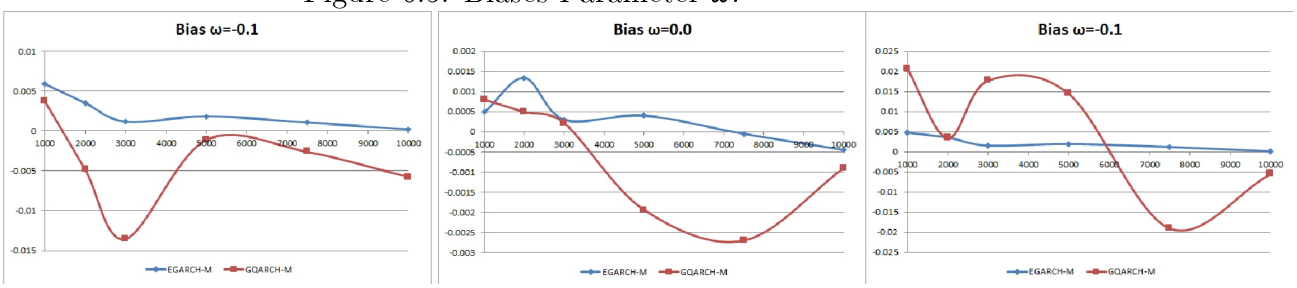


Figure 6.6: Root MSE Parameter ω.

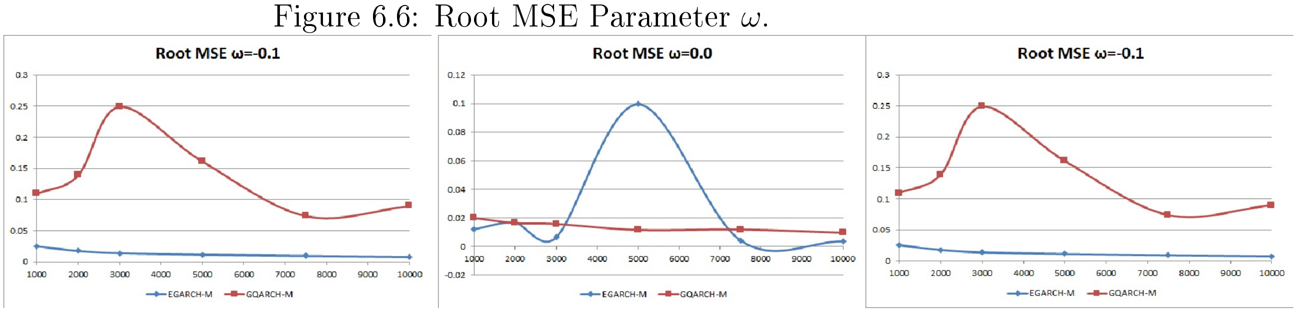


Figure 6.7: Biases Parameter ψ.

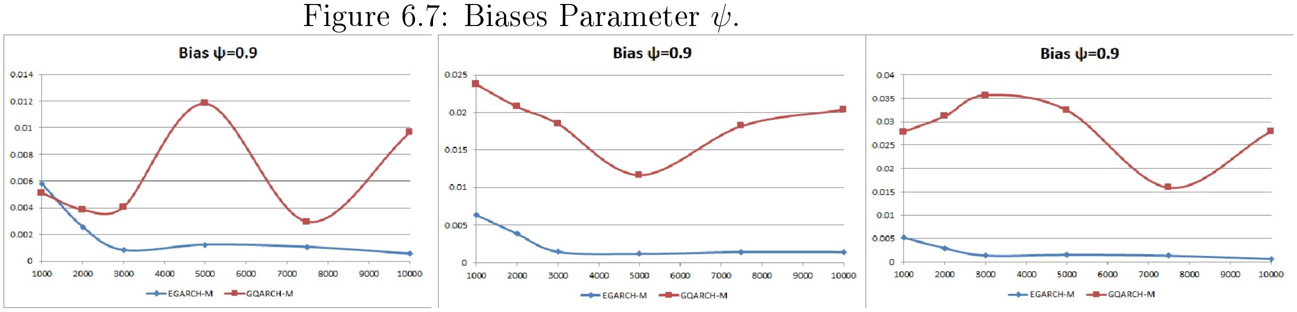


Figure 6.8: Root MSE Parameter ψ.

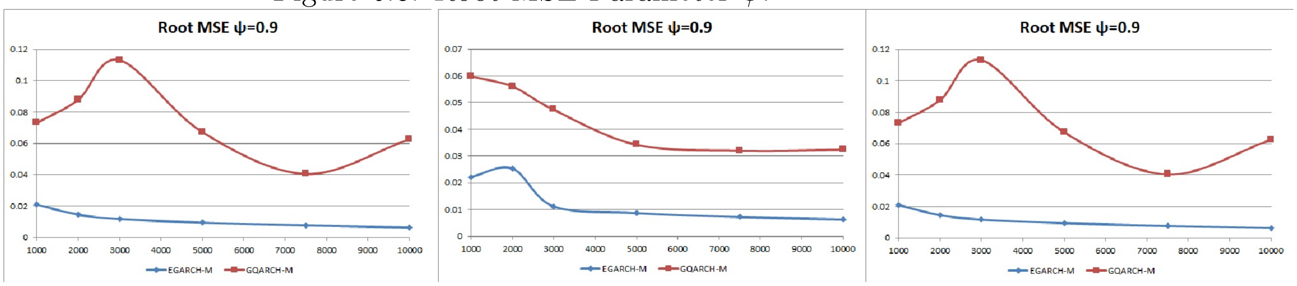


Figure 6.9: Bias Parameter ρ.

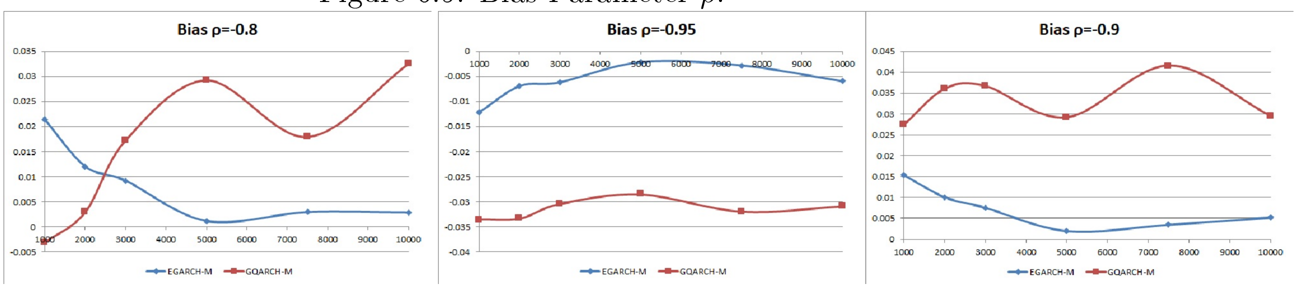


Figure 6.10: Root MSE Parameter ρ.

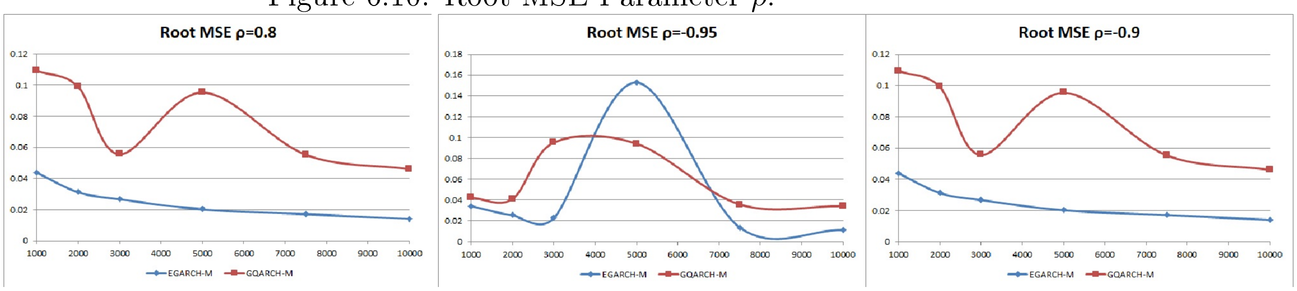


Figure 6.11: Bias Parameter σ_η.

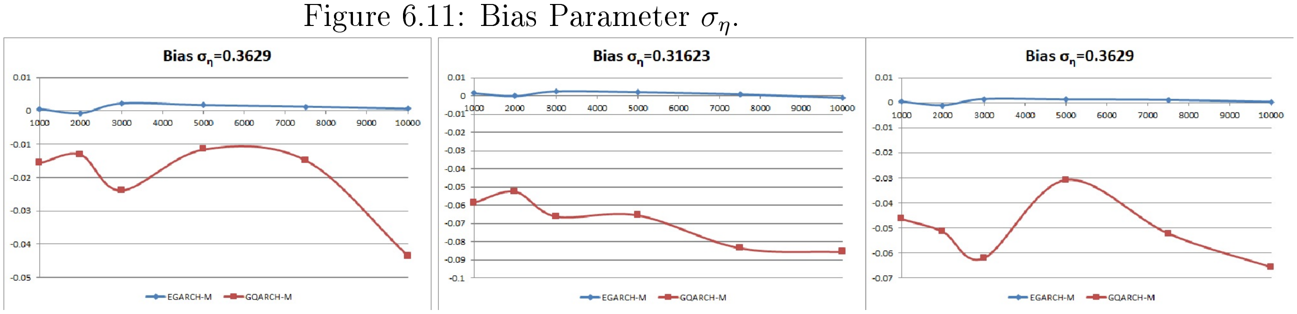
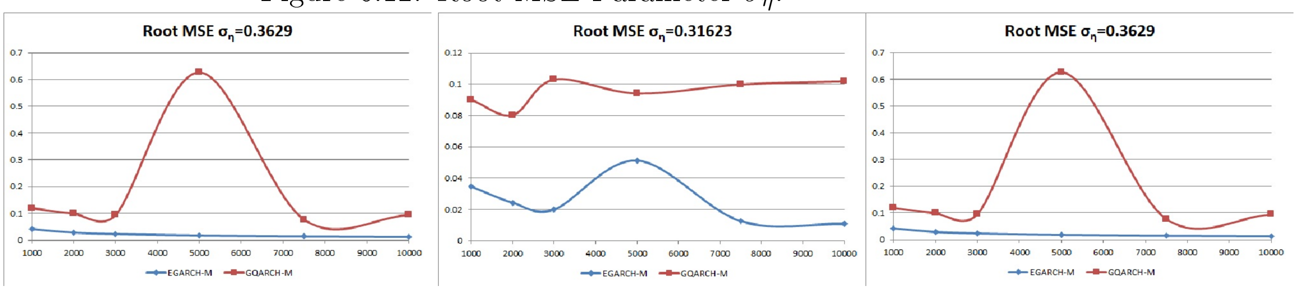


Figure 6.12: Root MSE Parameter σ_η.



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