DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

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Working Paper Series

24-02

January 2024

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January 16, 2024

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Abstract

This paper puts forth a growth model that takes into account the fact that the economy is embedded in a finite Earth. Economic activity causes greenhouse gas (GHG) emissions into the atmosphere and uses services from the biosphere. There are two main messages from the analysis: First, R&D in technologies that reduce GHG emissions and inputs from the biosphere must be ramped up rapidly. Second, in view of the fact that the top 10% of the world's inhabitants have roughly 76% of the world's wealth, consumption that causes emissions and uses inputs from the biosphere must decrease rapidly.

Keywords: growth, limits, biosphere, population dynamics, impact inequality, biosphere saving technology

JEL Classification: O44, J13, Q01

1 Introduction

The Dasgupta (2021) review is a comprehensive and in-depth review of the economics of biodiversity and ecosystem services which provides important insights about the current status of natural capital at a global level, and the policies which are necessary to reverse paths that have led ecosystems to their current conditions. In analyzing these issues, the Dasgupta review embeds the

*We would like to thank Partha Dasgupta for his very valuable comments on an earlier draft of this paper

economy in Nature.[1] This embedding causes an important issue to emerge, namely growth in a finite Earth, along with the accompanying issue of whether such growth has limits when embedded in a finite Earth.

A similar issue was raised by Solow (2009, page 1) when he pointed out that "It is possible that real demands on natural resources, and therefore on the natural environment, will be dramatically different in a world in which India and China, and other countries, too, grow at 8 or 10 percent a year, and need to pass through the material-goods-intensive phase of growth before they arrive at the service economy. The necessary process of (very material-intensive) urbanization is an outstanding example of what I mean. So it will probably be more important in the future to deal intellectually, quantitatively, as well as practically, with the mutual interdependence of economic growth, natural resource availability, and environmental constraints."

A central concept in the Dasgupta review (Dasgupta, 2021, Chapter 4; Dasgupta, 2022) is the impact inequality which compares the demand from the global economy for services provided by the biosphere or Nature,[2] to the supply of these services by Nature. The impact inequality can be written as

$$\frac{Py}{\alpha} > G(S). \tag{1}$$

In (1), the left-hand side represents global demand for Nature's services where P is the level of population, y is GDP per capita and α is an efficiency parameter reflecting the transformation of Nature's services to GDP, while the right-hand side represents the supply of Nature's services. In the impact inequality, S can be interpreted as natural capital which can be regenerated at a rate $G(\cdot)$. As pointed out in Dasgupta (2022, page 1021),

> "... the ratio of our demand for maintenance and regulating services (the left-hand side of inequality (1)) to Nature's ability to meet that demand on a *sustainable* basis (the right-hand side of inequality (1)) is today 1.7 ... The term "sustainable" is an all-important qualifier here, for it says that we are enjoying the overshoot at the expense of the health of the biosphere; that is, by depleting S. The number 1.7 is almost certainly an underestimate, which makes it even more a reason that inequality (1) be converted to an equality sooner rather than later. We are in a fire-fighting situation."

The supply side of the inequality can be increased by investing in Nature, which

[1] See also Dasgupta and Levin (2023).

[2] The terms biosphere and Nature are used interchangeably in the Dasgupta review. Biosphere is the part of the Earth that is occupied by living organisms. We will use these terms in the same way in this paper.

involves waiting for Nature to recover. The returns of such investment might be realized in slow time, or could even be zero for ecosystems that have been irreversibly damaged, or might be limited because of constraints imposed by physical processes. Furthermore, even if the biosphere regenerates, supported by environmental policies, the regeneration rate is expected to be bounded because of natural limits. Thus, conversion of the impact inequality to equality requires controlling the demand side, that is, controlling the evolution of population P, GDP per capita y, and "biosphere saving" technical progress α, which implies controlling the growth of P, y and α. Restricting the demand side to restore and maintain equality between demand and supply relates to a fundamental question with regard to economic growth, which seeks to explore whether there are limits to the growth of GDP per capita given the bounds imposed by Nature.[3]

The view that the economy is embedded in a finite planet and that there are "limits to growth" is not new. It has been put forward by writers such as Kenneth Boulding (1966), Barbara Ward (1966) and Herman Daly (1974), who stressed the point that Earth is not an unlimited reservoir for extraction or pollution. Similar ideas are embodied in the concept of planetary boundaries (Rockström et al., 2009) defining safe operating space for the earth system which, if transgressed, could generate unacceptable environmental change.

In this paper we develop a growth model in which growth is embedded in a finite Earth. We explore limits imposed by the impact inequality and conditions for their relaxation in the context of a welfare-maximizing, Ramsey-type social planner who is embedded in Nature S. In our model we consider a linear production function embedded in Nature that provides services which are consumed during the production processes. The planner faces a welfare cost if the demand on Nature's services exceeds the supply. In addition, output production generates climate damages. Output increases with population, so the model is AP rather than AK. A can be regarded as the level of per capita income, y, but it can also be interpreted as a productivity parameter driven by technical change (TFP). Thus, AP is aggregate output that generates demand for the services of the biosphere equal to $\frac{AP}{\alpha}$. Population evolves endogenously and population dynamics follow Galor (2012) by assuming that raising children is time intensive and requires a certain proportion of the parental unit-time endowment.[4] The rest of the time endowment of a representative individual is allocated to raising the educational level of children, technical change that

[3]See, for example, the special issue of *The Review of Economics Studies* (1974), the Daly vs Solow–Stiglitz controversy in the special issue of *Ecological Economics* (1997) or Stokey (1998).

[4]Population dynamics and the concept of optimum population have been analyzed by Partha Dasgupta (see, for example, Dasgupta, 1969, 1995, 2019 and Dasgupta et al., 2021). We formulate population dynamics in a way that can be regarded as complementary to this literature. This approach also fits our optimizing framework.

increases TFP associated with a linear production function, and "green R&D" or biosphere saving technical change that can reduce the stress of human demand on Nature's services.

In our main results, we provide a general economic growth model constrained by the biosphere of a finite Earth where population, time allocation divided among construction of GDP, work, consumption, R&D, child rearing and child educating, as well as the size of the biosphere are all endogenous. The level of education, called g, needed for children to be successful plus the time cost to parents of educating each child to the target level g is a major force for achieving sustainable population levels. The scaling, $R = rA$, of the level of biosphere saving technology and CO2 emission saving technology denoted by R, with A, is a major force in potentially turning the impact inequality into an equality if $r > 1$. Higher A and higher r lead to less pressure on our planet from the economy by raising R under the scaling. Besides these major take-home points, we provide quasi-analytical results for several special cases that illustrate the power of our unified framework in delivering economic insights even though our model has many moving parts.

The rest of the paper is organized as follows. Section 1 provides the introduction. Section 2 sets forth a general abstract growth model which features optimal population, optimal allocation of time across building GDP, work time, consumption time, time spent on R&D in emission reduction technology, and technology that reduces the impact on the biosphere from producing GDP as well as emissions, and time spent on child rearing and education of children. This general framework is used to organize analysis of the special cases that follow in Sections 3, 4, and 5. Section 3 treats a special case where GDP and R&D are fixed constants and time is allocated between consumption, and child rearing and child education. The general level of background education, g, needed to function well in the economy, is exogenously fixed in the Section 3 model. The higher g is, the more costly it is in terms of time to raise a child to level g. Thus, higher g reduces the number of children and, hence, leads to a smaller population and less emissions and less impact on the biosphere, all other things equal. In Section 3 we show that if the level of R&D scales with factor r with GDP, and if $r > 1$, then $r > 1$ relaxes the impact inequality and allows a larger GDP and a larger population without violating the impact inequality. Optimal dynamics and optimal steady states are studied in Section 3. Section 4 adds work time, time building level of GDP, and R&D time. Section 4 computes quasi-closed form solutions for optimal steady states. Section 5 studies a variation of the general model where there are two types of consumption: type 1 is modest in impact on the biosphere and emissions, while type 2 has very large emissions and a very large impact on the biosphere. Section 6 endogenizes the biosphere dynamics whereas previous sections fixed the biosphere at a constant to reduce the number of state variables. Fixing the size of the biosphere could be viewed as "loading the analysis" in favor of economic growth by ignoring the reduction in size of the biosphere due to pressure from economic activity. Evidence of a rapid fall in the cost of wind and solar power in the production of energy is suggestive of the potential fall in the cost of reducing the impact on the biosphere. We take the standard arguments as given, that subsidies both explicit and implicit for activities harmful to our planet (e.g., fossil fuel subsidies), subsidies for activities that degrade the biosphere, and maybe even subsidies for large families, should be eliminated.

Our paper offers a unified framework where population, earth system, and time allocation across many activities with different impacts on the earth system are all endogenous. This kind of model provides a useful framework for policy analysis, e.g., it helps expose unintended consequences to other parts of an economy when a policy targets one part of the economy. Humanity's current ecological footprint is more than one Earth can sustain. Our model helps locate sufficient conditions for a favorable outcome of the "race" to reduce humanity's impact on the biosphere fast enough to overcome the current speed of reduction of the biosphere's capacity to continue delivering the essential services needed for a good life, for both current and future generations. Section 7 closes the paper with a summary, conclusions, and suggestions for future research.

2 A unified model and sustainable development

Consider a representative individual with a unitary time endowment. This endowment is allocated to time devoted: to working; to leisure and consumption time for goods produced by work time, t_C; to raising and educating children, $\tau = \tau^q + \tau^e e$, where τ^q, τ^e, is time devoted to raising and educating children, respectively, and e is the education level; to research, t_R , that increases the stock of knowledge R, which is the green or biosphere saving R&D; and to the time, t_A, devoted to promoting technical change that increases work time productivity, TFP. Let C and P denote the total consumption and the level of population at time t respectively,[5] and let n denote the number of surviving children of the representative individual. Then the flow budget constraint of the representative individual is defined in terms of per capita consumption which is consumed during the leisure-consumption time as

$$\frac{C}{P} = t_C = 1 - t_w - n\tau - t_R - t_A. \tag{2}$$

[5] All the state and control variables have a time dimension. We omitted the time argument t to ease notation except where it is necessary for clarifications.

Let T denote the temperature anomaly and A the labor productivity parameter associated with a linear production function that produces the consumption good which is written as AP. Choosing units so that production denotes also GHG emissions, the dynamics for our model economy can be written as

$$\dot{P} = nP - \delta_P P\,,\; P(0) = P_0 \tag{3}$$

$$\dot{T} = \frac{AP}{\beta\alpha(R)} - \delta_T T\,,\; T(0) = 0 \tag{4}$$

$$\dot{A} = \beta_A P t_A - \delta_A A\,,\; A(0)\,given \tag{5}$$

$$\dot{R} = \beta_R P t_R - \delta_R R\,,\; R(0)\,given \tag{6}$$

$$\dot{S} = G(S) - \frac{AP}{\alpha(R)}\,,\; S(0) = S_0, \tag{7}$$

where $\delta_j\,, j = P, T, A, R$ denote appropriate depreciation rates and $\alpha(R)$ captures the effect of green R&D in reducing the impact of the economy on Nature, while $AP/\left(\beta\alpha(R)\right)$ captures the impact of knowledge in reducing GHG emissions. The underlying assumption for (5) and (6) is that each agent is accumulating the technology capital to produce A and R so that the total technology capital of each type in the world economy is obtained by multiplying by P. Utility flow is determined by per capita consumption, the ecosystem services provided by the biosphere and damages from climate change, and is defined as

$$\mathcal{U}(P, C, S, T) = U\left(\frac{C}{P}, S\right) - D(T),$$

where $D(T)$ is an increasing convex function representing damages from climate change. Extending Galor's (2012) approach, we define $U(\cdot, \cdot)$ as

$$U\left(\frac{C}{P}, S\right) = (1-\gamma)\ln\left[(At_w)\left(1 - t_w - n\tau - t_R - t_A\right)\right] + \gamma\left[\ln n + \beta h\left(g, e\right)\right] + \\ + B\left(S, \frac{AP}{\alpha(R)}\right), \tag{8}$$

where $(1-\gamma)\ln\left[(At_w)\, t_C\right]$ corresponds to the utility of consumption time of the goods produced by work time, with $\tau = \tau^q + \tau^e e$; $\gamma\left[\ln n + \beta h\left(g, e\right)\right]$ is the utility corresponding to the number of children n, with the level of education e chosen by the parent at a time cost τ^e, where $h\left(g, e\right)$ is increasing strictly concave in the education level and decreasing strictly convex at the level of education needed to function well in the economy, g, motivated by Galor (2012). Finally, $B(S, \frac{AP}{\alpha(R)})$ is a function reflecting welfare benefits or costs associated with excess supply or demand on Nature's services over the stress term $\frac{AP}{\alpha(R)}$. Then the social planner's problem can be defined as

$$\max_{\mathbf{x}} \int_{s=t}^{\infty} \mathrm{e}^{-\rho(\mathrm{s-t})} \left[PU\left(\frac{C}{P}, S\right) - D(T) \right] dt, \tag{9}$$

where $\mathbf{x} = (n, e, t_w, t_A, t_R)$ is the vector of controls that will determine the optimal number of children, n, and the optimal allocation of time to work t_w, TFP related technical change t_A, and R&D accumulation t_R. The vector of the state variables is $\mathbf{z} = (P, T, A, R, S)$. The maximization problem (9) is subject to constraints (3)–(7).

Let $\boldsymbol{\lambda} = (\lambda_P, \lambda_T, \lambda_A, \lambda_R, \lambda_S)$ be the vector of the the costate variables associated with the state variables. Then the current value Hamiltonian can be written as

$$\mathcal{H} = PU\left(\frac{C}{P}, S\right) - D(T) + \boldsymbol{\lambda} \cdot \mathbf{z}. \tag{10}$$

Pontryagin's maximum principle implies that the optimal controls are determined by

$$\frac{\partial \mathcal{H}}{\partial \mathbf{x}} \leq 0, \ \mathbf{u}^* \geq \mathbf{0}. \tag{11}$$

The optimality conditions characterizing the paths of the costate variables, which are interpreted as the shadow values of the state variables – population, temperature, TFP, green R&D and biosphere – along the socially optimal path, evaluated at the optimal controls defined by (11), are

$$\dot{\boldsymbol{\lambda}} = \rho \boldsymbol{\lambda} - \frac{\partial \mathcal{H}}{\partial \mathbf{z}}. \tag{12}$$

The Hamiltonian system is completed by (3)–(7), also evaluated at the optimal controls defined by (11). The optimality conditions characterize the solution of the social optimum and can provide some useful insights regarding sustainability optimization. Following Arrow et al. (2012), a definition of sustainable growth and development along the socially optimal path can be provided. Define the value function of problem (9) as

$$V(P_t, T_t, A_t, R_t, S_t) = \max \left\{ \int_{s=t}^{\infty} \mathrm{e}^{-\rho(\mathrm{s-t})} \left[PU\left(\frac{C}{P}, S\right) - D(T) \right] ds \right\}, \tag{13}$$

subject to (3)–(7).

Definition (sustainable growth and development): *We say that sustainable growth and development holds if*

$$\frac{dV}{dt} \geq 0 \, for \, all \, dates \, t \geq 0.$$

Totally differentiating (13), we obtain

$$\frac{dV}{dt} = \frac{dV(P_t, T_t, A_t, R_t, S_t)}{dt} = \frac{\partial V}{\partial P}\dot{P} + \frac{\partial V}{\partial T}\dot{T} + \frac{\partial V}{\partial A}\dot{A} + \frac{\partial V}{\partial R}\dot{R} + \frac{\partial V}{\partial S}\dot{S}.$$
$$= \lambda_p \dot{P} + \lambda_T \dot{T} + \lambda_A \dot{A} + \lambda_R \dot{R} + \lambda_S \dot{S} \tag{14}$$

The partial derivatives of the value function are the costate variables of the Hamiltonian system and have the interpretation of the accounting or shadow prices of the state variables, while the time derivatives of the state variables can be interpreted as the corresponding investments. Thus (14) can be interpreted as genuine or comprehensive savings along the optimal path. Development is sustainable, therefore, if the social welfare maximizing choice of time allocated to production, raising and educating children, and promoting productivity and knowledge accumulation results in non-declining comprehensive wealth or genuine savings (Arrow et al., 2014). This definition relates optimal population dynamics to sustainability. Note that in (14), the shadow value or the accounting price for the population is derived endogenously through the optimizing framework of (10).[6]

The optimality conditions characterize the solution of the problem; their complexity, however, does not allow tractability. Therefore, in the following sections we examine certain special cases which can provide useful insights regarding optimal population dynamics and growth embedded in finite Earth.

Before we do that, we point out in more detail how our model and approach relate to that of the Dasgupta review (Dasgupta, 2021). The unified model described by equations (2)–(9) and the models emerging after alternative simplifications which are discussed in the rest of this paper describe different versions of optimal growth models embedded in Nature. This embedding follows the central idea of the prototype model developed in detail in the Dasgupta review and in particular chapter 4*, which describes the structure of the model, and chapter 13*, which provides the optimality conditions and explores population evolution in the long run based on logistic growth. One of the basic conclusions of the Dasgupta (2021) model is that the efficiency parameter α of the impact inequality has an upper bound (see chapter 4*, pages 140–141). This result suggests that the economy cannot free itself from the bounded biosphere.

Our approach, which also embeds growth in a finite biosphere, takes a different modeling approach relative to the Dasgupta review. Using Galor's (2012) modelling, we associate the accumulation of TFP (A in our notation) and the biosphere saving R&D (R in our notation) with the optimal allocation time available to an individual, along with optimal time allocation for work, consumption

[6]See Arrow et al. (2003, 2014) for different approaches to dealing with population in the context of sustainability criteria.

and children rearing. Furthermore, we introduce damages from climate change and this models an important anthropogenic impact on the biosphere, along with a social cost in case the demand for nature's services exceeds the available supply. Having obtained the optimality conditions, we provide quasi-tractable solutions that could support a realistic calibration of the model. By assuming that the efficiency parameter of the Dasgupta review can be written as $\alpha = \alpha R$ and then scaling R with TFP growth as $R = rA$, we provide conditions under which the restrictions on growth by the finite biosphere can be relaxed. If the ratio R/A can be sustained forever at values greater that one as the economy grows, then the denominator on the demand side of the impact inequality will increase and it is possible to have growth without violating the constraints imposed by the biosphere.

The important question, however, is whether this is feasible. In a private communication, Partha Dasgupta put forward the argument that "to imagine that α can be made as large as we like so long as we invest enough in R&D/institutions is to imagine that no matter how large the economy, further enlargement will make vanishingly small demands on the biosphere, which is to say, the economy can free itself of the biosphere in the limit". Although a straightforward answer to this point might not be clear, the argument itself makes clear that the economy cannot untangle itself from the limits imposed by the bounded biosphere.

3 Slow population dynamics and the impact inequality

3.1 No work time: the "manna from heaven model"

We assume that the dynamics for climate, technical progress, and R&D accumulation converge in fast time to their steady states, while the evolution of population is driven by slower moving cultural as well as economic drivers. Furthermore, we consider Nature, S, as given, so that the flow of services supplied by Nature and denoted by $G(S)$ is also fixed. We assume that work time is equal to zero and that A and R are fixed and treated as "manna from heaven" so that time t_R which increases the stock of R&D R, and time t_A devoted to promoting technical change which increases work time productivity, are set equal to zero. We treat A and R as fixed and remove from the analysis (t_w, t_A, t_R) for simplicity. We do this exercise as an initial foray to gain insight into the economic forces that may ease limits on economic growth before tackling the more complicated model. We further assume that damages from climate change, T, are linear; that if the demand for nature services exceeds supply the social planner's utility is reduced, and that $\alpha(R) = \alpha R$, $A = y$, where y is the income of the representative household so A can also be regarded as determining the level of output.

The dynamics of the problem, setting $\beta = 1$ to simplify, become

$$\dot{P} = P(n - \delta_P) \tag{15}$$

$$\dot{T} = 0 \Rightarrow T = \frac{yP}{\delta_T \alpha R} \tag{16}$$

$$A = y \tag{17}$$

$$\text{Impact} : G(S) - \frac{yP}{\alpha R}. \tag{18}$$

The utility associated with the number of children and their education is represented in (8) by the term $\gamma[\ln n + \beta h(g, e)]$. We simplify this term to $\gamma \ln(yn\tau)$ by using the simplifying assumption $h(e, g) = \min\{e, g\} = g$. The underlying assumption is that, by interpreting g as the level of education needed to function well in the economy, we let g "pull up" the demand for education e via a Leontief function. Then $\tau = \tau^q + e\tau^e = \tau^q + g\tau^e$. Thus $n\tau = n(\tau^q + g\tau^e)$ is the total time spent in raising and educating children n with y being its opportunity cost in dollars per unit of time. Therefore $yn\tau$ is dollars of consumption spent in the form of children and their education. This assumption tries to capture the role of g in pulling up the demand for education, i.e., human capital, for each child, thus increasing τ which, in turn, raises the "price" of each child.

We assume that when the demand for Nature's services exceeds the supply, the planner's utility is reduced and the planner's Hamiltonian can be written as

$$\mathcal{H} = P\left[(1-\gamma)\ln(yt_C) + \gamma \ln(yn\tau) - D\frac{yP}{\delta_T \alpha R} + B\left(G(S) - \frac{yP}{\alpha R}\right)\right] + \lambda_P P(n - \delta_P), \tag{19}$$

where $\tau = \tau^q + g\tau^e$. The term $B\left(G(S) - \frac{yP_t}{\alpha R}\right)$, $B > 0$, indicates that a negative impact inequality will penalize the planner's objective, while $1 = t_C + n\tau$. The optimality condition for the choice of the number of children n derived from Pontryagin's maximum principle is

$$n^* : \frac{(1-\gamma)\tau}{1 - n\tau} = \frac{\gamma}{n} + \lambda_P, \tag{20}$$

while the Hamiltonian dynamic system is

$$\dot{P} = P\left(n^{*} - \delta_{p}\right) = H_{\lambda_P} \tag{21}$$

$$\dot{\lambda}_P = \rho\lambda_P - H_P(n^{*}, \lambda_P), \tag{22}$$

with n^{*} the solution for n of the optimality condition (20) which is obtained as

$$n^{*} = \frac{\lambda_P - \tau \pm \sqrt{\lambda_p^2 - 2\lambda_P\tau + 4\gamma\lambda_p\tau + \tau^2}}{2\lambda_P\tau}. \tag{23}$$

As is shown below, the correct choice between plus or minus the square root is the solution corresponding to the negative square root.

The optimal steady state (OSS) is obtained as the solution of $\left(\bar{P}, \bar{\lambda}_P\right)$ of (21) and (22) for $\dot{P} = 0, \dot{\lambda}_P = 0$. We can gain more insight into the structure of the OSS by considering the limiting case $\gamma = 1$ in which all preference weight is placed on children and none on consumption. In this case, from (20) and (21) we obtain

$$n^{*} = \frac{-1}{\lambda_P},\ \bar{\lambda}_P = \frac{-1}{\delta_P}, \tag{24}$$

while from (22),

$$\dot{\lambda}_P = \rho\lambda_P - \ln\left(\delta_P\tau y\right) - BG\left(S\right) + 2AP\left[\frac{BR}{\alpha} + \frac{\alpha DR}{\delta_T}\right]. \tag{25}$$

Then the following results are obtained:

1. For $\gamma = 1$, the shadow price λ_P of an extra person is negative from (24). This is to be expected since there is no preference weight on consumption.

2. When $\gamma = 1$, the negative square root of (23) results in $n^{*} = \frac{-1}{\lambda_P}$.

3. When $\gamma = 0$, $\bar{\lambda}_P = \frac{\tau}{(1-\delta_P\tau)} > 0$. When all preference weight is on consumption, the shadow price of an extra person is positive. Therefore, there is a critical value $\gamma_c \in (0,1)$ where $\lambda_P\left(\gamma_c\right) = 0$ and $\gamma > \gamma_c$ implies $\lambda_P\left(\gamma\right) < 0$, while $\gamma < \gamma_c$ implies $\lambda_P\left(\gamma\right) > 0$. Thus the shadow value for the population, which is relevant for sustainability assessment, depends on the preference weight placed on children.

4. Solving (20) for λ_P at the OSS where $n^{*} = \delta_p$, we obtain $\bar{\lambda}_P = \frac{\gamma - \delta_P\tau}{\delta_P(-1+\delta_P\tau)}$.

Proposition 1: *For $\gamma = 1$, the OSS is a saddle point. The stable manifold has a positive slope.*

Proof: The isocline $\dot{P} = 0$ is given by $P = 0$ and $\lambda_P = -1/\delta_P$, with $n^{*} = \delta_P$

at the OSS. The isocline for $\dot{\lambda}_P = 0$ is given by (25) as

$$0 = \rho\lambda_P - \ln(\delta_P \tau y) + 2Py\left(\frac{D}{\alpha\delta_T R} + \frac{B}{\alpha R}\right) - BG(S). \tag{26}$$

The Jacobian matrix at the OSS is

$$H(P, \lambda_P) = \begin{pmatrix} 0 & \dfrac{P}{\lambda_P^2} \\ \dfrac{2By}{\alpha R} + \dfrac{2Dy}{\alpha\delta_T R} & \rho \end{pmatrix},$$

with positive trace and negative determinant. Therefore the steady state is a saddle point. From (26), $(d\lambda_P/dP) < 0$. The OSS is depicted in Figure 1.■

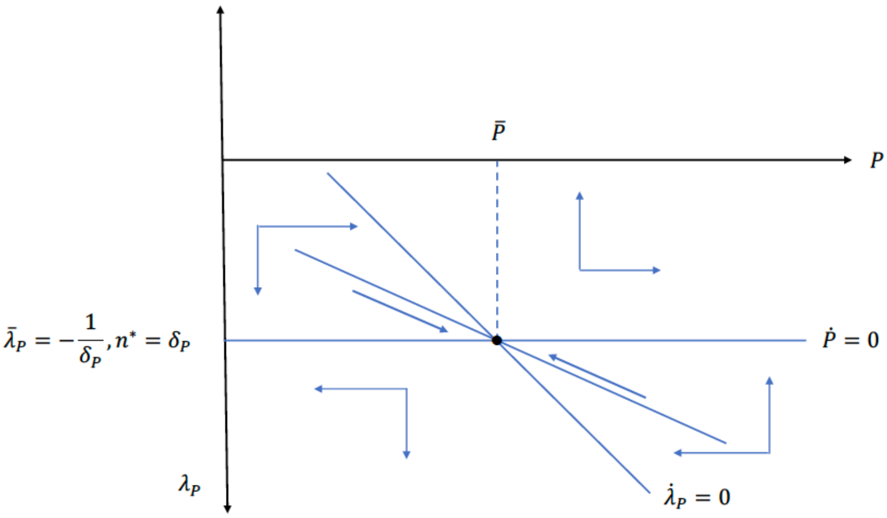

Figure 1. Saddle point OSS for $\gamma = 1$.

To obtain some insights into the potential limits constraining output augmentation, assume that R&D scales with the productivity parameter A as $R = rA$, with $y = A$ and AP interpreted as a linear production function. Then, at the OSS with $\gamma = 1$,

$$\bar{P} = \frac{\alpha r \delta_T \left[\frac{\rho}{\delta_P} + \ln(\delta_P \tau A) + BG(S)\right]}{2(D + \delta_T B)}. \tag{27}$$

For $\gamma \in (0, 1)$, the isocline for $\dot{\lambda}_P = 0$ is given, with $y = A$, by

$$0 = \rho\lambda_P - (1-\gamma)\ln(1 - \delta_P \tau A) - \gamma \ln(\delta_P \tau A) + 2PA\left(\frac{D}{\alpha\delta_T R} + \frac{B}{\alpha R}\right) - BG(S).$$

Assuming again that $R = rA$, we obtain for the population at the OSS

$$\bar{P} = \frac{\alpha r \delta_T \left[\frac{\rho}{\delta_P} + (1-\gamma)\ln\left(A(1-\delta_P\tau)\right) + \gamma \ln\left(\delta_P \tau A\right) + BG(S)\right]}{2\left(D + \delta_T B\right)}. \tag{28}$$

In (27), steady-state population increases with the logarithm of A. Thus the scaling of R&D biosphere saving R, with A, can ease demand on ecosystem services. Thus, $R = rA$ allows us to exceed the limits that constrain P while consumption is maintained at the per capita level. Indeed, if A increases, P can increase with $\ln A$ and thus aggregate output increases. Our simple example suggests the need to study whether the productivity of "green R&D", which relieves stress on the Earth System from population and consumption per capita, can increase at the same rate as total factor productivity in goods and services production. Increase in green R&D can be associated with evidence that renewable energy costs have fallen, as described by Moore's Law and Wright's Law,[7] while costs of fossil fuels have remained roughly constant (Way et al., 2022). The fall of renewable energy costs can ease the human stress on the Earth System. If we take $B \to \infty$, so that the utility cost of excess demand on ecosystem services is very high, then by l' Hôspital's rule,

$$\bar{P} \to \frac{\alpha r G(S)}{2}. \tag{29}$$

Proposition 2: *Conditions (27)–(29) indicate that if A or r tend to infinity, then the steady-state optimal population tends to infinity.*

While these conditions are unrealistic, they do show pathways to relaxation of constraints on aggregate output that can ease the stress on limits to growth in the "manna from heaven" model.

If $\gamma = 0$, then

$$\mathcal{H} = P_t \left[\ln\left(y t_C\right) - D\frac{yP_t}{\delta_T \alpha R} + B\left(G(S) - \frac{yP_t}{\alpha R}\right)\right] + \lambda_P P_t\left(n - \delta_P\right) \tag{30}$$

$$y = A,\ 1 = t_C. \tag{31}$$

In this case, $n = 0$ and the population P goes to zero.

[7] Moore's law says that costs drop exponentially as a function of time (i.e., at a fixed percentage per year), while Wright's law predicts that costs decline as a power law of cumulative production.

3.2 Spending time working, consuming and raising children

We assume now that the representative individual spends time working, consuming and raising children. This implies that:

$$y = t_w A,\ 1 = t_C + t_w + n\tau,\ \tau = \tau^q + \tau^e g,$$

with the associated Hamiltonian system

$$\mathcal{H} = \max P\left[(1-\gamma)\ln(yt_C) + \gamma\ln(yn\tau) - D\frac{yP_t}{\delta_T \alpha R} + B\left(G(S) - \frac{yP}{\alpha R}\right)\right] + \lambda_P P(n - \delta_P) \tag{32}$$

$$\dot{P} = P(n - \delta_P) = \mathcal{H}_{\lambda_P} \tag{33}$$

$$\dot{\lambda}_P = \rho\lambda_P - \mathcal{H}_P. \tag{34}$$

3.2.1 The Ramsey OSS, $\rho = 0$

At the Ramsey OSS (i.e., an OSS for $\rho = 0$), we have $n = \delta_P$, $\mathcal{H}_P = 0$. The first-order condition (FOC) for t_w is $\mathcal{H}_{t_w} = 0$ which results, after some algebra, in

$$\frac{(1-\gamma)(1-2t_w-\delta_P\tau)}{t_w(1-t_w-\delta_P\tau)} + \frac{\gamma}{t_w} = \left(\frac{AP}{\alpha R}\right)\left(\frac{D}{\delta_T} + B\right) \equiv kP \tag{35}$$

$$k \equiv \left(\frac{A}{\alpha R}\right)\left(\frac{D}{\delta_T} + B\right).$$

We multiply both sides of (35) by t_w and do some algebra to obtain

$$1 - \delta_P\tau + t_w(\gamma - 2) = t_w(1 - t_w - \delta_P\tau)kP \Rightarrow$$

$$Pk(t_w)^2 - t_w\left[Pk(1-\delta_P\tau) + 2 - \gamma\right] + 1 - \delta_P\tau = 0. \tag{36}$$

The solution of the quadratic (36) will provide two solutions for the optimal t_w as functions of population P.

We make the following intuitively reasonable assumption:

A1: $1 - \delta_P\tau > 0$**.**

Then, at $t_w = 0$, the quadratic (36) is positive with a negative slope, $-\left[Pk(1-\delta_P\tau) + 2 - \gamma\right]$, for $P > 0$. Notice that $1 - \delta_P\tau = t_C + t_w$ and that, as $1 - \delta_P\tau \to 0$, the smallest root of the quadratic (36) tends to zero. This suggests that under A1 there exist two positive solutions for the optimal working time $t_w(P)$. The acceptable solution will be the one that satisfies $\mathcal{H}_{t_w t_w} < 0$.

Define $j \equiv 1 - \delta_P \tau$, then the roots of (36) are:

$$t_w = \frac{Pkj + 2 - \gamma \pm D^{1/2}}{2Pk}$$
$$D \equiv (Pkj + 2 - \gamma)^2 - 4Pkj = (Pkj)^2 - 2\gamma Pkj + (2 - \gamma)^2 .$$

For $\gamma = 1$,the negative root gives:

$$t_w = \frac{1}{Pk},$$

while for $\gamma = 0$,

$$t_w = \frac{Pkj + 2 \pm D^{1/2}}{2Pk}$$
$$D = (Pkj)^2 + (2)^2 .$$

Using the FOC for the optimal population level, we obtain

$$\mathcal{H}_P = 0 \Rightarrow (1 - \gamma) \ln (t_w A t_C) + \gamma \ln (t_w A n \tau) + BG(S) - \frac{2 t_w AP}{\alpha R} \left(\frac{D}{\delta_T} + B \right) = 0.$$

The optimal population for the Ramsey problem can be obtained by substituting the solution for $t_w(P)$ and checking that $\mathcal{H}_{PP} < 0$.

3.2.2 The optimal steady state

The Hamiltonian for this problem is

$$\mathcal{H} = \max P \left[(1 - \gamma) \ln (y t_C) + \gamma \ln (y n \tau) - D \frac{yP}{\delta_T \alpha R} + B \left(G(S) - \frac{y P_t}{\alpha R} \right) \right] + \lambda_P P (n - \delta_P),$$
$$y = t_w A, \ 1 = t_C + t_w + n\tau, \ \tau = \tau^q + \tau^e g,$$

with FOC for the controls n, t_w, and a corresponding Hamiltonian dynamic system:

$$\begin{aligned}
n &: \frac{-(1-\gamma)\tau}{(1-t_w-\delta_P\tau)} + \frac{\gamma}{n} + \lambda_P = 0 \\
t_w &: \frac{(1-\gamma)(1-2t_w-\delta_P\tau)}{(1-t_w-\delta_P\tau)} + \frac{\gamma}{t_w} - \frac{DAP}{\delta_T\alpha R} - \frac{BAP}{\alpha R} = 0 \\
&\dot{P} = P(n-\delta_P) = \mathcal{H}_{\lambda_P} = 0 \Rightarrow \bar{n} = \delta_P \, or \, \bar{P} = 0 \\
&\dot{\lambda}_P = \rho\lambda_P - \mathcal{H}_P \Rightarrow \\
&\dot{\lambda}_P = \rho\lambda_P + (1-\gamma)\ln(A(1-t_w-\delta_P\tau)) + \gamma\ln(At_w) + \gamma\ln(\delta_P\tau) - \frac{2AP}{\alpha R}\left(\frac{D}{\delta_T}+B\right).
\end{aligned}$$

For OSS we have, for $0 < \gamma < 1$, $\bar{n} = \delta_p$ and $\bar{P} > 0$, three remaining unknowns – $\left(\bar{\tau}^w, \bar{\lambda}_P, \bar{P}\right)$ – which can be determined by solving the three steady-state equations for these three unknowns:

$$t_w : \frac{(1-\gamma)(1-2t_w-\delta_P\tau)}{(1-t_w-\delta_P\tau)} + \frac{\gamma}{t_w} = kP \tag{37}$$

$$k \equiv \frac{A}{\alpha R}\left(\frac{D}{\delta_T}+B\right)$$

$$n : \frac{-(1-\gamma)\tau}{(1-t_w-\delta_P\tau)} + \frac{\gamma}{n} + \lambda_P = 0 \tag{38}$$

$$\rho\lambda_P - \{(1-\gamma)\ln(A(1-t_w-\delta_P\tau)) + \gamma\ln(At_w) + \gamma\ln(\delta_P\tau) - 2kP\} = 0. \tag{39}$$

In principle, the system can be solved for the three unknowns by using (37) and (38) to obtain $P = \zeta(t_w)$ and $\lambda_P = \xi(t_w)$ respectively, and then substituting into (39) and solving for t_w.

To obtain closed form solutions, we consider the two polar cases of $\gamma = 1$ and $\gamma = 0$.

Special case $\gamma = 1$

For $\gamma = 1$, we have

$$t_w = \frac{1}{kP} \tag{40}$$

$$\lambda_P = \frac{-1}{\delta_p} \tag{41}$$

$$\frac{\rho}{\delta_P} + \ln A - \ln k + \ln(\delta_P\tau) = 2kP + \ln P. \tag{42}$$

Proposition 3: *For $\gamma = 1$, the OSS is a saddle point. The stable manifold has a positive slope.*

Proof: The isocline $\dot{P} = 0$ is given, for $\dot{P} = P(n - \delta_P)$, by $P = 0$ or $\bar{n} = \delta_P$ at the OSS. Using (41), we obtain $\lambda_P = -1/\delta_p$. Then, for the isocline $\dot{P} = 0$, $n = -1/\lambda_P$. The isocline for $\dot{\lambda}_P = 0$ is given by the solution for $\gamma = 1$ of (39) for the unknown λp, or

$$0 = \rho\lambda_P - \ln\left(\frac{A}{kP}\right) + 2kP. \tag{43}$$

The Jacobian matrix at the OSS is

$$J(P, \lambda_P) = \begin{pmatrix} 0 & \dfrac{P}{\lambda_P^2} \\ \dfrac{1}{P} + 2k & \rho \end{pmatrix},$$

with positive trace and negative determinant. Therefore, the steady state is a saddle point with a phase diagram similar to Figure 1.■

The solution for the population steady state is[8]

$$\bar{P}(A) = \frac{\text{ProductLog}\left[2\delta_P k\tau A e^{\frac{\rho}{\delta_P}}\right]}{2k}. \tag{44}$$

We set $R = rA$ so that $k \equiv \frac{1}{\alpha r}\left(\frac{D}{\delta_T} + B\right)$. Then, for $\bar{P} > 0$ at the OSS,

$$\frac{\partial\bar{P}(A)}{\partial A} = \frac{\text{ProductLog}\left[2\delta_P k\tau A e^{\frac{\rho}{\delta_P}}\right]}{2kA\left[1 + \text{ProductLog}\left[2\delta_P k\tau A e^{\frac{\rho}{\delta_P}}\right]\right]} > 0,$$

and a result similar to Proposition 2 can be stated. An increase in TFP will increase the OSS population, which again suggests a pathway to relaxing limits to growth.

Special case $\gamma = 0$

Using assumption A1, define $j \equiv 1 - \delta_P\tau > 0$. Then

$$\frac{1}{(j - t_w)} = kP \Longrightarrow t_w = j - \frac{1}{kP} \tag{45}$$

$$\frac{-\tau}{(j - t_w)} + \lambda_P = 0 \Rightarrow \lambda_P = \frac{\tau}{(j - t_w)} = \tau kP \tag{46}$$

$$\rho\lambda_P - \{\ln(A(j - t_w)) - 2kP\} = 0$$

$$\ln A - \ln k = (2 + \rho\tau)kP + \ln(P). \tag{47}$$

[8]The solution was obtained from Mathematica 13. ProductLog gives the primary (real) solution for w in $z = we^w$. The derivative of the solution is obtained using Mathematica 13.

Proposition 4: *For $\gamma = 0$, the OSS is a saddle point. The stable manifold has a positive slope.*

Proof: The isocline $\dot{P} = 0$ is given for $\dot{P} = P\,(n - \delta_P)$ with $\bar{n} = \delta_P$ at the OSS. Using (46) and the definition of j, we obtain $\lambda_P = -1/\,(1 - \delta_p \tau - t_w)\,.$ Then, on the isocline $\dot{P} = 0$,

$$n = \frac{-1}{\lambda_P} + \frac{1 - t_w}{\tau}.$$

The isocline for $\dot{\lambda}_P = 0$ is given by the solution λ_P of

$$0 = \rho\lambda_P - \left\{ \ln\left(\frac{A}{kP}\right) - 2kP \right\}.$$

.

The Jacobian matrix of the Hamiltonian dynamic system at the OSS is

$$J(P, \lambda_P) = \begin{pmatrix} 0 & \dfrac{P}{\lambda_P^2} \\ \dfrac{1}{P} + 2k & \rho \end{pmatrix},$$

with positive trace and negative determinant. Therefore, the OSS is a saddle point with a phase diagram shown in Figure 2.■

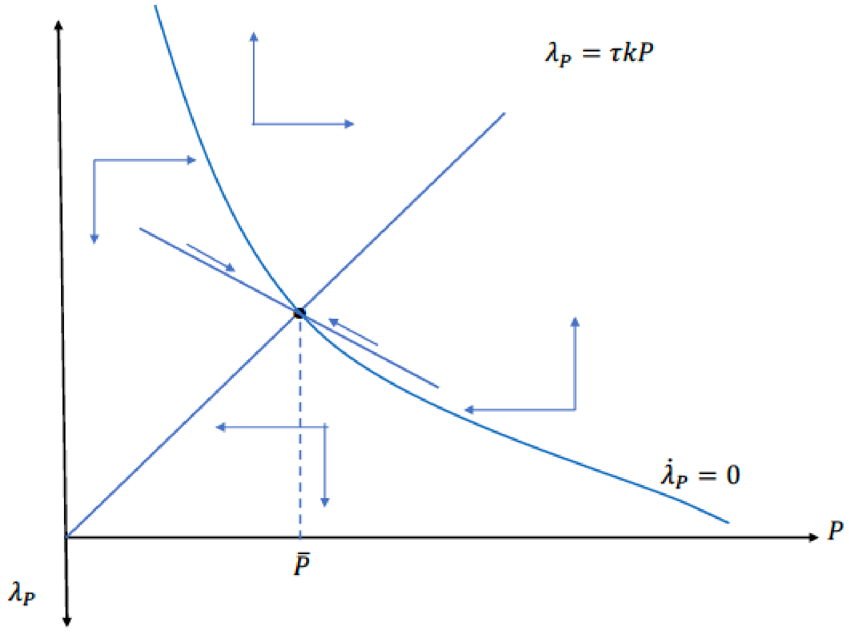

Figure 2. Saddle point OSS for $\gamma = 0$.

The solution for the OSS population implied by (47) has the same structure as (44), suggesting that an increase in the productivity parameters will increase the OSS population. Thus Proposition 2 holds in this model.

4 Endogenizing TFP and R&D

In the previous section we regarded A and R, the productivity parameters that affect output, and R&R that reduces the demand for Nature's services, as "manna from heaven". In this section, we endogenize these factors by assuming that they can be increased by devoting time to them. Thus, the total available time is allocated between consumption t_C, raising children $n\tau$, and working for production of output (t_w), TFP (t_A), and R&R (t_R). In this case, therefore, the budget constraint becomes

$$1 = t_C + t_A + \tau + t_w + t_R. \tag{48}$$

We maintain the assumption of Section 3.1 that the dynamics for climate, technical progress, and knowledge accumulation converge in fast time to their steady states, while the evolution of population is driven by slower moving cultural as well as economic drivers, and we consider Nature's services, $G(S)$, as fixed. Then the Hamiltonian system associated with the optimization problem becomes

$$\mathcal{H} = \max P\left[(1-\gamma)\ln\left(t_w A\left(t_A\right) t_C\right) + \gamma \ln\left(t_w A\left(t_A\right) n\tau\right) - D\frac{t_w A\left(t_A\right) P}{\delta_T \alpha R\left(t_R\right)} + B\left(G(S) - \frac{t_w A\left(t_A\right) P}{\alpha R\left(t_R\right)}\right)\right]$$

$$+ \lambda_P P\left(n - \delta_P\right) \tag{49}$$

$$y = t_w A\left(t_A\right),\ A\left(t_A\right) = \frac{A_0 t_A}{\delta_A} \equiv k_A t_A,\ R\left(t_R\right) = \frac{R_0 t_R}{\delta_R} \equiv k_R t_R \tag{50}$$

$$\dot{P} = P\left(n - \delta_P\right) = \mathcal{H}_{\lambda_P} \tag{51}$$

$$\dot{\lambda}_P = \rho\lambda_P - \mathcal{H}_P. \tag{52}$$

The impact inequality In this problem, the controls are $\boldsymbol{x} = (t_C, t_w, t_A, n\tau, t_R)$. Using (50), the condition to satisfy the impact inequality for all $t \geq 0$ implies control choice, given the rest of the parameters, such that

$$\frac{t_w k_A t_A P_t}{\alpha k_R t_R} \leq G(S).$$

4.1 Choosing optimal controls

We proceed to derive optimality conditions for the controls that can be used to explore routes to satisfy the impact inequality and thus satisfy the limits set by the biosphere. In this process, it is useful to write the Hamiltonian (49), with

$A(t_A), R(t_R)$ defined by (50), in the compact form:

$$\mathcal{H} = P\left[u(\boldsymbol{x}) - C(\boldsymbol{x})P\right] + \lambda_P P_t \left(n - \delta_P\right), \tag{53}$$

where $u(\boldsymbol{x})$ is the logarithmic utility function and $C(\boldsymbol{x})$ is a "cost function" corresponding to the cost terms of (49).[9]

The FOC for the controls resulting from (49) can be stated as follows:

Define $c \equiv \frac{k_A}{\alpha k_R}\left(\frac{D}{\delta_T} + B\right)$. Then

$$\begin{array}{cc} n: & \frac{(1-\gamma)}{t_C} = \frac{\gamma}{n\tau} + \frac{\lambda_P}{\tau} \\ t_A: & \frac{-(1-\gamma)}{t_C} + \frac{(1-\gamma)}{t_A} + \frac{\gamma}{t_A} - \frac{t_w}{t_R}cP = 0 \\ \Rightarrow & \frac{-(1-\gamma)}{t_C} + \frac{1}{t_A} = \frac{t_w}{t_R}cP \\ t_w & \frac{-(1-\gamma)}{t_C} + \frac{(1-\gamma)}{t_w} + \frac{\gamma}{t_w} - \frac{t_A}{t_R}cP = 0 \\ \Rightarrow & \frac{-(1-\gamma)}{t_C} + \frac{1}{t_w} = \frac{t_A}{t_R}cP \Rightarrow \\ & t_A = t_w = t \\ t_R: & \frac{-(1-\gamma)}{t_C} + \frac{t^2 cP}{t_R^2} = 0. \end{array} \tag{54}$$

From the FOC for t_A and t_w, it follows that $t_A = t_w = t$. Then the following route to satisfying the limits set by the biosphere can be characterized.

Satisfying the limits set by the biosphere Define $X \equiv \frac{\gamma}{n} + \lambda_P$. Choose $t = t_A = t_w$ and n such that $\frac{t^2 k_A P_t}{\alpha k_R t_R} \leq G(S)$. Since $P = P(0) = P_0$ initially, the planner could simply choose n, in order to satisfy the limit, $n(t) = \delta_P$ for all $t \geq 0$ and choose the controls $t = t_A = t_w, t_R$ such that

$$\frac{t^2 k_A P_0}{\alpha k_R t_R} \leq G(S)$$

for all dates greater than or equal to zero.

Another route to relaxing limits is biosphere reducing technical change scaling as $k_R = r k_A$ for some scaling factor $r > 1$. In this case

$$\frac{t^2 P_0}{\alpha r t_R} \leq G(S).$$

[9]Note that the log function $u(\boldsymbol{x})$ implies that it is a concave increasing function in the controls $\boldsymbol{x}$. Because of the product terms, the "cost function" $C(\boldsymbol{x})$ is not convex. We get left-hand Inada conditions from the log function for all controls except t_R, which only appears in the cost function . However, $t_R = 0$ together with the left-hand Inada conditions on the rest of the controls, implies a left hand-Inada condition for t_R too.

4.2 Optimal steady state

The Hamiltonian (49) can be written as:

$$\mathcal{H} = \max P\left[(1-\gamma)\ln\left(t_w k_A t_A t_C\right) + \gamma \ln\left(t_w k_A t_A n\tau\right) - D\frac{t_w k_A t_A P}{\delta_T \alpha R t_R}\right. \tag{55}$$
$$\left. +B\left(G(S) - \frac{t_w k_A t_A P}{\alpha R t_R}\right) + \lambda_P P_t\left(n - \delta_P\right)\right].$$

Define:

$$c \equiv \left(\frac{k_A}{\alpha k_R}\right)\left(\frac{D}{\delta_P} + B\right),\ X \equiv \frac{\gamma}{\tau n} + \frac{\lambda_P}{\tau}.$$

Using the optimality conditions (54) we obtain for the optimal controls:

$$t_w = t_A = t = \frac{1}{\left[\left[X\left(cP\right)\right]^{1/2} + X\right]} \tag{56}$$

$$t_R : -X + \frac{t^2 cP}{t_R^2} = 0 \Rightarrow \left(\frac{X}{cP}\right)^{1/2} = \frac{t}{t_R}. \tag{57}$$

At the OSS $n = \delta_P$, and X is a function of the costate variable λ_P which is the shadow value of population. Then, at the OSS

$$\frac{\gamma}{\tau\delta_P} + \left(\frac{(1/\rho)\mathcal{H}_P}{\tau}\right) = X,$$

where $\mathcal{H}_P$ is evaluated at steady state. To explore the existence on OSS and provide some insight for its computation we proceed as follows:

From the definition of X we have $t_C = \frac{(1-\gamma)}{X}$. Using the constraint on total available time we obtain,

$$1 = t_C + 2t + n\tau + t_R \ or$$
$$1 = \frac{(1-\gamma)}{X} + \frac{2}{\left[\left[X\left(cP\right)\right]^{1/2} + X\right]} + \tau\delta_P + \frac{1}{\left[\left[X\left(cP\right)\right]^{1/2} + X\right]\left(\frac{X}{cP}\right)^{1/2}}. \tag{58}$$

Note that increasing X causes the RHS of (58) to decrease., so X must decrease to balance the RHS to one. Looking at the definition of X we see that λ_P in OSS decreases as P increases. This makes economic sense.

Define the implicit function

$$X(P) \text{ by } 1 = h_1\left(X(P), P\right). \tag{59}$$

Note that $\frac{\partial h_1}{\partial X} \equiv h_{1X} < 0$ and we need $X \geq 0$ so that the square root in (58) is a real number. Thus, $X'(P) = \frac{-h_{1p}}{h_{1X}}$ and the implicit function $X(P)$ appears to be well defined on the set of non-negative X and non-negative P.

We turn now to the evaluation of $\mathcal{H}_P$ at OSS. From (52) we have:

$$\rho\lambda_P = \mathcal{H}_P = \ln(t_w) + \ln(k_A) + \ln(t_A) + (1-\gamma)\ln(t_C) + \gamma\ln(\delta_P\tau) + BG(S) - \left(\frac{t_w t_A}{t_R}\right)2cP$$
$$= 2\ln(t) + \ln(k_A) + (1-\gamma)\ln(t_C) + \gamma\ln(\delta_P\tau) + BG(S) - \left(\frac{t^2}{t_R}\right)2cP$$
$$= 2\ln\left\{\frac{2}{\left[X(cP)^{1/2} + X\right]}\right\} + \ln(k_A) + (1-\gamma)\ln\left[\frac{(1-\gamma)}{X}\right] + \gamma\ln(\delta_P\tau)$$
$$+ BG(S) - \left[\frac{4X^{1/2}(cP)^{-1/2}}{\left[X(cP)^{1/2} + X\right]}\right]2cP \tag{60}$$
$$\rho\left(\tau X - \frac{\gamma}{\delta_P}\right),$$

evaluated at OSS. Recall that at OSS $\lambda_P(n-\delta_P) = 0$, and that $\lambda_P = \tau X - \frac{\gamma}{\delta_P}$, and that after some algebra

$$\frac{t^2}{t_R} = \frac{4X^{1/2}(cP)^{-1/2}}{\left[X(cP)^{1/2} + X\right]}.$$

Equation (60) defines an implicit function,

$$h_2(X, P) = 0 = h_2(X(P), P), \tag{61}$$

where $X(P)$ was shown above - using the implicit function theorem - to be well defined from the constraint $1 = h_1(X(P), P)$. Solutions of the implicit functions will determine for the OSS. values $(\bar{P}, \bar{\lambda}_P)$, and the optimal controls $(\bar{t}, \bar{t}_R)$.

The impact inequality At the OSS, the impact inequality becomes

$$\frac{\bar{t}^2\delta_R A_0 \bar{P}}{\alpha\delta_A R_0 \bar{t}_R} \leq G(S). \tag{62}$$

where $\bar{t}$ and $\bar{t}_R$ are the steady state solutions for time spent in production and TFP, and time spent in R&D respectively. Assume that $R_0 \approx rA_0$ and $\delta_R = \delta_A$. Then the impact inequality can be written as

$$\frac{\bar{t}\bar{P}}{\alpha r\bar{t}_R} \leq G(S). \tag{63}$$

Therefore, increasing r, which is how the green R&D scales with TFP, eases the stress on Nature's services, and allows higher steady-state population and

output. The same source can reduce damages from climate change, which in this case can be written as
$$\frac{D\bar{t}\bar{P}}{\delta_T \alpha r \bar{t}_R}.$$

5 Limits with two types of consumption

In this section we introduce the concept of two types of consumption, "basic" and "profligate", with the profligate type exercising relatively more pressure on Nature. We return to the "manna from heaven" and no-work-time model in order to more clearly identify the impact from this differentiation. Damages to the biosphere and climate are associated with the types of consumption and the raising of children.

Let $t_{c_i}, i = 1, 2$ be the time devoted to basic and profligate consumption respectively, and let D_{T_i}, $D_{G_i}, i = 1, 2, n\tau$ be damages from climate change and damages to the biosphere from the two types of consumption and raising children respectively. The relevant Lagrangian can be written as

$$\begin{aligned}
\mathcal{L} = \max P \{ & \gamma_1 \ln (yt_{c_1}) + \gamma_2 \ln (yt_{c_2}) + \gamma_3 \ln (yn\tau) - \\
& (D_{T_1} t_{c_1} + D_{T_2} t_{c_2} + D_{T,nt} n\tau) \left(\frac{yP}{\delta_T \alpha R} \right) + \\
& B \left[G(S) - (D_{G_1} t_{c_1} + D_{G_2} t_{c_2} + D_{G,nt} n\tau) \left(\frac{yP}{\alpha R} \right) \right] + \\
& \lambda_P (n - \delta_P) + \mu (1 - t_{c_1} - t_{c_2} - n\tau) \} ,
\end{aligned} \tag{64}$$

with $y = A$, $1 = \gamma_1 + \gamma_2 + \gamma_3$.

Define

$$\begin{aligned}
K_1 &\equiv \left(\frac{D_{T_1}}{\delta_T} + BD_{G_1} \right) \left(\frac{yP}{\alpha R} \right) \\
K_2 &\equiv \left(\frac{D_{T_2}}{\delta_T} + BD_{G_2} \right) \left(\frac{yP}{\alpha R} \right) \\
K_{n\tau} &\equiv \left(\frac{D_{T,nt}}{\delta_T} + BD_{G,nt} \right) \left(\frac{yP}{\alpha R} \right).
\end{aligned}$$

The FOC for the controls (t_{c_1}, t_{c_2}, n) are

$$t_{c_1} : \frac{\gamma_1}{t_{c_1}} - K_1 - \mu = 0 \tag{65}$$

$$t_{c_2} : \frac{\gamma_2}{t_{c_2}} - K_2 - \mu = 0 \tag{66}$$

$$n : \frac{\gamma_3}{n} + \lambda_P - \tau K_{n\tau} - \mu = 0 \tag{67}$$

$$P : \mathcal{L}_P = 0.$$

At the OSS, $n = \delta_P$. Solving (65) and (66), we obtain (t_{c_1}, t_{c_2}) as a function of P.

Solving for t_{c_1}, we obtain:

$$t_{c_1}(P) = \frac{\left\{ - \left[(\gamma_1 + \gamma_2) - LPj \right] + \left[(\gamma_1 + \gamma_2 - LPj)^2 + 4LP\gamma_1 j \right]^{1/2} \right\}}{2LP}, \tag{68}$$

where $L \equiv K_2 - K_1 \gg 0$, since we expect that profligate consumption will cause more damages to climate and Nature, and by an earlier assumption $j \equiv 1 - \delta_P \tau > 0$. Then t_{c_2} is determined as a function of P by the constraint

$$1 - t_{c_1}(P) - t_{c_2}(P) - \delta_P \tau = 0. \tag{69}$$

The OSS population level and the corresponding shadow value of the population can be determined by (67) and the OSS condition evaluated at $n = \delta_P, t_{c_1}(P), t_{c_2}(P)$,

$$\rho \lambda_P = H_P, \tag{70}$$

where H is the Hamiltonian associated with (64)

Equation (68) provides a solution for t_{c_1} as a function of P. Then t_{c_2} can be determined as a function of P, by (69) and (P, λ_P) by (67) and (70).

Note that when $\gamma_1 = 0$, so that preferences concentrate on profligate consumption and child rearing, (68) results in

$$t_{c_1} = \frac{(LPj - \gamma_2)}{LP} > 0,$$

under the assumption $LPj - (\gamma_1 + \gamma_2) > 0$.

The impact inequality In the context of the model with two types of consumption, the impact inequality includes damages to climate change and damages to Nature and can be written at a steady state, recalling that $y = A$, as

$$(D_{T_1}t_{c_1} + D_{T_2}t_{c_2} + D_{T,nt}n\tau)\left(\frac{AP}{\delta_T \alpha R}\right) +$$
$$(D_{G_1}t_{c_1} + D_{G_2}t_{c_2} + D_{G,nt}n\tau)\left(\frac{AP}{\alpha R}\right) \le G(S).$$

Note that if all Ds are equal to 1, the demand on Nature is driven by $\frac{yP}{\alpha R}$. If $R = rA$, then the impact inequality becomes

$$(D_{T_1}t_{c_1} + D_{T_2}t_{c_2} + D_{T,nt}n\tau)\left(\frac{P}{\delta_T \alpha r}\right) +$$
$$(D_{G_1}t_{c_1} + D_{G_2}t_{c_2} + D_{G,nt}n\tau)\left(\frac{P}{\alpha r}\right) \le G(S).$$

If we use rates of cost decline per unit of energy produced by green technology relative to fossil fuel technology as, for example, in Way et al (2022), to proxy $R = rA$, then r is larger than one. In this case, a higher level of population and output could be sustained at the steady state without violating the impact inequality, that is, keeping the left-hand side lower than the right-hand side.

6 The dynamics of the finite biosphere

In the previous sections, we assumed that Nature's services $G(S)$ were constant and we introduced a penalty that reduced the planner's welfare when there was excess demand for these services. The purpose of this simplifying assumption was to provide a clearer picture of the sources that might ease the intensity of humans' demand for the services that the biosphere can provide. This, however, implies that under excess demand, the stock of natural capital S embedded in a finite biosphere will be declining, or $\frac{dS}{dt} < 0$. The decumulation of the stock of natural capital is pointed out in the Dasgupta Review (Dasgupta, 2021, page 44), and can be linked with the crossing of the planetary boundaries. As Richardson et al. (2023) point out, Earth is now beyond six of the nine planetary boundaries. This implies decumulation of natural capital or accumulation of pollutants, with boundary crossings implying in turn loss of Nature's services. To take into account biosphere dynamics, we simplify our model by assuming that only the number of children is a control variable and write the Hamiltonian of the problem as

$$\mathcal{H} = P\left[(1-\gamma)\ln(1-n\tau) + \gamma\ln(n\tau)\right] + \lambda_P P\,(n-\delta_P) + \mu P\left[G(S) - \frac{AP}{\alpha(R)}\right], \tag{71}$$

where R denotes biosphere savings R&D.

Optimality conditions at the OSS, where the steady-state conditions $n = \delta_P$, $G(S) - \frac{AP}{\alpha(R)} = 0$ are satisfied, imply:

$$n : \frac{-\tau(1-\gamma)}{(1-\delta_P\tau)} + \frac{\gamma}{\delta_P} + \lambda_P = 0 \tag{72}$$

$$P : \rho\lambda_P = \mathcal{H}_P = (1-\gamma)\ln(1-\delta_P\tau) + \gamma\ln(\delta_P\tau) - \mu\frac{AP}{\alpha(R)}$$

$$\Rightarrow P = \frac{[(1-\gamma)\ln(1-\delta_P\tau) + \gamma\ln(\delta_P\tau)]\,\alpha(R)}{\mu A} \tag{73}$$

$$S : \rho\mu = \mathcal{H}_S = \mu P G^{'}(S) \Rightarrow \mu = 0 \; or \; \rho = P G^{'}(S) \tag{74}$$

$$G(S) = \frac{AP}{\alpha(R)} \tag{75}$$

$$n = \delta_P.$$

At an OSS, λ_P is determined by (72), while (74) determines S as a function of P, or $S = S(P)$. Then (75) can be solved for P as function of A, R which are treated as parameters. This solution can be substituted into (73) to determine μ.

To provide some insight into the limits imposed by the biosphere, we assume that $\alpha(R) = \alpha R$ and that $R = rA$. Then (73) and (75) imply:

$$P = \frac{[(1-\gamma)\ln(1-\delta_P\tau) + \gamma\ln(\delta_P\tau)]\,\alpha r}{\mu} \tag{76}$$

$$G(S) = \frac{AP}{\alpha(R)} = \frac{P}{\alpha r}. \tag{77}$$

These conditions mean that A can be arbitrarily large and, with constant population at the OSS, we can have arbitrarily large output. However, this result depends on the assumption that our preferences are independent of A.[10] If we relax this assumption, the Hamiltonian will be

$$\mathcal{H} = P\left[(1-\gamma)\ln(A(1-n\tau)) + \gamma\ln(n\tau)\right] + \lambda_P P(n-\delta_P) + \mu P\left[G(S) - \frac{AP}{\alpha(R)}\right].$$

Then (76), under the assumption $\alpha(R) = \alpha R$ and $R = rA$ implies

$$P = \frac{[(1-\gamma)\ln A + (1-\gamma)\ln(1-\delta_P\tau) + \gamma\ln(\delta_P\tau)]\,\alpha r}{\mu},$$

which means that as A increases, the steady-state population will increase with $\ln A$. In this case, however, constraint (77) imposes limits even when R&D scales with A, since $G(S)$ is finite because the biosphere S is finite.

[10] The case in which an increase in A requires Nature's services can also be considered.

7 Conclusions and areas for further research

This paper built a biosphere-limited optimal growth model where population is endogenous following Galor (2012), and where the total time of each person is allocated across work, consumption, R&D, building GDP, child rearing and child educating. The economy is constrained by damages from GHG emissions and damages to the biosphere that increase with the scale of the economy. Since it is widely recognized that the world economy has overshot the capacity of the biosphere, our analysis suggests a rapid expansion of biosphere-saving technology, in addition to the current rapid ramping up of GHG emissions-saving technology. As Steffen et al. (2018) point out self-reinforcing feedbacks could prevent stabilization of the Earth's climate at intermediate temperature rises. To prevent destabilization, collective human action is required to stabilize the Earth System in a habitable state. This requires policies promoting decarbonization of the global economy, enhancement of biosphere carbon sinks, behavioral changes, technological innovations, new governance arrangements, and transformed social values. Our general model covers most of the avenues of Earth System damage associated with potential destabilization mentioned by Steffen et al (2018), and provides insights about the appropriate policies.

World wealth inequality is extreme, since the top 10% of the population have 76% of the world's wealth (World Bank (2022)). Since log consumption is roughly proportional to log wealth and log income (Jawadi and Sousa, 2014), our optimal biosphere-limited growth model implies that the heavy consumption of the world's wealth must be reduced to stay within the Earth's limits. Recent estimates by Burke et al. (2023) of loss and damages (L&D) from GHG emissions support this conclusion. Since our growth model is somewhat novel – although it builds on ideas from the Dasgupta Review, Galor's endogenous population theory, climate economics, and other work – many directions for future research are open. First, we need to explore the most effective policy instruments to reduce humanity's footprint overshoot and destruction of the biosphere flagged by the Dasgupta Review. Second, we need to explore ways of using revenue from perhaps a higher tax on "profligate" biosphere-damaging consumption and other biosphere-related taxes for the most effective restoration of our damaged Earth. Third, we need to do more on endogenizing the dynamics of the biosphere, $dS/dt = G(S)$ minus impact. This creates a system with a minimum of two state variables, S the biosphere stock and P population numbers. Fourth, in view of the fact that it takes emissions and increased impact on the biosphere to produce emission-saving technology and biosphere-saving technology, research is needed to compute the net savings of emissions and biosphere impact. Even if we try to find better ways to produce such green technology, that effort requires additional emissions and impacts on the biosphere. Taking actions to find better ways to economize on such additional emissions and impacts leads to yet more emissions and impacts, and so on, ad infinitum. Investigation is needed into the ultimate outcome in net emissions reduction and reduction of net impact on the biosphere. Fifth, extension to heterogeneous consumers stratified by emissions and impact on the biosphere is needed to better target policy interventions. Sixth, extension is needed to include spatial effects and optimal allocation of spatial economic activity to optimize world welfare. The Amazon, for example, is an important part of the biosphere. Assunçao et al. (2023) show how impact on the Amazon biosphere can be minimized by optimal allocation of impact-causing economic activity across space. Their study indicates large gains from optimal spatial allocation of economic activity world-wide.

We look forward to working on these extensions and ideas and hope others join us.

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