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This paper is dedicated to Professor A. Xepapadeas on the occasion of his retirement from Athens University of Economics and Business, with friendship and admiration.

Abstract

In this paper we propose a consensus group decision making scheme under model uncertainty consisting of a two-stage procedure and based on the concept of Fréchet barycenter. The first stage is a clustering procedure in the metric space of opinions leading to homogeneous groups, whereas the second stage consists of a proposal most likely to be accepted by all groups. An evolutionary learning scheme of proposal updates leading to consensus is also proposed. The schemes are illustrated in examples motivated from environmental economics.

Keywords: consensus group decision making; model uncertainty; environmental decision making; Fréchet barycenter;

Contents

1 Introduction

Group decision making is an important field in decision making with important applications in various disciplines, among which environmental economics. Group decision, often requires that all or the majority of agents in the group agree to a single proposal or opinion. This is particularly true in cases where there is no coercion involved on the decision made, so that the implementation of the decision depends on the good will, or rather the acceptance of the common decision by all members of the group.

To make the discussion more concrete we consider the following generic situation: Assume that a group of agents G has to reach a common decision concerning policies regarding a future contingency X. Policies may refer for instance to the cost of abatement measures for protection against X, which clearly require the acceptance of a commonly acceptable estimate for the value of X by every member of the group as well as the acceptance of a commonly acceptably discount factor. Typically, different member of the group will have different valuations for X, therefore report different costs for the adverse effects of X. Moreover, different members of the group will have different discount rates for calculating the present value of the future adverse effect X. As a result of the above, each member $i$of the group will report a different value for a reasonable cost C_i of abatement measures taken today so as to ease the future effect X. This means that unless the abatement cost C proposed by the policy maker (upon which the proposed policy measures are prices) carefully chosen so that it is finally acceptable by every member of the group (whose report of the cost C_i deviates from C) it will not be acceptable by all group members, therefore the policy '(unless coerced) will not be successful.

The above example, introduces the important notion of consensus, an important concept in group decision making, which essentially means choosing a proposal for the common decision, on which every member of the group (or its majority) will agree upon eventhough their initial anchor positions may deviate from that. Consensus decision making is very important in group decision making where coercion is not applicable, as an example one may consider climate change negotiations. An important role in group decision making is played by the mediator, an agent that introduces a proposal (e.g. based on some opinion different to those of each member of the group) places it to the attention of the group and hopes for consensus.

This widespread acceptance of a proposal upon which the final decision is made by all members of a group is made more difficult (if not impossible) by the following two factors:

(a) Group heterogeneity: If the group of agents that has to reach a common decision has a widespread spectrum of positions in opinion space, i.e. presents large "variance", with the concept of variance to be made concrete in Section 2.2 below, then the prospect of agreement to a common position is rather grim. A midpoint in position space has somehow to be proposed, so that bona fide agents willing to deviate from their initial positions in the interest of agreement, will not feel that their deviation is far larger than that of their counterparts.

(b) Model uncertainty: If there is not a single model for X, to which all the agents in the group abide, then each agent may adopt a different model for X and therefore report different estimates C_i for the cost of X (with a similar situation for the discount rates, see e.g. Section 4.2). Hence, model uncertainty may contribute even more to group inhomogeneity (see (a) above) and make group consensus even harder. These considerations introduce the need for choosing a commonly acceptable model for X, by the whole group, which will be subsequently used for valuation purposes, upon which policy making will be based. This is related again to the concept of the mean and variance in the space of models for X (which in turn can be considered as a position space for the agents in the group; see Section 2.2).

The aim of this paper is to address the question of group decision making with the above points and difficulties in mind. In particular, we propose a scheme for consensus group decision making, in the presence of group inhomogeneity and model uncertainty based on the modelling of the opinion space of the agents as an appropriate metric space, and the concept of the Fréchet mean (barycenter) and variance. We report a two stage group decision making process that first identifies almost homogeneous groups of agents (in terms of opinions) hereafter called clusters, and then uses the representative opinions in the clusters for a proposal which is a candidate for common acceptance, in terms of the barycenter of the representative opinions of each cluster. Moreover, we introduce the concept of learning i.e. we allow the agents to update their initial opinions (anchor points) as a result of interaction with their peers and propose an evolutionary process of opinion updating and proposal making (e.g. by the mediator) that results to consensus. While the proposed decision making process is of wider interest in group decision making, it is inspired and illustrated within the context of environmental economics, a field which accomodates all of the above mentioned features (i) a feeling that we must agree, (ii) compliance to an agreement is voluntary and non coercive, hence relies on proposals that will be widely acceptable by all members of the group (iii) contingencies X are subject to model uncertainty and (iv) the decisions to be made are subject to great inhomogeneity of the agents involved, due to their spatial scales.

The structure of the paper is as follows: In Section 2 we introduce the two stage group decision making process. In particular, in Section 2.2 we introduce the concept of the opinion space presenting the opinions of each member of the group as an appropriate metric space, illustrating it with various examples, and introduce the concepts of the Fréchet mean (barycenter) and variance. We then introduce stage 1 in Section 2.3 which results to a grouping of diverse agents into K homogeneous groups using an appropriately designed clustering procedure, and then introduce stage 2 in Section 2.4, where a proposal by the mediator that is likely to guarantee consensus is chosen in terms of the Fréchet barycenter of the clusters. We motivate the choice of the barycenter by the mediator using geometric arguments (based on duality results) in Section 2.4.1 or probabilistic arguments in Section 2.4.2. In Section 3 we propose an evolutionary algorithm, based on a learning scheme, that allows for modelling the process of reaching consensus for the agents in the group (or the clusters). The scheme allows us to introduce and assess the effects of various behavioural characteristics of the agents, such as inhomogeneity of opinions, agents impatience and propensity to deviate from anchor points etc, on the process of reaching consensus. Finally, in Section 4 we illustrate the proposed scheme with an application in environmental economics and in particular on the problem of determining a common social discount rate for a group of heterogeneous agents as well as a common probability model for a future contingency X, that will be subsequently used to evaluating the proposed cost of abatement measures and the design of policy measures.

2 A two stage group decision making process

In this section we propose a multi-stage process for group decision making involving diverse preferences and model uncertainty. The various stages correspond to different levels of governance or different scales of agents involved. For example, one may consider two distinct scales, the fine scale that corresponds to the individual agents level (presenting great variability), and the scale corresponding to different countries (which in some sense can be conceived as some sort of averaging over individuals – but more is about to come about that). Both scales are important in the policy making process. For instance, decision making at the country level requires a good understanding of the variability of preferences concerning an important issue at the agent level. As diverse groups are comprising, the society if a policy is based on the preferences of a single group then this policy is likely not to be implemented in practice by the other groups and get effectively cancelled

2.1 Spoiler: The two-stage process

We begin this section by summing up the general idea and intuition concerning the two-stage group decision process we propose in this paper. The details concerning the various stages and the technical aspects of the proposed process are introduced and studied in depth in the following sections.

1. Collect and map all opinions of the group as points z_i, $i \in G$ into the appropriate opinion space $\mathcal{M}$. This can be done in various ways, i.e. interviews, behavioural studies etc. For details on the concept of the opinion space as a metric space and various possible choices for the opinion space with motivating examples see Section 2.2

2. Perform a clustering procedure in opinion space as proposed in Section 2.3 Algorithm 1 to form K groups in opinion space. Within each cluster a degree of homogeneity of opinions is achieved, and hence each cluster k is adequately described in terms of the cluster's barycenter $z_{B,k}$. Naturally, there is not a degree of homogeneity in between different clusters.

3. We identify the group agreement point as a barycenter of the set of opinions $\mathbb{M} = \{z_{B,1}, \dots, z_{B,K}\}$, i.e. as a barycenter of barycenters. The choice of weights in the barycenter will depend either on the characteristics of the groups (e.g. their power or importance, or their characteristics and propensities towards deviating from their initial positions). The dynamics of reaching the consensus point, including the determination of weights is described in an evolutionary approach presented in Section 3

Remark 2.1. The homogenization stage, is considered important for the two reasons:

The first reason is that it allows to map general tendencies in opinion space by grouping the original opinions (which is a large heterogeneous group displaying large variance in opinion space) into K subsets consisting of more homogeneous content. These K subsets are to be considered as the tendencies of K rather homogeneous groups. As an example of that one may consider the original group G as the opinions of all countries in the world towards climate change effects, whereas the K groups determined after the clustering procedure may represent general tendencies of groups of countries e.g. the group of countries who are more susceptible to climate change effects in the short run, the group of countries that are more indifferent to such effects etc.

The second reason is that it accelerates the consensus achievement. If one tries to envisage an evolutionary approach in which agents that start from different anchor points in opinion space, in the interest of reaching a common decision are willing to update their anchor points and accept a new proposed point in opinion space, with some probability of acceptance of the new proposal, depending on the discrepancy of the proposal from the anchor points. Assuming that this procedure is repeated until a sufficient number of agents has agreed to the proposal, then the more heterogeneous the original group is in opinion space, the larger the number of repetitions (i.e. the longer the time required) until agreement is reached. This intuition will be made more concrete with the proposed evolutionary algorithm for reaching consensus presented in Section 3.

2.2 Opinion space as a metric space and the concept of the barycenter

In this section we make the following abstraction, which will be essential in what follows: We will try to model opinion space as a metric space i.e. a set endowed with an appropriate notion of distance or dissimilarity which will also allow for the quantification of variability between beliefs in the opinion space. In the abstract framework, to be made more concrete shortly, we assume that each agent i carries an opinion (stand point) concerning the issue under consideration that can be considered as a point x_i in some set M. The dissimilarity between different opinions can be quantified in terms of a metric on M, i.e. a function $d : M \times M \to \mathbb{R}_+$ such that for any points $x_i, x_j, x_k \in M$ it holds that

(i) $d(x_i, x_j) \geq 0$ with $d(x_i, x_j) = 0$ if and only if $x_i = x_j$.

(ii) $d(x_i, x_j) = d(x_j, x_i)$

(iii) $d(x_i, x_j) \leq d(x_i, x_k) + d(x_k, x_j)$.

Adopting such a dissimilarity measure d for any two opinions x_i, x_j in M, the larger the $d(x_i, x_j)$ is, the greater the difference between these two opinions will be.

Clearly, there are different ways in which the opinion space can be chosen, as well as the dissimilarity measure between opinions. This is also highly dependent on the type of situation we wish to model. The following examples may provide some of the possibilities.

Example 2.2. If opinions refer to the desired levels of expenditure for a bundle of possible causes (say d possible causes) then an opinion x_i can be expressed as a d dimensional vector with coordinates $x_i = (z_{1,i}, \dots, z_{d,i}) \in \mathbb{R}^d$ (or possibly $\mathbb{R}^d_+$ and any ℓ^p metric of the form $d(x_i, x_j) = (\sum_{m=1}^{d} |z_{m,i} - z_{m,j}|^p)^{1/p}$, $1 \leq p < \infty$ or appropriately weighted versions) can be used to quantify the dissimilarity between two opinions x_i and x_j in the opinion space $(M, d) = (\mathbb{R}^d, \ell^p)$.

Example 2.3. If opinions refer to approval or not of various issues (say d possible issues) then an opinion x_i can be expressed as a d dimensional vector containing 0 and 1 entries, with $x_{m,i} = 0$ denoting disapproval of agent i concerning issue m and $x_{m,i} = 1$ approval. A possible dissimilarity measure is the Hamming distance which counts the number of issues on which two agents disagree. This can be expressed as $d(x_i, x_j) = \sum_{m=1}^{d} |x_{m,i} - x_{m.j}|$. Then the opinion space can be understood as the Hamming metric space MacKay et al. (2003).

Example 2.4 (*The metric space of curves as an opinion space: Social discount rate term structure)*. An important example where this framework can be applied is in the valuation of future costs or income. Suppose that agents are to face a payoff (or loss) $X(t)$ at time t. The value $V(0)$ of $X(t)$ and time 0 is given by $V(0) = X(t)e^{-r(t)t}$ where $r(t)$ is the discount rate between the times instances 0 and t. The function $t \mapsto r(t)$ is important in the cost-benefit analysis of any project. This function $r(\cdot)$ can be understood as a curve called the discount rate term structure. The discount rate function quantifies the preferences of agents towards future payoffs (or costs). Different agents are expected to have different time preferences, hence different discount rate terms structures. In such cases, the opinion space is a space of curves $r(\cdot)$. Since term structure curves are expected to have certain characteristics (for example $r(t) \geq 0$ for all t or $r(\cdot)$ must be convex) they are not expected in general to be elements of a vector space but can be considered as elements of a metric space of curves of a suitable shape, so that they are acceptable discount rate curves.

Example 2.5. If opinions refer to evaluations of a future unknown risk or consequence, X, upon which decisions for policy making, e.g., decisions for expenses on public awareness campaigns, adoption of mitigation measures etc will be made. As a concrete example, consider as X the cost of adverse effects or damages resulting from climate change. Depending on the perception and estimation of this cost an agent may accept easily or not policies and their costs related to abatement of this damage. Since X concerns a future cost, it is in general unknown and can be treated as a random variable, for which at best we can infer its distribution. Different agents have in principle different distributions for the same X. The difference in the distributions for X may be attributed either to heterogeneity in preferences or even to model uncertainty effects (i.e. the inability of experts to produce a single and universally accepted probability law concerning the distribution of X on account of incomplete data or other effects). For such cases we may identify opinions by probability measures P_i, concerning the distribution of X, i.e. estimates for $Prob(X \geq x)$ for any $x \in \mathbb{R}$ and assume that the space of opinions is the space of probability measures (or probability distributions) endowed with an appropriate measure of dissimilarity between any two probability measures. Possible choices may be the Kullback-Leibler Divergence or relative entropy Kullback and Leibler (1951), the Wasserstein distances Santambrogio (2015); Villani (2021) or other appropriate metrics. The Wasserstein distances (see Section 6.1 in the Appendix for the relevant definitions) may be a better choice as they satisfy all the appropriate properties of a metric in the space of probability measures.

Example 2.6 (*The Wasserstein space as an opinion space)*. Consider diverse opinions concerning the future unknown risk or consequence $X \in \mathbb{R}^d$. As an example we may let X combine predictions about future values of quantities such as e.g. global temperature, temperature in particular regions of the globe, levels of economic activity etc, all combined into a single vector $X \in \mathbb{R}^d$. Being a random variable X can be interpreted by its probability distribution, $P(X \in A)$ where $A \subset \mathbb{R}^d$ is a Borel set. Clearly, adopting different probability models for the description of X, will lead to different predictions and valuations for X. We may turn the space of probability models for X (our opinion space) to a metric space using the Wasserstein metric, defined for any two probability models P_1, P_2 as

$$W_2(P_1, P_2) = \left\{ \inf_{X \sim P_1, Y \sim P_2} \mathbb{E}[|X - Y|^2] \right\}^{1/2},$$

which clearly indicates that it is related to the error of prediction of a random variable X due to model mispecification (i.e. if X is modelled using P_2 whereas the true model is P_1). For more details on the Wasserstein metric and its calculation see Section 6.1 in the appendix).

Example 2.7. If opinions refer to connectivity structures (i.e. referring to connections and interactions between various subgroups in a society or stakeholders, inter-dependencies between actors, etc) then we may consider as the opinion space M a space of appropriate matrices. One such example could be the space of positive definite and symmetric matrices (Gram matrices) which are matrices akin to covariance matrices in statistics, reporting pairwise similarities between the various elements in a group $\mathcal{G} = \{g_1, \ldots, g_N\}$. A Gram matrix is an $N \times N$ matrix $G = (g_{ij})$ such that $g_{ij} = \mathfrak{d}(g_i, g_j)$, where $\mathfrak{d}$ is a similarity measure between the elements g_i and g_j which can be defined arbitrarily based on the context under consideration. Then M can be chosen as the space of positive definite symmetric matrices G, representing the connectivity structures, and this space can be metrized in terms of a suitable metric, one such choice could be the Bures-Wasserstein metric Bhatia et al. (2019) (see Section 6.1 in the Appendix).

Having motivated the modelling of opinion space as a metric space with the above examples, we now return to our general abstract view of considering the opinions of a group of N agents as a collection of N points x_i of a metric space (M, d). The exact nature of the set M and the metric d depend on the nature of the situation we intend to model (see e.g. the above examples), and for any pair of agents (i, j) the quantity $d(x_i, x_j)$ represents the dissimilarity of their views. We will denote by $\mathbb{M} = \{x_1, \ldots, x_N\} \subset M$ the set of opinions for the group of agents. Given a choice of weights $w = (w_1, \ldots, w_N) \in \Delta^{N-1}$, (where by Δ^{N-1} we denote the N dimensional simplex[1], a measure of the variability of opinions in the set $\mathbb{M}$,

[1] Δ^{N-1} denotes the N-dimensional unit simplex, i.e. $\Delta^{N-1} := \{x \in \mathbb{R}^N \ : \ \sum_{i=1}^{N} x_i = 1, \ x_i \geq 0, \ \forall i\}$

can be given in terms of the function

$$F_{\mathbb{M}} : M \to \mathbb{R}, \;\; F_{\mathbb{M}}(z) := \sum_{i=1}^{N} w_i d^2(z, x_i)$$

which is called the Fréchet function of the set $\mathbb{M}$. In fact, the quantity

$$V_{\mathbb{M}} := \min_{z \in M} F_{\mathbb{M}}(z) \tag{1}$$

is called the Fréchet variance of the set $\mathbb{M}$, and its magnitude is a measure of the variability of elements contained in the set. The smaller $V_{\mathbb{M}}$ is the more homogeneous is the set, while larger values of $V_{\mathbb{M}}$ indicate high heterogeneity in the set. Moreover, the minimizer of $F_{\mathbb{M}}$,

$$z_{\mathbb{M}} := \arg\min_{z \in M} F_{\mathbb{M}}(z) = \arg\min_{z \in M} \sum_{i=1}^{N} w_i d^2(z, x_i), \tag{2}$$

is called the Fréchet mean Fréchet (1948) of $\mathbb{M}$, and is the analogue of the "mean" of $\mathbb{M}$, i.e. an element of M (not necessarily an element of $\mathbb{M}$) that can be understood as the best approximation of the elements in $\mathbb{M}$. Since M does not necessarily admit a linear structure (as e.g. in Examples 2.3 and 2.5) a linear estimator for the mean such as $\hat{z} := \sum_{i=1}^{N} w_i x_i$ may not be of much use or easily interpretable since $\hat{z}$ may not even be an element of M in the first place! On the other hand, if $M = \mathbb{R}^d$ and $d(z, x_j) = \|z - x_j\|$, the Euclidean distance, then choosing $w_i = 1/N$, it is an easy calculation to show that (2) yields that $z_{\mathbb{M}} = \hat{x} := \frac{1}{N} \sum_{i=1}^{N} x_i \in M$, and $V_M = \frac{1}{N} \sum_{i=1}^{N} \|x_i - \hat{x}\|^2$ coinciding with the standard estimators for the mean and variance. In this sense, $z_{\mathbb{M}} \in M$ and $V_{\mathbb{M}}$ as defined in (2) and (1) respectively can be understood as generalizations of the mean and variance for random elements taking values in general metric spaces such as for instance Examples (2.3) or (2.5). The Fréchet mean is also well defined in the case where the opinion space is considered as the space of probability models (equiv. probability measures) metrized in terms of the Wasserstein metric. This leads to the Wasserstein barycenter, a concept which has gained a lot of popularity both in statistical and machine learning (see e.g. Panaretos and Zemel (2020), Peyré et al. (2019)) as well as decision theory in the presence of model uncertainty (see e.g. Petracou et al. (2022) or Papayiannis and Yannacopoulos (2018a)). Moreover, the Wasserstein bacycenter admits closed form solutions for special cases (see Section 6.1 in the Appendix).

The mean is the element minimizing the square deviations, and alternatively can be considered as the solution of a least squares problem fitting an element of minimum distance for a set of elements $\mathbb{M}$. This interpretation offers various interesting possibilities.

- One such possibility is that we may consider a large group of agents presenting a diversity of opinions and then the elements in $\mathbb{M}$ can be considered as observations (a sampling of the opinions of the agents by choosing a representative sample of N of them) so that $z_{\mathbb{M}}$ is an estimate for the opinions of the large group. The choice of the weights w_i may represents in this case how the sampling procedure was designed (i.e. the reason for the choice of particular agents in the sample $\mathbb{M}$).
- Another possibility is to consider the agents $i \in \{1, \ldots, N\}$ as different groups, in which case w_i is a measure for the importance of the various opinions in the sample (for example as representatives of important pressure groups, etc).
- Another interesting perspective is to consider the agents $i \in \{1, \ldots, N\}$ as a group of experts reporting their opinion on quantity X of interest. In particular X is the quantity concerning which policy measures should be made. In this case w_i may quantify the credibility of the expert i, concerning her/his opinion (prediction) on the future evolution of the quantity X.

2.3 Stage 1: Clustering in the opinion space

As a first stage of our proposed decision making process we are concerned with the problem of allocating the opinions of a large group of agents, whose composition is heterogeneous, into as far as possible homogeneous sub-groups (clusters) that may designate the trends in the general group. For instance, the large group could be the general population of a country, whereas the clusters may correspond to tendencies within the country.

To this end we propose the following version of the celebrated K-means clustering algorithm. We will consider the opinions of the large groups as elements $\{x_1, \ldots, x_N\}$ of the opinion metric space (M, d). The idea is that like opinions will form clusters in this metric space. Upon being able to identify these clusters we can form a coarse graining of the group into sub-groups of like opinions, which can be treated as homogeneous groups for our level of coarse graining. Mathematically, this corresponds to breaking the large group G into k subgroups G_i, $i = 1, \ldots, k$, such that $G = \bigcup_{i=1}^{k} G_i$, and $G_i \cap G_j = \emptyset$ for $i \neq j$, with the opinions $x_\ell \in G_i$ being as homogeneous as possible. As discussed above, homogeneity of a subgroup will be understood in terms of the Fréchet function of the subgroup, whereas a relevant measure for the center of the group will be the Fréchet barycenter of the subgroup. This scheme can be applied for any relevant metrization of the opinion space M (see e.g. examples in previous section), for the case of the Wasserstein space see Papayiannis et al. (2021). The proposed clustering algorithm to be implemented in the opinion space is summarized in Algorithm 1.

Algorithm 1 K-Means Clustering Scheme in the Opinions Space

1. Choose a relevant metrization of the opinion space (M, d) and a number of clusters K with centers $\bar{x}_k^{(0)}$ for $k = 1, \ldots, K$.
2. At each step m, each of the opinions x_i for $i = 1, \ldots, N$ is assigned to one of the K clusters where the cluster membership $k(i) \in \{1, \ldots, K\}$ is determined according to the rule
$$k(i) \in \arg \min_{k \in \{1,\ldots,K\}} d(x_i, \bar{x}_k^{(m)})$$
3. Cluster centers are updated through the rule
$$\bar{x}_k^{(m+1)} = \arg \min_{z \in M} \frac{1}{n_k^{(m)}} \sum_{i=1}^{n_k^{(m)}} d^2(x, x_{k,i}^{(m)}), \quad k = 1, \ldots, K,$$
where $n_k^{(m)}$ is the number of points that have been assigned to cluster k and by $x_{k,i}^{(m)}$, $i = 1, \ldots, n_k^{(m)}$ we denote the points that have been assigned to cluster k, at step m of the algorithm.
4. Steps 2-3 are repeated until the cluster centers do not change significantly.

At the convergence of the algorithm, K clusters of opinions are determined, centered at the points $\bar{x}_k$, $k = 1, \ldots, K$ in opinion space (M, d). Each of these clusters can be understood as a more or less "homogeneous" group of agents in terms of opinions. Denoting the groups by G_k, $k = 1, \ldots, K$, we expect our clustering algorithm to perform well in segregating the general group of agents G into subgroups if the Fréchet variance of each subgroup $V_k := \min_{z \in M} F_{G_k}(z)$ is comparatively low. Recall that the Fréchet variance of a subset $G_k \subset M$ can be also understood as an indicator of its homogeneity.

Note that the above algorithm can be expressed in terms of an optimization problem of the form

$$\min_{\substack{\bar{x}_k \in M, \\ k=1,\ldots,K}} \sum_{j=1}^{K} \sum_{i=1}^{N} a_{ij} d^2(x_i, \bar{x}_k), \tag{3}$$

$$\text{where } a_{ij} = \begin{cases} 1 & \text{if} \quad j = \arg\min_\ell d(x_i, \bar{x}_\ell) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

In other words, the elements a_{ij} provide information as to the membership of the point i to the cluster j, taking the value 1 if i belongs to cluster j and 0 otherwise. The K-means algorithm solves this problem by the following two-step procedure iterating Steps A and B till convergence:

A. Given the centers $\bar{x}_j$, calculate a_{ij} solving the minimization problem (4). This generates a membership matrix $A = (a_{ij}) \in \mathbb{R}^{N \times K}$ containing binary entries, with each column k of A denoting the composition of the group G_k.

B. Given the solution for a_{ij} from Step A, the new centers are determined by solving (3). Note that this step breaks down into K decoupled problems, each one involving the minimization of the Fréchet function for each G_k, or equivalently finding the Fréchet mean of the group, which is recognized as the center of the corresponding cluster. The objective's value at the minimum will then be the sum of the Fréchet variances of the clusters $\sum_{k=1}^{K} V_{G_k}$.

2.4 Stage 2: Reaching a consensus for the K groups

Once the clustering procedure has been completed and the K groups $G_1, \dots, G_K$ along with their barycenters (representative opinions) $\bar{x}_1, \dots \bar{x}_K$ have been identified, we may look for this opinion $x \in M$ that will be the most likely to be acceptable by all of the K groups. We argue that that this will be the Fréchet barycenter of the set $\mathcal{G} = \{\bar{x}_1, \dots, \bar{x}_K\}$, for an appropriate choice of weights $\lambda = \{\lambda_1, \dots, \lambda_K\} \in \Delta^{K-1}$ (where by Δ^{K-1} we denote the unit simplex in K-dimensions). We propose two alternative approaches, a geometric one (see Section 2.4.1) and a probabilistic one (see Section 2.4.2) , both of which highlight the barycenter as the appropriate choice, indicating different important aspects related to the group decision making process.

2.4.1 A geometric characterization of the consensus

Consider the representative opinions $\mathcal{G} = \{\bar{x}_1, \dots, \bar{x}_K\} \subset (M, d)$ of the groups $G_1, \cdots, G_K$. Each group k has a tendency to deviate around its central opinion (anchor), which can be modelled geometrically as follows: An opinion $x \in (M, d)$ will be considered as acceptable by group k as long as $d(x, \bar{x}_k) \leq \epsilon_k$, for some $\epsilon_k \geq 0$. The larger ϵ_k is the more likely is the group k to accept an opinion far from its anchor opinion. Similarly, a group with very small ϵ_k is very strict concerning its anchor opinion, and will not accept an opinion which is far from its anchor. Geometrically speaking, an opinion x will be acceptable by a group k if it lies within a ball in (M, d) of radius ϵ_k centered at $\bar{x}_k \in (M, d)$. If we wish to find an opinion which will be acceptable by all, then we need to search for opinions in the intersection of all the relevant balls for each group. Denoting by $B_k := \{z \in M \ : \ d(z, \bar{x}_k) \leq \epsilon_k\}$ the ball in M containing the acceptable opinions by group G_k, we can characterize the set of acceptable opinions by all groups in $\mathcal{G}$ as the intersection $\bigcap_{k=1}^K B_k$. This is equivalent to finding a solution to the set of inequalities

$$d(x, \bar{x}_k) \leq \epsilon_k, \ \ k = 1, \dots, K. \tag{5}$$

We will show that the solution to the set of inequalities (5) corresponds to a Fréchet barycenter $x^* = Bar(\mathcal{G}, \lambda)$ for a set of weights $\lambda = (\lambda_1, \cdots, \lambda_K)$ solving the problem

$$\max_{\lambda \in \Delta^{K-1}} V_{\mathcal{G}}(\lambda) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2 \tag{6}$$

where $V_{\mathcal{G}}(\lambda)$ is the Fréchet variance for the group $\mathcal{G}$ for a Fréchet function defined by assigning each member i of the group $\mathcal{G}$ the weight λ_i, and setting $\lambda = (\lambda_1, \cdots, \lambda_K)$. This result follows from a duality argument which is presented in Section 6.2 in the Appendix. The minimum $\epsilon > 0$ under which an agreement can be reached is chosen by the rule

$$\epsilon_* := \arg \max_{k \in \{1,2,\dots,K\}} d^2(Bar(\mathcal{G}, \lambda_*), \bar{x}_k)$$

where $\lambda_* \in \Delta^{K-1}$ minimizes the nontrivial dual problem. In particular, $\epsilon_* > 0$ provides the non-empty set property, i.e. if $d^2(Bar(\mathcal{G}, \lambda_*), \bar{x}_k) \leq \epsilon_*$ for all $k = 1, 2, ..., K$ then there exist agreement points. For details on this result see Section 6.3 in the Appendix.

Choosing the weights in the opinion barycenter as the solution to this dual problem we see that the weights must be chosen so as to maximize the Fréchet variance of the group, penalized by the weighted sum $\sum_{k=1}^K \lambda_k \epsilon_k^2$, which takes into account the propensity of the certain groups to deviate from their anchor position $\bar{x}_i$. This implies that the allocation of weights λ in the barycenter should be chosen so that we get the maximum possible variability of opinions.

Example 2.8 ($M = \mathbb{R}^d$ with the Euclidean metric)**.** If $M = \mathbb{R}^d$ endowed with the Euclidean metric (see e.g. Example 2.2) then the above calculations can be made explicit. In this case upon choosing a set of weights $\lambda \in \Delta^{K-1}$, the Fréchet barycenter of the group $\mathcal{G} = \{\bar{x}_1, \dots, \bar{x}_K\} \subset \mathbb{R}^d$ reduces to the standard notion of the weighted average

$$x^*(\lambda) = x_{\mathcal{G}}(\lambda) = \sum_{k=1}^{K} \lambda_k \bar{x}_k,$$

with the weight vector $\lambda = (\lambda_1, \dots, \lambda_K)$ given by

$$\lambda = \frac{1}{2}\mathbb{G}^{-1}(b - \frac{1}{K}\langle \mathbb{G}^{-1}b, \mathbf{1}\rangle \mathbf{1}), \tag{7}$$

with

$$\mathbb{G} = (\langle \bar{x}_k, \bar{x}_{k'} \rangle\ )_{k,k'=1,\dots,K} \in \mathbb{R}^{K\times K},$$
$$b = (\|\bar{x}_k\|^2 - \epsilon_k^2)_{k=1,\dots,K} \in \mathbb{R}^K.$$

For details, please see Section 6.4 in the Appendix.

Example 2.9 ($M = \mathcal{P}(\mathbb{R})$ with the Wasserstein 2 metric)**.** If M is the space of probability measures on $\mathbb{R}$ endowed with the Wasserstein metric (see Section 6.1 in the Appendix). Then, the Fréchet barycenter becomes the probability measure $x_{\mathcal{G}} = P_{\mathcal{G}}$ represented by the quantile average

$$q_{\mathcal{G}} = \sum_{k=1}^{K} \lambda_k q_k,$$

with the weights $\lambda = (\lambda_1, \dots, \lambda_K)$ determined by (7), with the sole exception that now the Gram matrix $\mathbb{G}$ and the vector b are defined in terms of the relevant quantities involving the quantile functions q_k and the L^2 inner product. For details see Section 6.5 in the Appendix.

Example 2.10 ($M = \mathcal{P}_N(\mathbb{R}^d)$ with the Wasserstein 2 metric)**.** Let M be the space of probability measures on $\mathbb{R}^d$ corresponding to random variables $X = (X_1, \dots, X_d)$ that may be modelled by the Gaussian family, denoted by $\mathcal{P}_N(\mathbb{R}^d)$. This will correspond to an opinion space, modelling diverse views concerning a quantity of interest $X \in \mathbb{R}^d$ (i.e. multiple scalar quantities such as e.g. future temperature, future level of economic activity etc) where agents can only infer concerning the probability distribution of the vector valued random variable X. In this case the barycenter P_B of a set of measures $\mathbb{M} = \{P_1, \dots, P_K\}$ is also a Gaussian measure $P_B \sim N(m_{\mathcal{G}}, S_{\mathcal{G}})$ with

$$m_{\mathcal{G}} = m_{\mathcal{G}}(\lambda) = \sum_{k=1}^{K} \lambda_k m_k,$$
$$S_{\mathcal{G}} = S_{\mathcal{G}}(\lambda) \text{ solving } S_{\mathcal{G}} = \sum_{k=1}^{K} \lambda_k (S_{\mathcal{G}}^{1/2} S_k S_{\mathcal{G}}^{1/2})^{1/2},$$

where $\lambda = (\lambda_1, \dots, \lambda_K)$ solving the dual problem

$$\max_{\lambda \geq 0} \sum_{k=1}^{K} \lambda_k \left(\|m_k - m_{\mathcal{G}}\|_2^2 + Tr\left(S_{\mathcal{G}} + S_k - 2(S_{\mathcal{G}}^{1/2} S_k S_{\mathcal{G}}^{1/2})^{1/2} \right) \right) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2$$

This is a well posed problem that can be solved numerically. For details see Section 6.6 in the Appendix.

2.4.2 Barycenters as maximizers of probability of agreement

Let us now provide an alternative argument for the choice of the barycenter as the most likely agreement point of the groups $\mathcal{G}$ generated by the clustering procedure. Given the clusters and their representative opinions $\mathcal{G} = \{\bar{x}_k, \,:\, k = 1, \dots, K\} \subset M$, we assume that the probability of the group k agreeing with an opinion $x \in M$ depends on the distance $d(x, \bar{x}_k)$ of this opinion from the center of the cluster of opinions forming group k. This probability p_k is expected to be a decreasing function of the distance, i.e. $p_k = \phi_k(d^2(x, \bar{x}_k))$, where $s \mapsto \phi_k(s)$ is a decreasing function. Since the function ϕ_k may be different for each agent (or group of agents, depending on the framework), it should contain each side's preferences concerning the aversion from the anchor position. We will not delve into details on the choice of the functions ϕ_k (however the evolutionary scheme presented in the next section provides an intuition concerning the aspects that may shape this function), but as an indicative example we offer the function $\phi_k(x) = \frac{1}{2}\frac{1}{1+e^{\alpha_k d^2(x,\bar{x}_k)}}$, $\alpha_k \geq 0$ which resembles the logistic model.

Assuming that the representative agents of each group are independent, we see that the probability of acceptance of proposed opinion x by all the groups is equal to

$$P = p_1 \cdots p_K = \prod_{k=1}^{K} \phi_k(d^2(x, \bar{x}_k)).$$

A reasonable choice for x, if common acceptance is required, is that x which maximizes the probability of acceptance, i.e. the solution of the optimization problem

$$\max_{x \in M} P(x) = \max_{x \in M} \prod_{k=1}^{K} \phi_k(d^2(x, \bar{x}_k)) \tag{8}$$

The maximizer of the above problem is the same as of

$$\max_{x \in M} \ln P(x) = \max_{x \in M} \sum_{i=1}^{K} \ln \phi_k(d^2(x, \bar{x}_k)) \tag{9}$$

which resembles the problem of the barycenter. In fact, we will see that the solution of this problem can be expressed in terms of a Fréchet mean (barycenter) for an appropriate choice of λ depending on the properties of the functions ϕ_k. Note also that the above problem is formally equivalent to a Nash bargaining game in the opinion space M. This result can be proved for certain special cases explicitly.

Example 2.11 ($M = \mathbb{R}^d$ with Euclidean distance)**.** In this case the solution of (9) is

$$x = \sum_{k=1}^{K} \lambda_k \bar{x}_k, \quad \lambda_k = \frac{\Lambda_k}{\sum_{k'=1}^{K} \Lambda_{k'}}, \quad \Lambda_k := 2\psi_k'(\|x - \bar{x}_k\|^2).$$

The above shows that x is a barycenter, in $M = \mathbb{R}$, for an appropriate choice of weights $\lambda = (\lambda_1, \ldots, \lambda_K)$ with the λ_k given as above, reflecting the preferences of the agents towards deviating from their anchor positions as modeled by the functions ψ_k and their elasticities. For details see Section 6.7 in the Appendix.

Example 2.12 ($M = \mathcal{P}(\mathbb{R})$ with the 2-Wasserstein distance)**.** For that we get a similar result - see Petracou et al. (2022) for a full discussion. In particular, following similar steps as above, the minimizer can be characterized as the quantile average

$$g(s) = \sum_{k=1}^{K} \lambda_k \bar{g}_k(s), \quad \lambda_k = \frac{\Lambda_k}{\sum_{k'=1}^{K} \Lambda_{k'}}, \quad s \in [0, 1]$$

where $\bar{g}_k$ for $k = 1, ..., K$ denotes the quantile function related to the probability model of group k (representing the group's opinion) and $\Lambda_k := 2\psi_k'(\|g - \bar{g}_k\|^2)$.

Generalizing the result of Example 2.12 for general probability measures on $\mathbb{R}^d$ with the space of measures metrized in terms of the Wasserstein distance is not straightforward, and beyond the scope of the present paper. Here we present a partial result, which shows that for the case of multivariate normal families the measure maximizing the probability of agreement is also a barycenter for the set of probability models.

Proposition 2.13. *Let $\mathbb{M} = \{P_1, \ldots, P_K\}$ with $P_k \in \mathcal{P}(\mathbb{R}^d)$, and $P_k \sim N(\mu_k, S_k)$, $\mu_k \in \mathbb{R}^d$, $S_k \in \mathbb{R}_+^{d \times d}$, $k = 1, \ldots, k$. Then, a solution of problem (8) (or equivalently of (9)) coincides with a barycenter of $\mathbb{M}$ with the weights $w = (w_1, \ldots, w_K)$ inherently determined by the anchor points and the preferences of the agents towards deviating from them.*

Proof. For the proof see Section 6.8 in the Appendix. □

Remark 2.14. Notice, that in all the aforementioned cases the effects from the functions ϕ_k are introduced to the consensus through the weights Λ_k and λ_k which characterize the solution. Clearly, depending on these functions the final weight vector may vary significantly from the uniform weighting.

2.5 Summing up: The two-stage process

We close this section by summing up the two-stage group decision process to the following steps:

1. Collect and map all opinions of the group as points z_i, $i \in G$ into the appropriate opinion space $\mathcal{M}$.

2. Perform a clustering procedure in opinion space as proposed in Section 2.3 Algorithm 1 to form K groups in opinion space.

3. Identify the group agreement point as a barycenter of the set of opinions $\mathbb{M} = \{z_{B,1}, \ldots, z_{B,K}\}$, i.e. as a barycenter of barycenters.

Remark 2.15. The choice of weights in the barycenter will depend either on the characteristics of the groups (e.g. their power or importance), or their characteristics and propensities towards deviating from their initial positions. The dynamics of reaching the consensus point, including the determination of weights is described in an evolutionary approach presented in Section 3.

3 An evolutionary approach for reaching a consensus under the multiple-agents learning framework

Having established in the previous section the relevance of the barycenter as a possible consensus point in opinion space for a group of heterogeneous agents, we turn our attention to the dynamics of the group decision making process and the mechanics of reaching to a consensus. This issue is related primarily with Stage 2 of the decision making procedure described in the previous section, that of reaching a consensus between the K groups consisting of more or less homogeneous opinions, each represented by an appropriate barycenter, obtained after the clustering procedure of Stage 1. In this section, an evolutionary framework is proposed for the description of the behaviour for a number of agents when a consensus need to be reached. In particular, the notion of barycenter is employed for assessing the situation from the perspective of each agent, taking into account her/his subjective beliefs concerning the time of the agreement and aversion preferences from her/his anchor beliefs. The whole task is considered that it takes place on a network which edges represent the connections between the various agents and the respective weights are updated at time progresses. The section is concluded with an illustration of some particular examples of interest where the achieved consensus and the effect to their determination by the agents' preferences are illustrated.

3.1 Evolutionary schemes for consensus

We now propose an evolutionary scheme for consensus achievement that may fully take into account interactions and dependencies between the various agents. The scheme is inspired by Bishop and Doucet (2021).

Informal presentation of the evolutionary scheme Before stating the technical details of the scheme we start by presenting the fundamental ideas and the motivation behind the scheme:

0 **Setting up a neighbourhood structure:** We first have to propose a network structure for the group of agents, which may model possible interactions and dependencies between them. To this end, we first consider a group of agents $G = \{1, 2, ..., N\}$ and a set of time-varying edges (links) $\mathcal{E}(t)$ formulating the time-varying graph $\Gamma(t)(G, \mathcal{E}(t))$. Note here, that by agents we may either mean individual agents, or groups of agents, for example the clusters in opinion space $\mathcal{M}$ obtained by the clustering procedure proposed in Section 2.3 (in which case each agent is identified with a cluster, so that $N = K$).

The neighbor set of any agent $i = 1, 2, ..., N$ is denoted by $\mathcal{N}_i(t) = \{j \in G \ : \ (i, j) \in \mathcal{E}(t)\}$. The agents connect with each other through an also time-varying graph adjacency matrix $A(t) \in \mathbb{R}^{N \times N}$ where for any agent i its elements are defined as follows

$$\alpha_{ij}(t) = \begin{cases} 1, & i = j \\ 1, & (i, j) \in \mathcal{E}(t) \\ 0, & (i, j) \notin \mathcal{E}(t) \end{cases}$$

while $W(t) \in \mathbb{R}^{N \times N}$ denotes the corresponding weighted adjacency matrix representing the link intensity between the various agents. Standard assumptions that are made are: (i) $w_{ij}(t) \geq 0$ for any pair (i, j) and any $t \in \mathbb{N}$ and (ii) $\sum_{j \in \mathcal{N}_i(t)} w_{ij}(t) = 1$. Note that the graph $\Gamma(t)$ models possible dependencies or affinities between the various groups participating in the decision making process. Such an affinity may affect the probability that a given agent or group i accepts a particular proposal in the opinion space, depending on whether members of the group in the same clicque have accepted the proposal or not.

1 **Local updating of opinion for the agents:** Assuming that at time t the agents have not reached to a consensus and their opinions are identified by the set $\mathbb{M}(t) := \{\mu_i(t), \ \forall i \in G\}$ being a subset of an appropriate opinion space $\mathcal{M}$ whose nature depends on how the belief is represented, e.g. by a vector on $\mathbb{R}^d$, by a probability model, etc (see Section 2.2). At time t, when the information concerning the current position of all agents is revealed, the agents re-allocate their beliefs in order to reach to a consensus in the future. The time horizon in which each agent would like to reach consensus (so that the agreement is finalized) is subject to each agents time preferences and needs. Given that no consensus has been reached at time t, the agents enter a new round of negotiations, in which they enter after renewing their positions original positions $\mu_i(t)$ in opinion space $\mathcal{M}$ to a new position $\mu_i(t+1)$. In this position updating procedure, each agent i is affected by her/his immediate neighbours $\mathcal{N}_i(t)$ as these are quantified by the dependencies and connectivities between agents by the time varying graph $\Gamma(t)$. This is a reasonable assumption since an agent's opinion is likely to be more affected by her/his immediate dependencies and/or pressure/interest groups. As we have already provided ample evidence in Sections 2.4.1 and 2.5 for the Fréchet barycenter as a feasible new position in $\mathcal{M}$, we propose that the new position $\mu_i(t+1)$ for each agent i, will be a local Fréchet barycenter of the points in opinion space in its neighbourhood $\mathcal{N}_i(t)$, with an appropriate choice of weight for eacn point in $\mathcal{N}_i(t)$. This choice, models the interaction between the agents in $\mathcal{N}_i$, and the effect they may have on i, (i.e. due to influence or coercion etc). This effect is modelled by the choice of weights for the local barycenter, with the weight w_j assigned to each agent j in $\mathcal{N}_i$ reflecting the relative influence of j on i. The selection of weights will be made by a weight update mechanism (see e.g. (11)).

2 **Checking for consensus:** After this opinion updating mechanism has been completed, the agents may check for consensus for their new positions. This is done as follows: Given the new positions of the agents $\mu_i(t+1)$, $i = 1, \cdots, N$, in opinion space, we form the global barycenter (with homogeneous weights) $\mu_B(t+1)$ and form the probability of agreement P_i of each agent i with $\mu_B(t+1)$, so that it depends on the distance of $\mu_B(t+1)$ with the new anchor point $\mu_i(t+1)$. If the probability of agreement for all agents is sufficiently high then consensus is achieved, otherwise it is not. The probability of agreement for all agents can be either be calculated treating the agreement of the different agents as independent events, or by assuming a dependency structure between the agents similar to the one modelled by the graph $\Gamma(t)$. If consensus is reached we stop, otherwise we continue to next iteration.

The aspects that mostly affect the convergence to an agreement are expected to be: (a) the heterogeneity of beliefs and/or tendencies (propensities) of agents to update their anchor positions among the groups of agents, (b) the intensity of the connectivities and dependencies among the agents and (c) the level of impatience of each agent towards reaching consensus, related to discounting. These features, will have to be introduced in the opinion and weight update procedure in terms of properly selected parameters.

The evolutionary scheme: Technical details: We now present the technical details of the evolutionary algorithm, for convenience keeping the same numbering as above.

1. **Local updating of opinion for the agents:** The i-th agent's opinion re-allocation can is described through the i-th node local barycenter problem with respect to the weight vector $w_i(t+1) \in \Delta^{N-1}$:

$$\mu_i(t+1) = \arg\min_{\nu \in \mathcal{M}} \sum_{j \in \mathcal{N}_i(t)} w_{ij}(t+1) d^2(\nu, \mu_j(t)), \quad \forall i \in G \tag{10}$$

where $\mathcal{M}$ denotes the opinion space, and the weights are determined in terms of the reallocation scheme

$$w_{ij}(t+1) = \theta_i w_{ij}(t) + (1-\theta_i) \frac{\exp\{-r_i d^2(\mu_i(t), \mu_j(t))\}}{\sum_{k \in \mathcal{N}_i(t)} \exp\{-r_i d^2(\mu_i(t), \mu_k(t))\}}, \quad \forall (i,j) \in \mathcal{E}(t) \tag{11}$$

where $\theta_i \in [0, 1]$ and $r_i > 0$ represents the i-th agent's inertia preferences and her/his preferences for a quick convergence to a consensus, respectively.

2. **Checking for consensus:** Given the new positions in opinion space, $\mu_i(t+1) \in \mathcal{M}$, find the global barycenter $\mu_B(t+1)$ (e.g. with homogeneous weights $w_i = 1/N$) and compute the probability of acceptance of $\mu_B(t+1)$ by agent i in terms of

$$q_i^{accept}(t) := P(\text{Agent i accepts the barycentric opinion } \mu_B(t+1) \text{ at time t+1}) = e^{-\rho_i \; t \; d^2(\mu_i(t+1), \bar{\mu}_B(t+1))} \tag{12}$$

where $\rho_i > 0$ is a parameter modelling agent's i propensity of deviating from her/his anchor position μ_i.

The probability of acceptance of $\mu_B(t+1)$ by all agents of the group is calculated using q_i^{accept}, either assuming independence of the agents or taking into account dependencies (that may be modelled in terms of a graph structure similar to that of $\Gamma(t)$). If the total probability of acceptance is above a certain threshold, consensus is reached and we stop, otherwise we move to step 1.

The evolutionary scheme is summarized in Algorithm 2 below:

When a large number of agents needs to reach a consensus, it is often more efficient to first cluster the agents into groups, determine the dominant opinion in each group and then explore for the consensus point among the clusters' most representative opinions. Such a procedure could be performed through the following three steps:

1. Distinguish the n agents into $K << n$ groups (clusters) using the clustering algorithm 1
2. Apply consensus learning algorithm 2 for each cluster to determine the consensus opinion on cluster $k = 1, 2, ..., K$.
3. Formulate the opinion set $\mathbb{M}$ using the K local consensus points (opinions) obtained in Step 2 and apply once more the consensus learning algorithm 2 to derive the consensus point of all groups.

Algorithm 2 The Evolutionary Consensus Learning Scheme

Step 0 (Initialization):	Set $t = 0$ and provide the initial beliefs $\mathbb{M}(0)$, the connectivity structure $W(0)$ and the preference parameter vectors $\{\psi_i\}_{i\in G} = \{(\theta_i, \rho_i, \epsilon_i\}_{i\in G}$.
Step 1 (Iteration Update):	Set $t := t + 1$ and repeat Steps 2–5 till a consensus is reached.
Step 2 (Time-perspective update):	Each agent updates her/his time preferences through criterion (14).
Step 3 (Connectivity Update):	Each agent updates her/his local connectivity structure through criterion (11).
Step 4 (Opinion Update):	Each agent updates her/his opinion (probability measure) through criterion (10).
Step 5 (Acceptance condition):	Each agent accepts the barycenter of the updated opinion set $\mathbb{M}(t)$ with probability of acceptance as determined in (12).

Remark 3.1. The following comments are in order:

1. Clearly, the updated position $\mu_i(t + 1)$ that is adopted at time $t + 1$ by each agent (see (10)), depends directly on the intensity of connections of the current agent with the rest agents in the network. Moreover, as soon as the states of each node are revealed (i.e. the position of each agent at time t), the weighted contingency matrix $W(t)$ will be updated to $W(t + 1)$ with respect to each agent preferences and perspective. The optimal weight selection problem in such type of learning schemes has been discussed in Papayiannis and Yannacopoulos (2018b), and a variant to the typical weight optimization problem is introduced which allows for modelling different agent characteristics and behaviours. Here we consider (11), which is a modification of this modelling approach, where a distinct weight updating rule for each agent is introduced, allowing for different levels of inertia with respect to the previous connection structure (at the previous time step). Moreover, different approaches in weight reallocation is performed by the agents, where the weight reallocation speed is tuned by a separate sensitivity parameter according to each agent preferences and time-perspective.

2. As already mentioned (11) depends on some behavioural parameters for the agents: $\theta_i \in [0, 1]$ and $r_i > 0$ represents the i-th agent's inertia preferences and her/his preferences for a quick convergence to a consensus, respectively. Inertia parameter values close to 1 indicate a side that is willing to strongly remain to its initial connections (coalitions) while values close to 0 indicate a side that is quite flexible in drastically alternating its connectivity. Moreover, sensitivity parameter r_i models the strategy concerning the intensity of the side in reallocating its connectivity. Small values indicate reallocation-averse behaviour concerning the alternation of the local connectivity structure while high values indicate the exact opposite.

3. Possible extensions of the scheme where the sensitivity parameter r_i could be time-varying can be conceived. For example a possible evolution scheme for the parameter could be described as

$$r_i(t+1) = L_i(t, \mathbb{M}(t)), \quad \forall i \in G \tag{13}$$

considering L_i as the loss function of the agent i depending on the states of the current opinion set $\mathbb{M}(t)$, i.e. indicating the loss (under the assumption that $L_i \geq 0$) taking into account the time t and the level of homogeneity in the opinion set $\mathbb{M}(t)$. For instance, a possible choice could be the subjective rule

$$r_i(t+1) = \begin{cases} 0, & \text{if } d^2(\mu_i(0), \mu_j(t)) \leq \epsilon_i, \ \forall j \\ e^{-\rho_i t} \sum_{j \in \mathcal{N}_i} w_{ij}(t) d^2(\mu_i(0), \mu_j(t)), & \text{if } d^2(\mu_i(0), \mu_j(t)) > \epsilon_i \text{ for any } j \end{cases} \tag{14}$$

where $\rho_i > 0$ expresses the agent's preferences concerning a fast resolution of the problem while $\epsilon_i > 0$ denotes the agents desire to deviate from her/his anchor preferences $\mu_i(0)$. In this setting, the preferences concerning the time upon which a consensus should be reached governs the determination of the general time-preferences parameter as t grows.

4. The scheme converges to a consensus point if all agents accept the barycenter of their current opinions, i.e. the barycenter of the set $\mathbb{M}(t)$. We assume that each agent accepts the barycentric opinion $\bar{\mu}(t)$ with a certain probability taking into account the distance from her/his opinion discounted by her/his time preferences factor $\rho_i t$. As a result, the probability of acceptance for the agent is determined as in (12) which is quite close to the probability of agreement maximization problem (8) discussed in Section 2.4.2. In fact, the current learning scheme can be considered as a variant of this problem where the preferences functions ϕ_i are represented in terms of the parameter vector $\psi_i = (\theta_i, \rho_i, \epsilon_i)$ for each agent i specifying what her/his preferences are and how the consensus problem is affected by them. So the convergence condition in this scheme (all agents should accept at some time step t the barycentric opinion) is somehow equivalent to maximizing the weighted by the preference functions $\{\phi_i\}_{i=1}^n$ probability of agreement to some opinion $\nu \in \mathcal{M}$.

5. Depending on the agents' anchor points and the homogeneity levels of the agent's groups (if applicable) the modified two step learning scheme, which is performed to the K groups obtained after the initial (stage 1) clustering procedure (see Section 2.3) in opinion space has been performed, could be much faster in deriving the consensus point. For an opinion set $\mathbb{M}$ where the opinions are uniformly distributed and the agents' preferences are quite similar there is not expected much difference between the two schemes performance with respect to the time that a consensus is achieved. In fact, for this case the one-step scheme described in 2 is maybe a better idea. However, when several groupings appear in the opinion set and especially when these groups display different homogeneity levels concerning the agents' preferences, then the two-step scheme is much more preferable and quite faster. A non trivial issue concerning the implementation of the two-stage scheme arises when one has to perform the Step 3 of the related procedure. In this step, each group is considered as a single agent (K-fictitious agents instead of n) with anchor point the related local consensus $\bar{\mu}_k$ for $k = 1, 2, ..., K$. However, which are the preferences that represent the group? Clearly, this is not a straightforward question to be answered and this topic of preferences aggregation has attracted the interest of the economists (see e.g. Gollier and Zeckhauser (2005); Jouini and Napp (2014); Chambers and Echenique (2018); Zuber (2011)). Since, it is beyond the scope of this work to investigate this matter, we provide two different perspectives in preferences aggregation: (a) the averaging approach (avg) and (b) the group's most conservative preferences (worst) which may lead to the less flexible behaviour and higher times till the agreement. In particular the preferences

vector for each group k are determined for the averaging approach through the rule

$$\psi_k^{avg} = \left(\frac{1}{|G_k|} \sum_{i \in G_k} \theta_i, \frac{1}{|G_k|} \sum_{i \in G_k} \rho_i, \frac{1}{|G_k|} \sum_{i \in G_k} \epsilon_i, \right), \quad k = 1, 2, ..., K \tag{15}$$

with G_k denoting the set of vertices constituting the k group (it holds that $G = \cup_{k=1}^K G_k$), and for the worst case approach through the rule

$$\psi_k^{worst} = \left(\max_{i \in G_k} \theta_i, \max_{i \in G_k} \rho_i, \min_{i \in G_k} \epsilon_i \right), \quad k = 1, 2, ..., K. \tag{16}$$

The latter case can be realized as a worst case bound concerning the time that a consensus needs to be reached.

3.2 A numerical experiment

In this subsection we provide a numerical experiment employing the two consensus learning schemes described in the previous section to better understand and illustrate their behaviour and characteristics. Three different cases are considered concerning the agents' preferences and in particular are considered: (a) agents with similar aversion and time-discounting preferences, (b) agents with ordered preferences and (c) agents with different types of time-discounting preferences. To compare the required time for the one-stage and two-stage procedures we consider four different groups of agents where within each group there exist a homogeneity concerning the agents' preferences while between the groups the homogeneity level depends on the scenario. The one-stage scheme will handle all groups as one, while the two-step approach will first recover the groupings and then will apply the evolutionary method first within the groups and then globally to determine the consensus point. We consider elliptical groups with respect to the anchor opinions while the agents preferences within each group are generated by the model

$$\psi_{k,i} = (\theta_{k,i}, \rho_{k,i}, \epsilon_{k,i}), \quad \theta_{k,i} \sim U([\theta_{L,k}, \theta_{U_k}]), \quad \rho_{k,i} \sim U([\rho_{L,k}, \rho_{U,k}]), \quad \epsilon_{k,i} \sim U([\epsilon_{L,k}, \epsilon_{U,k}])$$

for any $i \in G_k$ for $k = 1, 2, 3, 4$. The lower parameter values $\theta_{L,k}, \rho_{L,k}, \epsilon_{L,k}$ and the upper ones $\theta_{U,k}, \rho_{U,k}, \epsilon_{U,k}$ differ per group k depending the scenario that is chosen. In Table (1) are briefly summarized the scenarios to be considered in the simulation experiments and the preferences specification for each group. An illustration of the initial anchor preferences of all agents and the obtained consensus points by the one-stage and two-stage schemes are presented in Figure (1) while the required time steps till the derivation of the consensus points by all methods are displayed in Table 2.

Scenario	**Agents' Preferences**	**Group A**	**Group B**	**Group C**	**Group D**
Similar preferences	Anchor opinion aversion	medium	medium	medium	medium
	Time-discounting type	indifferent	indifferent	indifferent	indifferent
	Weighting Inertia effect	medium	medium	medium	medium
Ordered preferences	Anchor opinion aversion	low	medium	medium	high
	Time-discounting type	patient	patient	impatient	impatient
	Weighting Inertia effect	high	medium	medium	low
Patience VS Impatience	Anchor opinion aversion	medium	medium	medium	medium
	Time-discounting type	patient	patient	impatient	impatient
	Weighting Inertia effect	high	high	medium	low

Table 1: Description of each scenario considered for all agents and for each group

Scenario	One-Stage Scheme	Two-Stage Scheme (avg)	Two-Stage Scheme (worst)
Similar Preferences	89	85 (58)	81 (54)
Ordered Preferences	127	34 (27)	57 (50)
Patience VS Impatience	79	45 (19)	65 (39)

Table 2: Time steps required for each scheme to derive the consensus point. In parentheses are displayed for the two-step schemes the time steps required to reach the local (cluster) consensus points.

The employed methods seems to provide quite close consensus points in all scenarios considered. It is also evident that the two-step procedures are quite faster and since a part of the total steps are performed only with the K-fictitious agents, the complexity is quite lower than the appeared one. The pure barycenter is depicted in all three scenarios to realize the effect of the agents' preferences in the

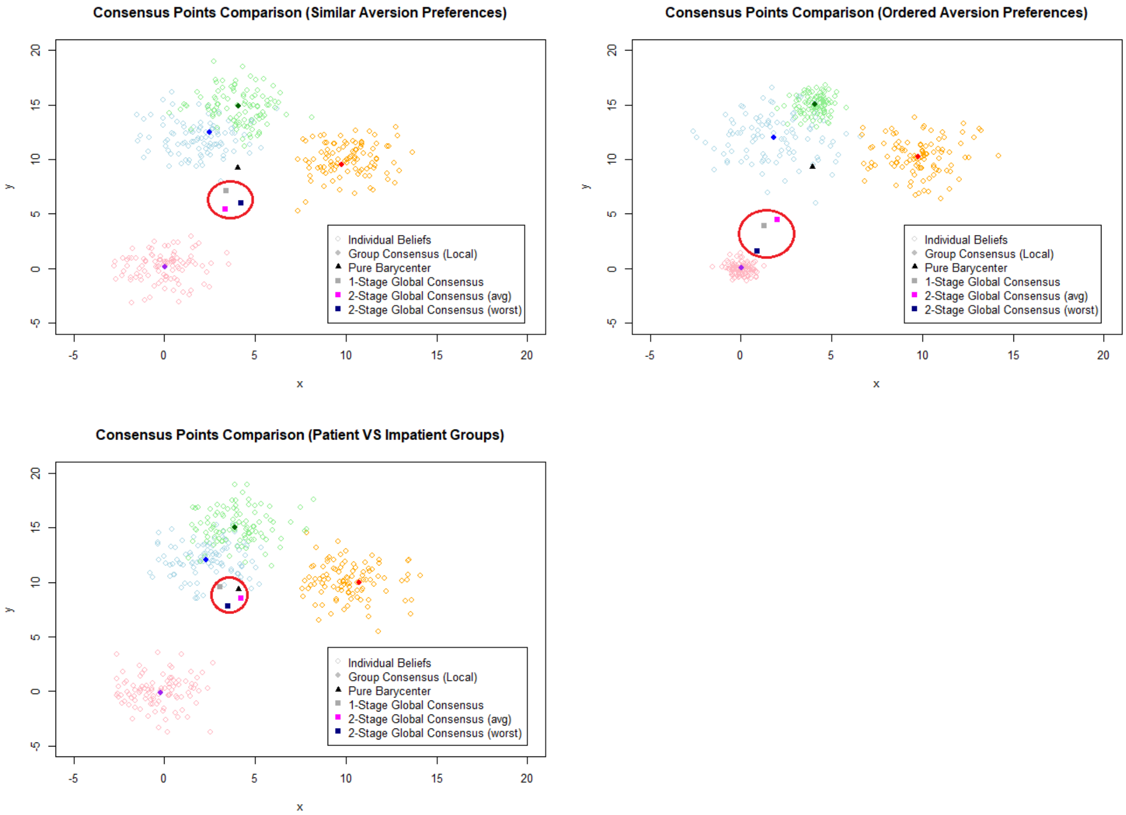

Figure 1: Illustration of the agents' anchor opinions (different colour indicates different cluster), the local consensus points and the derived consensus points (marked in red) with the proposed evolutionary learning schemes for all three scenarios considered.

final agreement point. This is quite obvious in the second scenario (ordered preferences) where the pure barycenter is quite distant from the calculated consensus points by the methods.

4 Application in Environmental Economics: Convergence to a Common Social Discount Rate

4.1 Motivation

Climate change seems to be a common threat and consequently a dominant scientific and political concern and in high priority in the global agenda. It constitutes one of the most crucial problems that needs urgent cooperative negotiations and solutions in order to achieve agreements dealing with various bad consequences of our ways of life as well as production and consumption! The United Nations Framework Convention of Climate Change, the Kyoto Protocol and Paris Agreement are indicators for international political actions and negotiations to deal with impact of climate change. Scientific knowledge for causes and effects of climate change and climate change's economic and social impact worldwide are closely connected in the terms of Intergovernmental Panel on Climate Change with the goal to assess the glocal situation and recommend potential adoption of policies. Climate change is a multifaceted and complicated (it is not the only!) phenomenon which among others is related to international relations, global governance in a geographically different and unequal world. Furthermore, it affects individuals and collectivities with uneven ways and with different levels of responsibility. In additions to power relations, climate change itself but also introduction and implementation of policies are related with present and future situations. Consequently, there is a need for common action! Actually, causes, conditions, impact are different spatially and timely but they are assembled under the processing of capitalist organizing of way of life. On various issues such as responsibility, justice, recommendation of policies and from whom and what are few issues of debates. One of the important aspect of these climate change negotiations is whether we have achieved consensus and for what- scientifically – politically and on what. Consensus is a wide issue/ element of negotiations and refers to different levels such as social, political, economical, technical etc. as well as the time of intervention such as how urgent must be the actions, when, where, which are the institutional arrangements and in which direction – market, technology... However, we have to consider about climate change's causes and crisis in order to identify potential conflicts and ways that we can overcome them. Besides debates and disagreements scientifically, geographically and politically, consensus is important but also mediator is a convenient way to overcome disagreement, scientifically and most importantly politically! If we would like to define process of decision making, we must take into consideration procedural injusticies in the climate negotiations.

4.2 Gollier's model for social discounting

The social discount rate (SDR) is one of most fundamental parameters in cost-benefit analysis and its determination is of crucial importance in any valuation study or for policy making (see e.g. Stern and Stern (2007); Nordhaus (2007), Gollier (2002); Weitzman (2007); Dasgupta (2008); Heal (2009); Groom et al. (2005) for area of climate change). The results of any valuation study are very sensitive to the choice of the social discount rate, and this sensitivity becomes more pronounced when longer horizon projects (such as for example environmental projects) are considered. Moreover, there is not unanimous agreement concerning the choice of the SDR, even when its calculation is based on widely accepted models, such as for example the Ramsey discounting formula.

As an example of how controversies concerning the determination of the SDR may arise between different agents, even when a single model is used, and its effects on the term structure of the discount rate we present the well known model for the determination of the SDR by Golier Gollier (2013), based on the classical Ramsey discounting formula. This formula connects the SDR with expected utility of consumption in the future in terms of

$$r(t) = \delta - \frac{1}{t} \ln \frac{\mathbb{E}[u'(C(t))]}{u'(C(0))}.$$

In the above,

- $r(t)$ is the discount rate at time 0 for any contingency X to be faced at time t
- δ is the utility discount rate,
- $C(t)$ denotes consumption at time t (a random variable unknown at time 0) and
- $C(0)$ denotes today's consumption.

From this formula, a term structure for r is derived (i.e. the dependence $t \mapsto r(t)$), and is a crucial parameter in standard cost-benefit analysis (Gollier (2013)). For example, given the term structure, the cost at time 0 of any contingency $X(t)$ to be faced at time t, is to be evaluated at $K(0,t) = \mathbb{E}[e^{-r(t)t}X(t)]$, a formula which clearly indicates the sensitivity of the estimated cost, and hence any valuation or cost-benefit analysis based policy, on the discount rate.

However, the future consumption at time t, $C(t)$, is unknown at time 0 that $r(t)$ is to be determined. Hence, the determination of $r(t)$ requires estimates of future consumption, a quantity which may well be subject to the effects of model uncertainty. Consequently, this uncertainty is moved on to the discount rate term structure, and from that to any valuation. As a result of such uncertainty it is conceivable that for a group of agents, possibly having different beliefs concerning $C(t)$, there will be different opinions regarding $r(t)$ and for any valuation for contingencies X.

To make the arguments more concrete, let us follow Gollier's model (Gollier (2013)) for the determination of the terms structure $t \mapsto r(t)$. We assume that a standard CRRA utility function with relative risk aversion γ is used to value consumption. Moreover, the consumption process $C(t)$ follows a single factor (autoregressive) model of the form

$$\begin{aligned} C(t+1) &= C(t)\exp(x(t)), \\ x(t+1) &= \mu + y(t) + \varepsilon_x(t), \\ y(t) &= \phi y(t-1) + \varepsilon_y(t), \end{aligned} \tag{17}$$

where $\varepsilon_x(t), \varepsilon_y(t)$ are independent and serially independent with $\mathbb{E}[\varepsilon_x(t)] = \mathbb{E}[\varepsilon_y(t)] = 0$ and $Var(\varepsilon_x(t)) = \sigma_x^2$, $Var(\varepsilon_y(t)) = \sigma_y^2$, y_{-1} is some initial state, and $\phi \in [0,1]$ is a parameter representing the degree of persistency (mean reversion) of y. This model is supported by empirical data (see e.g. Bansal and Yaron (2004)). Depending on the value of ϕ the model can be either reduced to a standard random walk model which is a discretization of a Wiener process ($\phi = 0$) or correspond to a discretization of an Ornstein-Uhlenbeck process ($\phi \neq 0$). Typically, $\{y(t)\}$ is an unobserved stochastic factor, which affects the observed growth rate $\{x(t)\}$ of the consumption process $\{C(t)\}$. Given values for ϕ and y_{-1}, the stochastic consumption process $\{C(t)\}$ is lognormally distributed and in particular

$$\ln C(t) - \ln C(0) \sim N(\mu_t, \sigma_t^2),$$

where

$$\mu_t = \mu\, t + y_{-1}\frac{1-\phi^t}{1-\phi},$$

$$\sigma_t^2 = \frac{\sigma_y^2}{(1-\phi)^2}\left[t - 2\phi\frac{\phi^t - 1}{\phi - 1} + \phi^2\frac{\phi^{2t}-1}{\phi^2-1}\right] + \sigma_y^2 t. \tag{18}$$

Using the general class of CRRA utilities, Gollier produces an analytic formula for the term structure of the discount rate as

$$r(t) = \delta + \gamma\frac{1}{t}\mu_t - \frac{1}{2}\gamma^2\frac{1}{t}\sigma_t^2. \tag{19}$$

Note that in the above formula, the term structure is increasing or decreasing depending on the sign of y_{-1}. Moreover, in the case where $\phi = 0$, the term structure is flat whereas for certain values it may have a convex structure. When all the parameters involved in model(17) are fully known the Ramsey formula can be used to produce a term structure for the SDR. However, even in this case the quantitative and qualitative (e.g. shape) properties of the term structure depend on the values of the parameters of the model, which themselves are not uniquely determined in terms of the available data. A calibration was performed in Bansal and Yaron (2004) for the factor model (17) for consumption using annual USA data from the period 1929-1998, yielding the estimated parameters (monthly estimates)

$$\mu = 0.0015, \;\; \sigma_x = 0.0078, \;\; \sigma_y = 0.00034.$$

On the same work, the mean-reversion parameter was estimated to $\phi = 0.979$. Of course, these estimations are subject to statistical errors which allow for other valued of these parameters, compatible with the available data, that may lead to different models for $C(t)$ and subsequently different (both in a quantitative and qualitative sense) models for the term structure of the discount rate as provided by (19).

4.3 Consensus achievement on the SDR and the probability model concerning the contingency: A numerical study

Motivated by the discussion in the previous section, we devise the following gedanken experiment concerning consensus achievement on the SDR (and hence on the valuation of any contingency) by a group of agents who albeit all abiding to model (19) (with $C(t)$ provided by (17)). The agents may have as anchor points versions of the model with different parameter values, hence resulting to differrent term structures for the discount factor and as a result different valuations of the same contingency X. The difference in the parameter values adopted by different agents in the group may arise from various reasons, among which being choice of different parameter values within the confidence interval for the US data, or the fact that different agents reflect differrent spatial locations and interests, i.e. are forming their time preferences for $r(t)$ in terms of future consumption for economies differrent than the US (hence leading to alternative calibrations for model (17)).

For the simulation study, a group G of N agents is considered, with each agent reporting a different term structure curve for the discount rate $t \mapsto r_i(t)$, all collected in a set $\mathbb{M} = \{r_1(\cdot), \dots, r_N(\cdot)\}$ of term structure curves. The set of curves $\mathbb{M}$ can be considered as a subset of a suitable metric space M, which will be chosen as a space of curves on $[0, T]$. This metric space of curves will serve as the opinion space M in the context of Section 2.2. The agents in G need to reach to a consensus towards the adoption of a commonly acceptable discount rate curve $r(\cdot)$ that will serve as the common instrument for the valuation of future contingencies X. Moreover, the group G consists of three different subgroups (i.e. $G = G_1 \cup G_2 \cup G_3$) with each subgroup introducing different homogeneity levels concerning the agents' preferences. For instance, each subgroup could be realised as a different region of the world where different range of elasticities related to consumption are observed due to cultural divergencies. The evolutionary algorithm introduced in Section 3 is employed for exploring potential consensus points in the metric space M of term structure curves, and investigate the dynamics of reaching consensus under the various heterogeneity levels between the agents with respect to the discount rate curves. The consensus discount rate curve, once and if reached, will be chosen as the SDR curve, used for the group G and will be the outcome of the agreement, henceforth chosen for evaluating a certain contingency X.

The generation of the SDR curves $r_i(\cdot)$, $i = 1, \ldots, N$, that the set $\mathbb{M}$ consists of, is made in accordance to the model of Gollier, presented in Section 4.2 by assuming that all agents abide to model (19) (with $C(t)$ provided by (17)), but adopting different values for the relevant parameters. To generate the opinion set $\mathbb{M}$ we sample a distribution of parameters for model (17), and then use (19) to generate the relevant discount rate curves. In particular, three different ranges for the elasticity parameter γ are considered, and specifically $\gamma_1 \sim \mathcal{U}([0.8, 1.5])$, $\gamma_2 \sim \mathcal{U}([0.4, 1.7])$ and $\gamma_3 \sim \mathcal{U}([0.3, 2.0])$ (where subscript denotes the subgroup) representing different behaviours and heterogeneity on the agents' perspectives, while the parameters δ, ϕ, y_{-1} are kept close to the calibration performed in Bansal and Yaron (2004), to capture more general behaviours. Specifically, these parameters are Uniformly and independently sampled as

$$\delta \sim \mathcal{U}([0.029, 0.031]), \quad \phi \sim \mathcal{U}([0.977, 0.981]), \quad y_{-1} \sim \mathcal{U}([-0.001, 0.001]).$$

According to this simulation scheme, each agent in G will report a discount rate curve corresponding to (19), with $C(t)$ generated by (17) (equiv. (18)) with parameters γ, δ, ϕ and y_{-1} chosen as a sample point from the above distribution. This concludes the construction of the opinion set $\mathbb{M}$. The parameters related to the consensus process, i.e. the parameters determining the agents' determination and impatience to reach a consensus, are generated according to the simulation scheme described in Section 3.2. For the simulation task a total number of $N = 90$ agents is generated, with each subgroup consisting of 30 agents. For the consensus determination task, both the one-stage and the two-stage processes are employed to illustrate and discuss the potential differences between the achieved consensus points. Moreover, two different scenarios are considered concerning the agents' preferences: (a) the *Uniform Beliefs* scenario, under which the agents in all groups are assumed to display uniformly distributed preferences in reaching a consensus, and (b) the *Impatient Agents* scenario, under which agents of different subgroup display different patience levels on reaching a consensus.

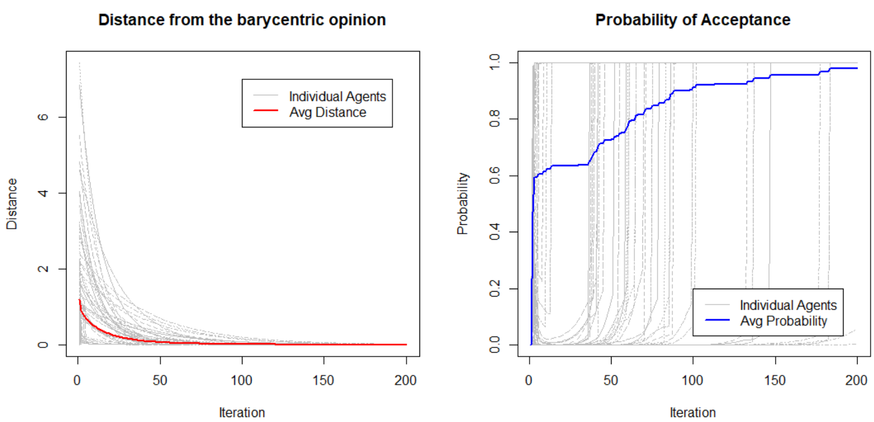

Figure 2: Convergence illustration to the barycenter by the one-stage process depicting for all agents: (a) distance from the consensus curve (left) and (b) acceptance probabilities with respect to the running barycentric curve

In Figure 2 is illustrated a case of the one-stage scheme where a SDR-consensus is achieved for understanding the convergence of the scheme. On the left plot, each agent's divergence from the achieved consensus curve is illustrated. The red line, indicating the average distance of all agents from the consensus curve at each iteration, displays purely decreasing tendency. On the right plot, each agent's acceptance probability of the running consensus curve is illustrated, with the blue line indicating the average acceptance probability for all agents. It is also evident that the average acceptance probability displays purely increasing tendency to 1 as iteration number grows indicating converging behaviour to a consensus. In general, for any scenario considered, convergence is expected with potential differences in the convergence rates to be explained by the special characteristics of the scenario under study (different time-preferences of the involved agents).

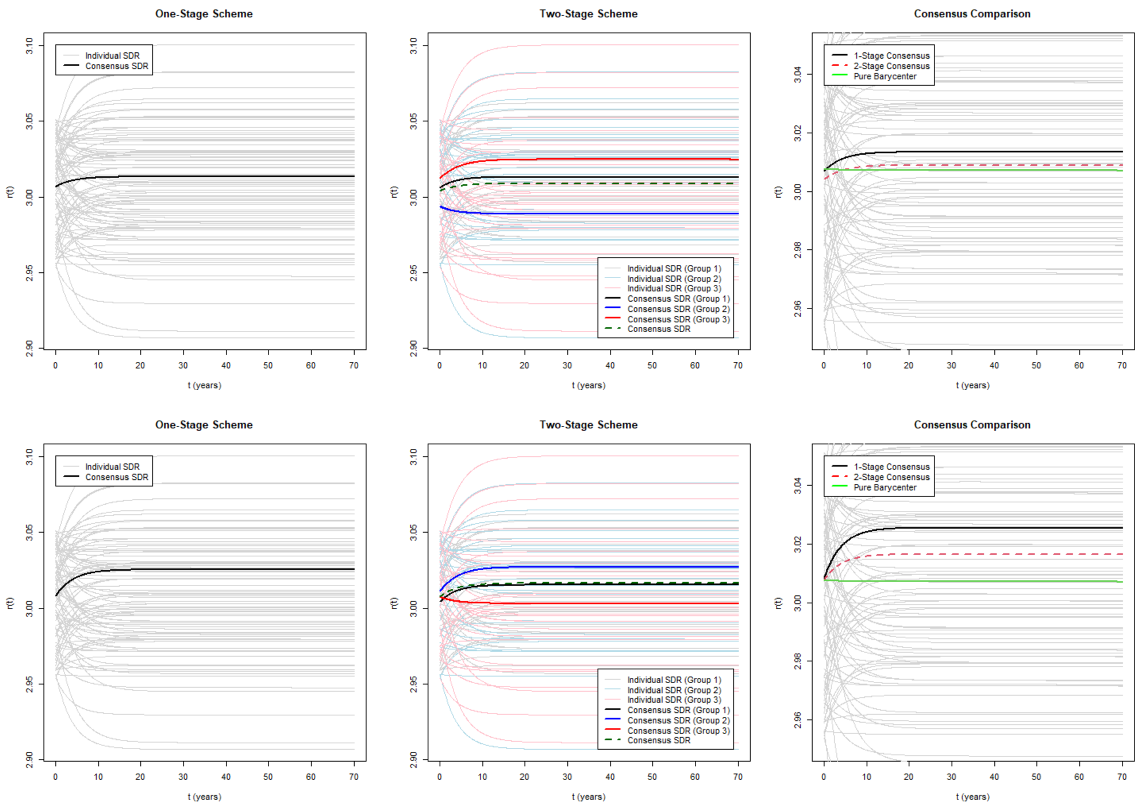

Figure 3: The achieved SDR-consensus curves achieved by the one-stage scheme (left), the two-stage process (center) and their comparison (right).

In Figure 3 are illustrated the sampled SDR curves, their classification to the different subgroups (distinguished by different colours on the middle plot) and the obtained consensus curves by the two schemes for the Uniform Beliefs scenario (upper panel) and the Impatient Agents scenario (lower panel). For both scenarios are also illustrated the barycentric curves (no preferences taken into account) for comparison reasons. In both cases, the obtained consensus curve from the two-stage scheme seems to be less affected by the agents' preferences since it is closer to the pure barycentric curve than the one-stage consensus. However, both achieved consensus curves in both scenarios do not differ that much, and since the two-stage scheme is computationally cheaper should be preferred.

At a second step, a consensus for the model describing the random behaviour (probability distribution) of the contingency X at a future time T is explored under both approaches and the two scenarios. Let us assume that all agents agree to the type of the model that could best describe the contingency distribution and in fact they consider the Generalized Extreme Value (GEV) distribution, which probability density function is

$$f(x) = \frac{1}{\sigma} t(x)^{\xi+1} e^{-t(x)}$$

with

$$t(x) = \begin{cases} \left(1 + \xi \left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}, & \text{if } \xi \neq 0 \\ e^{-(x-\mu)/\sigma}, & \text{if } \xi = 0 \end{cases}$$

where the parameters $\mu, \sigma > 0, \xi$ capture the location, scale and shape characteristics, respectively. The difference in the agents beliefs are introduced through different estimates concerning the true parameter values. In particular we consider that within subgroups there is a short of homogeneity in the respective estimates (however not of the same level for all groups) while across the subgroups the heterogeneity level higher. An illustration of the scenario under consideration for the contingency probability model with respect to the parameter values is provided by Figure 4.

Different considerations on the parameter vector $\theta = (\mu, \sigma\ \xi)'$ induce a different probability model P

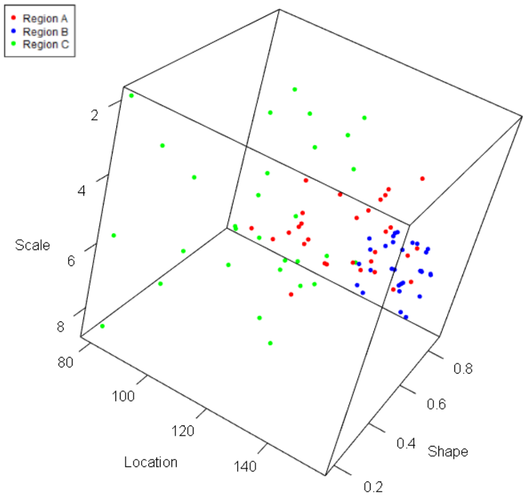

Figure 4: The simulated scenario for the beliefs concerning the probability model for the contingency

describing the contingency X. As a result, the current set of opinions in this case is $\mathbb{M} = \{P_1, ..., P_N\}$ which can be considered as a subset of the space of probability models in the real line, i.e. $\mathcal{M} = \mathcal{P}(\mathbb{R})$. Since, this is the metric space under which the consensus needs to be investigated, for the sake of simplicity, we assume that each provided P_i is independent from the SDR curve $r_i(\cdot)$ provided by each agent. In Figure 5 are illustrated both scenarios and the achieved consensus models by the two schemes.

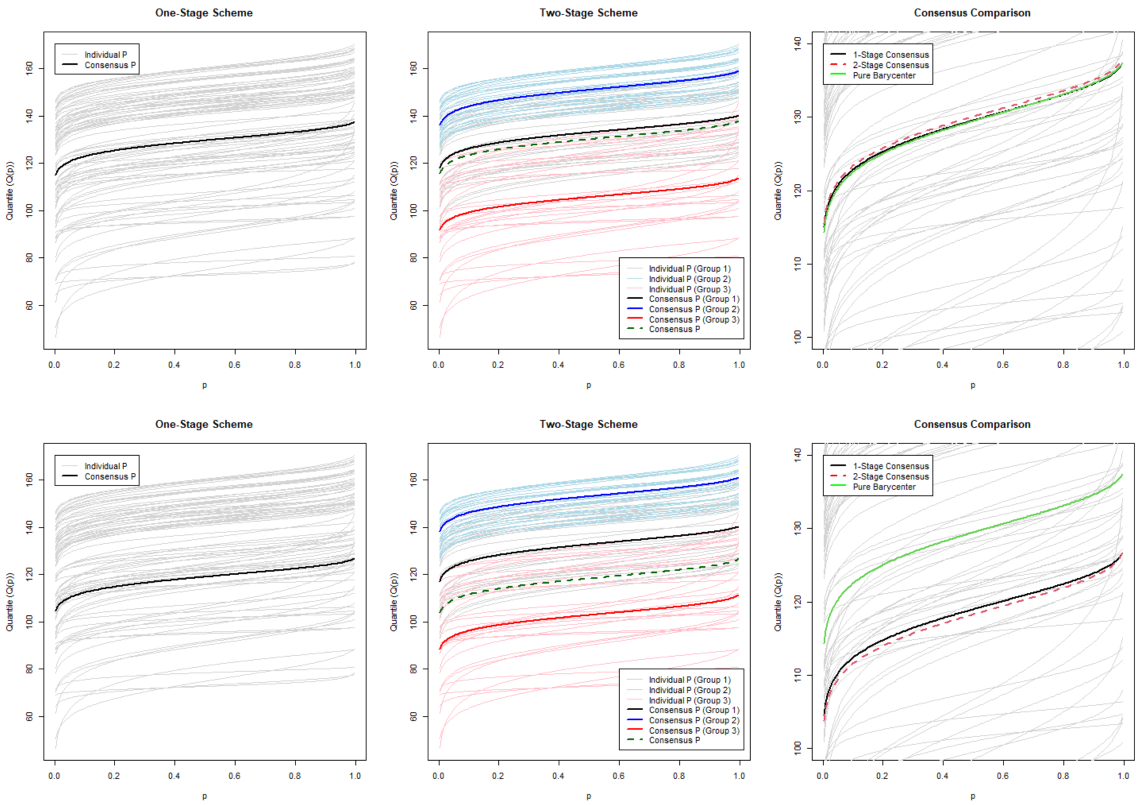

Figure 5: Achieved consensus from the one-stage scheme (left), the two-stage scheme (center) and their comparison (right) concerning the probability model that describes the contingency (in terms of quantiles) from the Uniform Beliefs scenario (upper panel) and the Impatient Agents scenario (lower panel).

The consensus models obtained by both schemes for the two scenarios are quite close, however, the pure barycenter (direct quantile average in the initial beliefs) in the Impatient Agents scenario is quite far from the consensus indicating the effect of the agents' preferences in the derivation of the consensus. Combining the derived consensus opinions by both schemes, evaluation for the contingency under consideration is provided in Table 3 under the two scenarios, accompanied by some descriptive statistics to better quantify the differences in the estimation. The contingency evaluation is provided in present values discounted by the obtained SDR-curves by each scheme and the related consensus probability model. Clearly, the estimates obtained in each scenario are quite close between the different approaches, however across the two scenarios, a significant difference is observed to the contingency valuation on account of the effect concerning different time-preferences of the involved agents.

	Scenario			
Descriptive	**Uniform Beliefs**		**Impatient Agents**	
Statistic	**1-Stage**	**2-Stage**	**1-Stage**	**2-Stage**
Mean	125.765	125.260	114.348	114.971
Std. Deviation	4.349	4.344	4.425	4.284
1st-Percentile	114.578	114.082	103.126	103.922
5th-Percentile	117.923	117.426	106.446	107.241
10th-Percentile	119.856	119.358	108.369	109.153
Median	126.241	125.737	114.777	115.443
90th-Percentile	131.041	130.531	119.760	120.162
95th-Percentile	131.988	131.477	120.794	121.098
99th-Percentile	133.198	132.686	122.177	122.305

Table 3: Decriptive statistics of the achieved consensus from the 1-Stage and the 2-Stage schemes for the contingency value for the two scenarios considered.

5 Conclusions

In this paper we have considered the problem of group decision making under the effects of agents heterogeneity and model uncertainty. Our approach is partly motivated by situations commonly encountered in environmental economics, but the methodological framework has wider applicability.

We propose a two stage procedure towards consensus group decision making, based on the concept of the Fréchet barycenter, which first partly homogenizes the agents by constructing clusters of fairly homogeneous agents in opinion space by an appropriate clustering procedure and second makes a proposal to members of the group for possible acceptance based on the Fréchet barycenter of the representative opinions of the clusters. Moreover, an evolutionary process for this proposal making process, which eventually leads to consensus is provided. This process clarifies the effect of the behavioural characteristics of the agents on the effectiveness of the decision making process, the probability of reaching consensus and the expected time required for consensus. The use of our proposed method is illustrated in a characteristic problem of environmental economics, that of deciding on a common social discount factor and a common probabilistic model for future contingencies, which is to be used for pricing abatement measures and policy making in such a way as to be widely acceptable by the group, hence effective.

6 Appendix: Technical details and proofs

6.1 The metric space of probability models: The Wasserstein space

In this section we collect some fundamental results concerning the Wasserstein metric and the Wasserstein space, as a metric space.

Definition 6.1 (The Wasserstein metric W_2)**.** Consider two probability measures μ, ν on $\mathbb{R}^d$. Their 2-Wasserstein distance $W_2(\mu, \nu)$ is defined as

$$W_2(\mu,\nu) = \left\{ \inf_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y) \right\}^{1/2} = \left\{ \inf_{X\sim\mu, Y\sim\nu} \mathbb{E}[|X-Y|^2] \right\}^{1/2},$$

where $\Pi(\mu, \nu)$ is the set of transport plans (i.e. measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν).

The 2-Wasserstein metric defines a true metric on the space of probability measures (or probability models) μ such that $\int_{\mathbb{R}^d} x^2 d\mu(x) \leq \infty$. That is the mapping $(\mu, \nu) \to W_2(\mu, \nu) =: d(\mu, \nu)$ satisfies the 3 properties required by a metric i.e. positivity, symmetry and the triangle inequality) thus making it a natural choice as a quantifier of discrepancy between different probability models. The metric space of such measures, endowed with the 2-Wasserstein metric will be denoted by $\mathcal{P}_2(\mathbb{R}^d$. While there are other choices for the Wasserstein metric, in terms of different costs than the cost function $(x, y) \mapsto |x - y|^2$, we will focus here on the 2-Wasserstein metric on account of some of its useful properties, and simply refer to is as the Wasserstein metric.

It is interesting to note that the Wasserstein metric provides some estimate of mispecification of a random variable is this is modelled using two different probability models μ, ν. This should be apparent from the very definition of W_2 or from the following estimate

$$|\mathbb{E}_P[U(X)] - \mathbb{E}_Q[U(X)]| \leq C W_2(P, Q).$$

This estimate models the mispecification error for any contingency X, as quantified by an expected utility, if X is modelled by two alternative probability models P, Q (corresponding to two different probability measures $\mu = P$, $\nu = Q$).

Even though the Wasserstein metric has very useful properties is is in principle very difficult to calculate. However, there are certain special cases where the Wasserstein metric can be calculated in closed form or almost closed form. We report two such cases which are of great interest.

1. Probability measures on $\mathbb{R}$: Probability measures on $\mathbb{R}$ are uniquely determined by distribution functions $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = P(X \leq x) = P((-\infty, x])$ or their (generalized) inverse F^{-1} called the quantile functions. The 2-Wasserstein distance between two probability measures P_1, P_2 on $\mathbb{R}$, can be obtained in terms of their corresponding quantile F_1^{-1}, F_2^{-1}, in terms of

$$W_2(P, Q) = \left\{ \int_0^1 (F_1^{-1}(s) - F_2^{-1}(s))^2 ds \right\}.$$

2. Normal measures on $\mathbb{R}^d$. Normal measures on $\mathbb{R}^d$ are fully determined in terms of a mean vector $\mu \in \mathbb{R}^d$ and a symmetric positive definite matrix $S \in \mathbb{R}_+^{d \times d}$, The Wasserstein distance between two measures P_1, P_2, such that $P_i \sim N(\mu_i, S_i)$, is given by

$$W_2(P_1, P_2) = \|\mu_1 - \mu_2\|^2 + Tr\left(S_1 + S_2 - 2(S_1^{1/2} S_2 S_1^{1/2})^{1/2} \right)$$

The Wasserstein distance in this particular context is often refered to as the Bures-Wasserstein distance, which (upon dropping the first contribution depending on μ_1, μ_2) also constitutes a metric in the space of symmetric positive definite matrices $\mathbb{R}_+^{d \times d}$.

The Wasserstein barycenter is a well defined quantity for various general classes of probability measures (see e.g. Agueh and Carlier (2011)). It computation is in general not an easy task, however in various special cases of particular interest in applications, closed forms or almost closed forms for the Wasserstein barycenter are available.

1. Probability measures on $\mathbb{R}$: If $\mathbb{M} = \{P_1, \ldots, , P_N\} = \{F_1, \ldots, F_N\}$ is a set of probability measures on $\mathbb{R}$, expressed in terms of the corresponding distribution functions F_i, $i = 1, \ldots, N$, then the Wasserstein barycenter corresponding to the weights vector $w = (w_1, \ldots, w_N) \in \Delta^{N-1}$, is the probability measure on $\mathbb{R}$ P_B, related to the quantile average function

$$F_B^{-1} = \sum_{i=1}^{N} w_i F_i^{-1}$$

2. Gaussian measures on $\mathbb{R}^d$: If If $\mathbb{M} = \{P_1, \ldots, , P_N\} = \{F_1, \ldots, F_N\}$ is a set of probability measures on $\mathbb{R}^d$, such that $P_i \sim N(\mu_i, S_i)$, $\mu_i \in \mathbb{R}^d$, $S_i \in \mathbb{R}_+^{d \times d}$, then the Wasserstein barycenter corresponding to the weights vector $w = (w_1, \cdots, w_N) \in \Delta^{N-1}$, is the probability measure on $\mathbb{R}^d$ $P_B \sim N(\mu_B, S_B)$, with $\mu_B = \sum_{i=1}^{N} w_i \mu_i$ and S_B being the solution of the matrix equation

$$S = \sum_{k=1}^{N} w_k (S^{1/2} S_k S^{1/2})^{1/2}$$

While this equation cannot be solved in closed form, the matrix S_B can be approximated in terms of a well behaved fixed point scheme.

6.2 Connection between problems (5) and (6) via duality

To treat problem (5), we now introduce the slack variable $c \in \mathbb{R}$ and replace the set of inequalities (5) by the optimization problem

$$\begin{aligned} \max_{c\in\mathbb{R}_+, x\in M} c \iff \min_{c\in\mathbb{R}_+, x\in M} -c \\ \text{subject to} \\ d^2(x,\bar{x}_k) \leq \epsilon_k^2 - c, \;\; k=1,\dots,K \end{aligned} \tag{20}$$

This problem corresponds to setting a margin c and choosing a point $x \in M$ which lies within each ball. If $c \in \mathbb{R}_+$ then x is in the interior of all the balls, meaning that it is safely acceptable by all groups. By maximizing c or equivalently minimizing $-c$ (for the case where $c \in \mathbb{R}_+$) we choose the point which is most likely to be acceptable by all the groups.

Notice that the slack variables c are allowed to take also negative values. Typically, slack variables are restricted to $\mathbb{R}_+^K$, i.e. selecting the optimal values for x, c that satisfy the equality constraints in (20) by placing x as "deeper" as it gets to the interior of the balls B_k. However, this would be feasible only if it is a priori known that there exist solutions for this problem, i.e. given the certain choices of the thresholds ϵ_k, there exist x that lies within the intersection of all balls. In the case that there is not a feasible solution x for the given thresholds, then allowing c to take negative values, then the negative components of c are interpreted as the less amount that certain groups should relax their preferences in order to reach an agreement. This is the case in situations where there is not common ground, either some sides should relax their anchor positions (i.e. provide greater radius in terms of our formulation) otherwise a consensus is not feasible. However, one could try to solve the problem first by restricting $c \geq 0$ and if the solution set is empty, then to relax the problem by allowing $c \in \mathbb{R}$.

Let us examine the case with the typical slack variable setting, i.e. assuming that a consensus is feasible. We first define the Lagrangian for problem (20) by choosing the respective multipliers $\lambda = (\lambda_1, \dots, \lambda_K) \in \mathbb{R}^K$ and defining the function

$$\begin{aligned} L(x,c;\lambda,\mu) &= -c + \sum_{k=1}^K \lambda_k(d^2(x,\bar{x}_k) - \epsilon_k^2 + c) + \mu c \\ &= -(1 - \sum_{k=1}^K \lambda_k)c + \sum_{k=1}^K \lambda_k d^2(x,\bar{x}_k) - \sum_{k=1}^K \lambda_k \epsilon_k^2 + \mu c \end{aligned}$$

We intend to write the optimization problem (20) in the saddle point formulation

$$\min_{c\in\mathbb{R}_+, x\in M} \left\{ -c \;:\; d^2(x,\bar{x}_k) \leq \epsilon_k^2 - c, \;\; k=1,\dots,K \right\} = \min_{c\in\mathbb{R}_+, x\in M} \; \max_{\lambda\in\mathbb{R}_+^K, \mu\in\mathbb{R}_+} L(x,c;\lambda,\mu) \tag{21}$$

$$(?) = \max_{\lambda\in\mathbb{R}_+^K, \mu\in\mathbb{R}_+} \; \min_{c\in\mathbb{R}_+, x\in M} L(x,c;\lambda,\mu) \tag{22}$$

with the question mark meaning that this equality is subject to applicability of the minimax theorem. We now look at the dual problem. Let us first consider the dual function

$$\mathbb{D}(\lambda,\mu) = \min_{c\in\mathbb{R}_+, x\in M} L(x,c;\lambda,\mu) = \min_{c\in\mathbb{R}_+} \left\{ -(1 - \sum_{k=1}^K \lambda_k)c + \mu c \right\} + \min_{x\in M} \sum_{k=1}^K \lambda_k d^2(x,\bar{x}_k) - \sum_{k=1}^K \lambda_k \epsilon_k^2$$

Upon inspection of the Lagrangian function, which is linear in c we see that

$$\mathbb{D}(\lambda,\mu) = \begin{cases} -\infty & \text{if } \sum_{k=1}^K \lambda_k \neq 1, \;\; \mu = 0 \\ \min_{x\in M} \sum_{k=1}^K \lambda_k d^2(x,\bar{x}_k) - \sum_{k=1}^K \lambda_k \epsilon_k^2 & \text{if } \sum_{k=1}^K \lambda_k = 1, \;\; \mu = 0 \end{cases}$$

Note that when $\mu = 0$ is the only interesting case, otherwise necessarily holds that $c = 0$, i.e. the common agreement point can be captured only on the boundaries of the preference balls for all groups. This immediately implies that the minimizer can be characterized in terms of the Lagrange multipliers as

$$x^* = \arg\min_{x \in M} \sum_{k=1}^{K} \lambda_k d^2(x, \bar{x}_k) \iff x^* = Bar(\mathcal{G}, \lambda), \tag{23}$$

i.e. x^* is the Fréchet mean (barycenter) for the group of clusters $\mathcal{G}$ with the choice of weights $\lambda \in \Delta^{K-1}$. The weights are yet unknown but are recognized as the Lagrange multipliers $\lambda \in \mathbb{R}_+^K$ for the above problem. We now, formally, write down the dual problem. The solution to this problem will provide us with the values of the Lagrange multipliers λ^* at optimality. By the above observation the dual function becomes (omitting the uninteresting case where $c = 0$ i.e. $\mu > 0$):

$$\mathbb{D}(\lambda, \mu) = \begin{cases} -\infty & \text{if } \sum_{k=1}^{K} \lambda_k \neq 1 \\ V_{\mathcal{G}}(\lambda) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2 & \text{if } \sum_{k=1}^{K} \lambda_k = 1 \end{cases}$$

where by $V_{\mathcal{G}}(\lambda)$ we denote the Fréchet variance of the group of clusters $\mathcal{G}$, is the set of weights (Lagrange multipliers) $\lambda = (\lambda_1, \ldots, \lambda_K) \in \mathbb{R}_+^K$, $\sum_{k=1}^{K} \lambda_k = 1$ is chosen. We will use the notation $\lambda \in \Delta^{K-1}$ to emphasize the fact that the Lagrange multipliers are positive and add to 1, hence playing the role of weights. Then, the dual problem becomes

$$\max_{\lambda \in \mathbb{R}_+^K} \mathbb{D}(\lambda) = \max_{\lambda \in \Delta^{K-1}} V_{\mathcal{G}}(\lambda) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2$$

which coincides with (6).

6.3 A feasibility condition given anchor opinion aversion preferences

We seek now for a condition under which the set of inequalities (5) has a non-empty feasible set. In this perspective, we need to seek for the minimum common radius $\epsilon > 0$ around all groups centers that should allow the problem to have feasible solutions. Therefore, we seek for the minimum $\epsilon > 0$ that all the equations are satisfied

$$d^2(x, \bar{x}_k) \leq \epsilon, \quad k = 1, ..., K \tag{24}$$

or equivalently, in terms of an optimization problem:

$$\begin{gathered} \min_{\epsilon \in \mathbb{R}_+, x \in M} \epsilon \\ \text{subject to} \\ d^2(x, \bar{x}_k) \leq \epsilon, \quad k = 1, \ldots, K \end{gathered} \tag{25}$$

The Lagrangian of the above problem is defined as

$$L(x, \epsilon; \lambda, \mu) = \left(1 - \sum_{k=1}^{K} \lambda_k\right) \epsilon - \mu\epsilon + \sum_{k=1}^{K} \lambda_k d^2(x, \bar{x}_k) \tag{26}$$

where $x \in M$, $\epsilon \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+^K$ and $\mu \in \mathbb{R}_+$. Following the same steps as above, we examine the dual problem. The related dual function defined as

$$\mathbb{D}(\lambda, \mu) = \min_{\epsilon \in \mathbb{R}_+} \left\{ \left(1 - \sum_{k=1}^{K} \lambda_k\right) \epsilon - \mu\epsilon \right\} + \min_{x \in M} \sum_{k=1}^{K} \lambda_k d^2(x, \bar{x}_k)$$

which due to linearity with respect to ϵ and upon inspection is obtained as

$$\mathbb{D}(\lambda, \mu) = \begin{cases} -\infty & \text{if } \sum_{k=1}^{K} \lambda_k \neq 1, \ \mu = 0 \\ \min_{x \in M} \sum_{k=1}^{K} \lambda_k d^2(x, \bar{x}_k) & \text{if } \sum_{k=1}^{K} \lambda_k = 1, \ \mu = 0 \end{cases}$$

containing the only interesting cases (i.e. where $\epsilon > 0$ and $\mu = 0$ otherwise the only feasible solution is when $\bar{x}_1 = \bar{x}_2 = ... = \bar{x}_K$). Then, this implies that the minimizer can be characterized in terms of the

Lagrange multipliers as

$$x^* = \arg\min_{x\in M} \sum_{k=1}^{K} \lambda_k d^2(x, \bar{x}_k) \iff x^* = Bar(\mathcal{G}, \lambda), \tag{27}$$

i.e. x^* is the Fréchet mean (barycenter) for the group of clusters $\mathcal{G}$ with the choice of weights $\lambda \in \Delta^{K-1}$. Then, the dual function becomes (omitting the uninteresting case where $\epsilon = 0$):

$$\mathbb{D}(\lambda, \mu) = \begin{cases} -\infty & \text{if } \sum_{k=1}^K \lambda_k \neq 1 \\ V_{\mathcal{G}}(\lambda) & \text{if } \sum_{k=1}^K \lambda_k = 1 \end{cases}$$

where by $V_{\mathcal{G}}(\lambda)$ we denote the Fréchet variance of the group of clusters $\mathcal{G}$, is the set of weights (Lagrange multipliers) $\lambda = (\lambda_1, \ldots, \lambda_K) \in \mathbb{R}_+^K$, $\sum_{k=1}^K \lambda_k = 1$ is chosen. The dual problem becomes

$$\max_{\lambda\in\mathbb{R}_+^K} \mathbb{D}(\lambda) = \max_{\lambda\in\Delta^{K-1}} V_{\mathcal{G}}(\lambda)$$

Then, the minimum $\epsilon > 0$ under which an agreement can be reached is chosen by the rule

$$\epsilon_* := \arg\max_{k\in\{1,2,\ldots,K\}} d^2(Bar(\mathcal{G}, \lambda_*), \bar{x}_k)$$

where $\lambda_* \in \Delta^{K-1}$ minimizes the nontrivial dual problem. In particular, $\epsilon_* > 0$ provides the non-empty set property, i.e. if $d^2(Bar(\mathcal{G}, \lambda_*), \bar{x}_k) \leq \epsilon_*$ for all $k = 1, 2, ..., K$ then there exist agreement points.

6.4 Details of the duality result in Example 2.8

In this case upon choosing a set of weights $\lambda \in \Delta^{K-1}$, the Fréchet barycenter of the group $\mathcal{G} = \{\bar{x}_1, \ldots, \bar{x}_K\} \subset \mathbb{R}^d$ reduces to the standard notion of the weighted average

$$x^*(\lambda) = x_{\mathcal{G}}(\lambda) = \sum_{k=1}^K \lambda_k \bar{x}_k,$$

whereas, the Fréchet variance reduces to

$$\begin{aligned} V_{\mathcal{G}}(\lambda) = \sum_{k=1}^K \lambda_k \|\bar{x}_k - x_{\mathcal{G}}(\lambda)\|^2 &= \sum_{k=1}^K \lambda_k \|\bar{x}_k\|^2 - \|x_{\mathcal{G}}(\lambda)\|^2 \\ &= \sum_{k=1}^K \lambda_k \|\bar{x}_k\|^2 - \sum_{k=1}^K \sum_{k'=1}^K \lambda_k \lambda_{k'} \langle \bar{x}_k, \bar{x}_{k'} \rangle \end{aligned}$$

where in the above we have already used the fact that $\lambda \in \Delta^{K-1}$. Hence, the dual function becomes

$$\mathbb{D}(\lambda) = \sum_{k=1}^K \lambda_k (\|\bar{x}_k\|^2 - \epsilon_k^2) - \sum_{k=1}^K \sum_{k'=1}^K \lambda_k \lambda_{k'} \langle \bar{x}_k, \bar{x}_{k'} \rangle.$$

The dual problem then becomes

$$\begin{aligned} \max_{\lambda_k\geq 0} \mathbb{D}(\lambda) = \max_{\lambda_k\geq 0} \sum_{k=1}^K \lambda_k (\|\bar{x}_k\|^2 - \epsilon_k^2) - \sum_{k=1}^K \sum_{k'=1}^K \lambda_k \lambda_{k'} \langle \bar{x}_k, \bar{x}_{k'} \rangle \\ \text{subject to} \sum_{k=1}^K \lambda_k = 1 \end{aligned}$$

which may by treated in terms of the Lagrangian function

$$\Lambda(\lambda, \nu) = \sum_{k=1}^K \lambda_k (\|\bar{x}_k\|^2 - \epsilon_k^2) - \sum_{k=1}^K \sum_{k'=1}^K \lambda_k \lambda_{k'} \langle \bar{x}_k, \bar{x}_{k'} \rangle + \nu(\sum_{k=1}^K \lambda_k - 1), \;\; \nu \in \mathbb{R}$$

This can be expressed in compact form by defining the Gram matrix $\mathbb{G} \in \mathbb{R}^{K\times K}$ and the vector $b \in \mathbb{R}^K$,

$$\mathbb{G} = (\langle \bar{x}_k, \bar{x}_{k'} \rangle\ )_{k,k'=1,\dots,K} \in \mathbb{R}^{K\times K},$$
$$b = (\|\bar{x}_k\|^2 - \epsilon_k^2)_{k=1,\dots,K} \in \mathbb{R}^K$$

so that the dual problem and its Lagrangian become

$$\max_{\lambda>0} -\langle \lambda, \mathbb{G}\lambda\rangle + \langle b, \lambda\rangle, \text{ subject to } \sum_{k=1}^K \lambda_k = 1,$$
$$\Lambda(\lambda,\nu) = -\langle \lambda, \mathbb{G}\lambda\rangle + \langle b, \lambda\rangle + \langle \nu\mathbf{1}, \lambda\rangle - \nu$$

Since the Gram matrix is a positive definite matrix the above problem is a well defined quadratic optimization problem that may be solved very easily to provide the optimal weights. Note that the optimal weights will be given in a parametric fashion (in terms of ν) by the solution of the first order conditions

$$2\mathbb{G}\lambda - b + \nu\mathbf{1} = 0$$

which indicates the dependence of the weight vector λ on the characteristics of the clusters (i.e. $\bar{x}_k$ and ϵ_k) in terms of

$$\lambda = \frac{1}{2}\mathbb{G}^{-1}(b - \nu\mathbf{1}),$$

where ν can be chosen by the condition

$$\frac{1}{2}\langle \mathbb{G}^{-1}b, \mathbf{1}\rangle - \frac{K}{2}\nu = 1 \iff \nu = \frac{1}{K}\langle \mathbb{G}^{-1}b, \mathbf{1}\rangle$$

The positivity of the λ could either be treated by projection on the positive cone of $\mathbb{R}^K$, or by imposing positivity in terms of a second set of multipliers $\mu \in \mathbb{R}^K$, using also the KKT conditions.

6.5 Details of the duality result in Example 2.9

If M is the space of probability measures on $\mathbb{R}$ endowed with the Wasserstein metric,

$$d(x_1,x_2)^2 = W_2^2(P_1,P_2) = \int_0^1 (q_1(s) - q_2(s))^2 ds,$$

where $x_i = P_i$, $i = 1,2$ are probability measures concerning the quantity of interest X and q_i, $i = 1,2$ are the quantile functions representing these measures. Then the Fréchet barycenter becomes the probability measure $x_{\mathcal{G}} = P_{\mathcal{G}}$ represented by the quantile average

$$q_{\mathcal{G}} = \sum_{k=1}^K \lambda_k q_k$$

and the Fréchet variance becomes

$$V_{\mathcal{G}}(\lambda) = \sum_{k=1}^K \lambda_k \int_0^1 |q_k(s) - q_{\mathcal{G}}(s)|^2 ds$$
$$= \sum_{k=1}^K \lambda_k \|q_k\|^2 - \sum_{k=1}^K \sum_{k'=1}^K \lambda_k \lambda_{k'} \langle q_k, q_{k'}\rangle$$

where in the specific context of this metric space we use the notation

$$\|q_k\|^2 = \int_0^1 |q_k(s)|^2 ds, \quad \langle q_k, q_{k'}\rangle = \int_0^1 q_k(s) q_{k'}(s) ds$$

Note that $\langle q_k, q_{k'} \rangle$ can be through of as the covariance between the probability measures $\bar{x}_k$ and $\bar{x}_{k'}$, representing the centres of the clusters k and k' respectively. Upon defining the Gram matrix $\mathbb{G}$ and the vector b by

$$\mathbb{G} = (\langle q_k, q_{k'} \rangle\ )_{k,k'=1,\dots,K} \in \mathbb{R}^{K\times K},$$
$$b = (\|q_k\|^2 - \epsilon_k^2)_{k=1,\dots,K} \in \mathbb{R}^K,$$

the dual problem and its Lagrangian become

$$\max_{\lambda>0} -\langle \lambda, \mathbb{G}\lambda \rangle + \langle b, \lambda \rangle, \text{ subject to } \sum_{k=1}^{K} \lambda_k = 1,$$
$$\Lambda(\lambda, \nu) = -\langle \lambda, \mathbb{G}\lambda \rangle + \langle b, \lambda \rangle + \langle \nu \mathbf{1}, \lambda \rangle - \nu$$

which is of the same form as the one in the previous example (eventhough the interpretation of the Gram matrix and the vector b is clearly different). From this point onwards the analysis of the previous example follows with the sole exception that now the Gram matrix $\mathbb{G}$ and the vector b are defined in terms of the relevant quantities involving the quantile functions q_k and the L^2 inner product (i.e. $\langle q_k, q_{k'} \rangle = \int_0^1 q_k(s) q_{k'}(s) ds$, and $\|q_k\|^2 = \int_0^1 |q_k(s)|^2 ds$).

6.6 Details of the duality result in Example 2.10

The barycenter of Gaussian measures in $\mathbb{R}^d$ is also a Gaussian measure. Hence, assuming that the barycenters of each cluster $\bar{x}_k$, comprising the set of clusters $\mathcal{G}$, are probability measures in $\mathcal{P}_N$ represented for each k by the pairs (m_k, S_k) (where $m_k \in \mathbb{R}^d$ is the mean vector and $S_k \in \mathbb{S}_+^d \subset \mathbb{R}^{d\times d}$ the positive definite covariance matrix) then the Fréchet mean of $\mathcal{G}$ is also a Gaussian probability measure parameterized by the pair $(m_{\mathcal{G}}, S_{\mathcal{G}})$ where

$$m_{\mathcal{G}} = m_{\mathcal{G}}(\lambda) = \sum_{k=1}^{K} \lambda_k m_k,$$
$$S_{\mathcal{G}} = S_{\mathcal{G}}(\lambda) \text{ solving } S_{\mathcal{G}} = \sum_{k=1}^{K} \lambda_k (S_{\mathcal{G}}^{1/2} S_k S_{\mathcal{G}}^{1/2})^{1/2},$$

with the notation $(m_{\mathcal{G}}(\lambda), S_{\mathcal{G}}(\lambda))$ used to emphasize the dependence of the parameters of the barycenter on the choice of λ). Unfortunately, here there is no explicit solution for the matrix equation providing $S_{\mathcal{G}}(\lambda)$, nor do we expect $S_{\mathcal{G}}(\lambda)$ to depend linearly on λ (or put better to be a convex combination of the S_k). The matrix equation

$$S_{\mathcal{G}} = \sum_{k=1}^{K} \lambda_k (S_{\mathcal{G}}^{1/2} S_k S_{\mathcal{G}}^{1/2})^{1/2}$$

can only be solved numerically. In fact, it may be approximated in terms of the fixed point scheme (see Álvarez-Esteban et al. (2016))

$$C_{n+1} = C_n^{-1/2} \left(\sum_{k=1}^{K} \lambda_k (C_n^{1/2} S_k C_n^{1/2})^{1/2} \right)^2 C_n^{-1/2}, \ \ n \in \mathbb{N}$$

where as $n \to \infty$ it holds that $C_n \to S_{\mathcal{G}}$. Having obtained the parameters for the Fréchet mean we may then obtain the value for the Fréchet variance of $\mathcal{G}$. This is the solution of the minimization problem

$$V_{\mathcal{G}}(\lambda) = \min_{m\in\mathbb{R}^d, S\in\mathbb{S}_+^d} \sum_{k=1}^{K} \lambda_k \left(\|m_k - m\|_2^2 + Tr(S + S_k - 2(S^{1/2} S_k S^{1/2})^{1/2}) \right)$$
$$= \sum_{k=1}^{K} \lambda_k \left(\|m_k - m_{\mathcal{G}}\|_2^2 + Tr(S_{\mathcal{G}} + S_k - 2(S_{\mathcal{G}}^{1/2} S_k S_{\mathcal{G}}^{1/2})^{1/2}) \right)$$

Eventhough we cannot obtain the explicit dependence of $V_{\mathcal{G}}(\lambda)$ on λ, we can easily show that $\lambda \mapsto V_{\mathcal{G}}(\lambda)$ is a concave function (as is true for the Fréchet function in general). Hence, the dual function

$$\mathbb{D}(\lambda) = V_{\mathcal{G}}(\lambda) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2,$$

is a concave function of λ and hence the dual problem,

$$\max_{\lambda \geq 0} \mathbb{D}(\lambda) = \max_{\lambda \geq 0} V_{\mathcal{G}}(\lambda) - \sum_{k=1}^{K} \lambda_k \epsilon_k^2$$

the solution of which will provide the choice of the weights in the barycenter is well posed. However, in this case it may only be solved numerically.

6.7 Details in Example 2.11

For this case, (9) reduces to the problem

$$\max_{x \in M} \sum_{k=1}^{K} \psi_k(\|x - \bar{x}_k\|^2)$$

with first order conditions (assuming differentiability of $\psi_k := \log \phi_k$)

$$\sum_{k=1}^{K} 2\psi_k'(\|x - \bar{x}_k\|^2)(x - \bar{x}_k) = 0.$$

Setting $\Lambda_k := 2\psi_k'(\|x - \bar{x}_k\|^2)$ the first order condition becomes

$$\sum_{k=1}^{K} \Lambda_k (x - \bar{x}_k) = 0$$

which immediately yields that

$$x = \sum_{k=1}^{K} \lambda_k \bar{x}_k, \quad \lambda_k = \frac{\Lambda_k}{\sum_{k'=1}^{K} \Lambda_{k'}}$$

The above shows that x is a barycenter, in $M = \mathbb{R}$, for an appropriate choice of weights $\lambda = (\lambda_1, \ldots, \lambda_K)$ with the λ_k given as above. The existence of the weights λ can be shown using Brouwer fixed point theorem, but the important observation here is that the opinion with the highest acceptance probability is a Fréchet mean with weights related to the derivatives of the functions ϕ_k which correspond to the rigidity of group k to its initial standpoint.

6.8 Proof of Proposition 2.13

Proof. We recall (see e.g. Bhatia et al. (2019)) that between two measures $P_i \sim N(\mu_i, S_i)$, $i = 1, 2$, the Wasserstein distance $W_2^2(P_1, P_2)$, admits the closed form

$$W_2^2(P_1, P_2) = \|\mu_1 - \mu_2\|^2 + Tr\Big(S_1 + S_2 - 2(S_1^{1/2} S_2 S_1^{1/2})^{1/2} \Big) \tag{28}$$

Moreover, given a set of probability measures $\mathbb{M}$ consisting of Gaussian measures P_i, $i = 1, \ldots, M$, and a weight vector $(w_1, \ldots, w_K)$, the corresponding Wasserstein barycenter P_B is a Gaussian measure $P_B \sim N(\mu, S)$ with $\mu = \sum_{i=1}^{K} w_i \mu_i$, and S being a matrix that satisfies the equation

$$0 = I - \sum_{k=1}^{K} w_k (S_k \# S^{-1}) \iff S = \sum_{k=1}^{K} w_k (S^{1/2} S_k S^{1/2})^{1/2}, \tag{29}$$

where the notation $A\#B$ is used to denote the geometric mean between two positive definite symmetric matrices given by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} = B\#A.$$

Without loss of generality we will assume that $\mu_k = 0$, $k = 1, \ldots, K$ (else simply center the measures). We will also consider problem (9) on $\mathcal{N}(\mathbb{R}^d) \subset \mathcal{P}(\mathbb{R}^d)$, the subset of Gaussian measures on $\mathbb{R}^d$. With the above information problem (8) (or equivalently of (9)) can be expressed as

$$\max_S \Psi(S) := \max_S \sum_{k=1}^{K} \psi_k \Big(Tr(S_k) + Tr(S - 2g_k(S)) \Big), \tag{30}$$

where $\psi_k = \ln \phi_k$ and $g_k(S) := (S_k^{1/2} S S_k^{1/2})^{1/2}$. Problem (30) is an optimization problem on the set of positive definite symmetric matrices. It can be treated by considering the Fréchet derivative of the functional in (30) with respect to S. Using the rules of Fréchet differentiation and assuming sufficient smoothness for the functions ψ_k we have that for any deviation $S + \epsilon Z$ from the matrix S the action of the Fréchet derivative $D\Psi(S)$ on any matrix Z yields

$$[D\Psi(S)]Z = \sum_{k=1}^{K} \psi_k'(W_k) Tr(Z - [Dg_k(S)]Z), \tag{31}$$

where we use the simplified notation

$$W_k = Tr(S_k) + Tr(S - 2g_k(S)).$$

Moreover, define the quantities

$$\Lambda_k = \psi_k'(W_k) \in \mathbb{R}_+,$$

where the positivity of Λ_k is guaranteed by the properties of the functions ψ_k. Following Bhatia et al. (2019), we can compute

$$Tr(Dg_k(S)Z) = Tr((S_k \# S^{-1})Z),$$

so that (31) yields (using the linearity of trace) that

$$[D\Psi(S)]Z = Tr[(\sum_{k=1}^{K} \Lambda_k)I - \sum_{k=1}^{K} \Lambda_k (S_k \# S^{-1}))Z]$$

The first order condition for the solution of (30) is $[D\Psi(S)]Z = 0$, for all possible perturbations Z of the covariance matrix S. Upon defining

$$w_k = \frac{\Lambda_k}{\sum_{j=1}^{K} \Lambda_j} \in [0, 1], \quad k = 1, \ldots, K,$$

the first order condition becomes

$$Tr[(I - \sum_{k=1}^{K} w_k (S_k \# S^{-1}))Z] = 0, \quad \forall\, Z,$$

which implies that the solution of(30) corresponds to a Gaussian measure with covariance matrix S such that

$$I - \sum_{k=1}^{K} w_k (S_k \# S^{-1}) = 0 \iff S = \sum_{k=1}^{K} w_k (S^{1/2} S_k S^{1/2})^{1/2}, \tag{32}$$

i.e. P^* is the barycenter of $\mathbb{M}$ with a selection of weights w_k, endogenously obtained by the preferences on the agents towards their anchor point (in other words their bargaining power).

Note that equation (32), although formally the same as equation (29) has a fundamental difference from (29). In (32) the coefficients $w_k = w_k(S)$, i.e. are depending on S, whereas in (29) the coefficients w_k are constants. It remains to show that equation (32) admits a solution. To show that we define the operator T, by $S \mapsto T(S) := \sum_{k=1}^{K} w_k(S^{1/2} S_k S^{1/2})^{1/2}$. It can be shown that this operator maps the closed convex set $\mathcal{K} = \{S \in \mathbb{R}_+^{d\times d} \mid c_1 I \leq S \leq c_2 I\}$, where $c_1, c_2 \geq 0$ and by $\leq$ we denote the natural ordering $S_1 \leq S_2 \iff S_1 - S_2 \geq 0$ (meaning $S_1 - S_2$ positive definite) onto itself. The set $\mathcal{K}$ is convex, and the map T is continuous, so by the Brouwer fixed point theorem T has a fixed point, therefore (32) admits a solution. □

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