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Growth embedded in a finite Earth: The role of using and producing ideas *

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Abstract

This paper puts forth a growth model that takes into account the fact that the economy is embedded in a finite Earth. Economic activity uses services which are provided by the biosphere; however, this supply is finite. The question we explore in this paper is whether ideas that drive the accumulation of "brown" and "green" R&D that produces material goods which could be biosphere using or biosphere saving can provide persistent growth when the whole system is embedded in a finite Earth. Or, to put it differently, whether it is possible to have persistent growth supported by idea-driven technical change without violating the impact inequality proposed by Dasgupta (2021), which compares global demand for services provided by the biosphere to the supply of these services. We develop optimal time allocation models and provide conditions that support the feasibility of growth when the net impact on biosphere is zero.

Keywords: growth, limits, biosphere, impact inequality, biosphere saving technology, combination of ideas, spillovers.

JEL Classification: O44, J13, Q01

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1 Introduction

The view that the economy is embedded within a finite planet and that there are limits to growth was advanced in the sixties and seventies by writers such as Kenneth Boulding (1966), Barbara Ward (1966), and Herman Daly (1974), who emphasized that Earth is not an unlimited reservoir for resource extraction or pollution. Similar concerns are reflected in the concept of planetary boundaries (Rockström et al., 2009) which define a safe operating space for the Earth system thresholds that, if crossed, could result in unacceptable environmental changes.

The Dasgupta Review (2021), a comprehensive and in-depth review of the economics of biodiversity and ecosystem services, embeds the economy in Nature,[1] which causes an important issue to emerge: namely, growth in a finite Earth and whether such growth has limits when embedded in a finite Earth. A similar issue was raised by Solow (2009) who stressed that real demands on the natural environment will be dramatically different in a world in which developing countries need to pass through the material-goods-intensive phase of growth before they arrive at the service economy. This evolution creates a mutual interdependence of economic growth, natural resource availability, and environmental constraints.

A central concept in Dasgupta's work (Dasgupta, 2021, Chapter 4; Dasgupta, 2022) is the impact inequality which compares the demand from the global economy for services provided by the biosphere or Nature,[2] to the supply of these services by Nature. The impact inequality can be written as

$$\frac{Py}{\alpha} > G\left(S\right). \tag{1}$$

In (1), the left-hand side represents global demand for Nature's services where P is the level of population, y is GDP per capita and α is an efficiency parameter reflecting the transformation of Nature's services to GDP, while the right-hand side represents the supply of Nature's services. In the impact inequality, S can be interpreted as natural capital which can be regenerated at a rate $G(\cdot)$. As pointed out in Dasgupta (2022, page 1021),

> "... the ratio of our demand for maintenance and regulating services (the left-hand side of inequality (1)) to Nature's ability to meet that demand on a *sustainable* basis (the right-hand side of inequality (1)) is today 1.7 ... . The term "sustainable" is an all-important qualifier here, for it says that we are enjoying the overshoot at the

[1] See also Dasgupta and Levin (2023).

[2] The terms biosphere – the part of the Earth that is occupied by living organisms – and Nature are used interchangeably in the Dasgupta Review. We use these terms in the same way in this paper.

> expense of the health of the biosphere; that is, by depleting S. The number 1.7 is almost certainly an underestimate, which makes it even more a reason that inequality (1) be converted to an equality sooner rather than later. We are in a fire-fighting situation."

The supply side of the impact inequality can be increased by investing in Nature, which involves waiting for Nature to recover. The returns of such investment might be realized in slow time, or could even be zero for ecosystems that have been irreversibly damaged, or might be limited because of constraints imposed by physical processes. Furthermore, even if the biosphere regenerates, supported by environmental policies, the regeneration rate is expected to be bounded because of natural limits. Thus, conversion of the impact inequality to equality requires controlling the demand side, that is, controlling the evolution of population P, GDP per capita y, and "biosphere saving" technical progress α, which implies controlling the growth of P, y and α. Restricting the demand side to restore and maintain equality between demand and supply relates to a fundamental question with regard to economic growth, which seeks to explore whether there are limits to the growth of GDP per capita given the bounds imposed by Nature.[3]

In this paper, we explore this issue by developing a growth model in which economic growth is explicitly situated within a finite Earth. We position our model in the very long run – a timescale longer than that typically considered in growth theory, and far longer than the frequency of business cycles. Over such extended horizons, all human-made capital goods are ultimately derived from human labor time. This idea traces back to classical economic thought. Ricardo (1821, p. 1) states that "The value of a commodity, or the quantity of any other commodity for which it will exchange, depends on the relative quantity of labor which is necessary for its production, and not on the greater or less compensation which is paid for that labor." Stigler (1958, p. 361) points out, regarding Ricardo on the labor theory of value, that "... there is no doubt that he (Ricardo) held what may be called an *empirical* labor theory of value, that is, a theory that the relative quantities of labor required in production are the dominant determinants of relative values." Similar ideas are found in Simon (1996), who links long-run growth to population growth. He notes that "Adding more people causes problems, but people are also the means to solve these problems. The main fuel to speed our progress is our stock of knowledge, and the brake is our lack of imagination. The ultimate resource is people – skilled, spirited, and hopeful people – who will exert their wills and imaginations for

[3]See, for example, the special issue of *The Review of Economics Studies* (1974), the Daly vs Solow–Stiglitz controversy in the special issue of *Ecological Economics* (1997), or Stokey (1998).

their own benefit, and inevitably they will benefit not only themselves but the rest of us as well" (Simon, 1996, page 589). In the framework of a Cobb-Douglas production function, $Y = F(K, Labor, Land)$, the factor share of land is small relative to labor and capital. Over a very long time horizon exceeding the average lifetime of capital, capital can be viewed as congealed labor, implying that the combined labor share dominates relative to fixed factors like land. This paper therefore treats labor (inclusive of congealed labor) as the primary driver of output.

The growth theory we develop in this paper, in this context, can be regarded as a two-factor long-run theory. One factor is labor/population and the other factor is Nature, which provides humans with ecosystem services that augment the ability of humans to produce and innovate. In the context of this model, we explore limits imposed by the impact inequality and also explore conditions for their relaxation by considering a welfare-maximizing, Ramsey-type social planner model and a rational expectations equilibrium model (REE) which are embedded in the biosphere denoted by S. In our model, production of material goods that use Nature and deplete natural capital – "the brown goods" – requires labor input and services by Nature. These services are consumed during the production and consumption processes. The demand for Nature's services increases with population. The increased demand for its services can be counterbalanced by the development of Nature or biosphere-saving technologies, which also use labor and Nature's services, and produce "green" material goods.

In our model, a representative individual allocates a unit of time to consumption and leisure, and production of brown and green material goods. We analyze the dynamic evolution of brown and green material goods within the boundaries set by Nature in the context of a model in which ideas are regarded as economic goods which are important for growth and development. Paul Romer (1998, page 212) points out that:

"In a world with physical limits, it is discoveries of big ideas (for example, how to make high-temperature superconductors), together with the discovery of millions of little ideas (better ways to sew a shirt), that make persistent economic growth possible. Ideas are the instructions that let us combine limited physical resources in arrangements that are ever more valuable."

With growth driven by ideas, each person in a population $P(t)$ allocates time to the creation of new ideas that can produce brown and green goods. According to Jones (2023), there are two "cookbooks": one with recipes which support the growth of material goods that increase the demand for biosphere services and the other with recipes that produce material goods that reduce the demand on biosphere services such as renewable energy. Allocating time t_A to increase the recipes in the first cookbook increases total factor productivity (TFP) in producing biosphere-using material goods, while allocating time t_R to increase the recipes in the second cookbook increases TFP in producing biosphere-saving material goods.

Romer (2016) refers to the sharing of discoveries with others as the "nonrivalry of knowledge" and stresses that there is a very large number of discoveries yet to be found which induces a "combinatorial explosion." Through the nonrivalry of ideas, the creation of new ideas generates Romer-type positive spillovers (e.g., Romer, 1990). Furthermore the combination of existing ideas could lead to production of new ideas and combinatorial exponential growth (Weitzman, 1998; Jones, 2023). Thus, an ideas-aggregate, or the total recipes in each cookbook at each point of time, can be defined as $A(t)P(t)$ and $R(t)P(t)$ for brown and green recipes respectively. In our model, the aggregate $A(t)P(t)$ produces materials and demands services from the biosphere, while the aggregate $R(t)P(t)$ produces biosphere-saving materials. The Romer-type spillovers emerging from the ideas-aggregate can be captured by a spillover function (see also Kortum, 1997; Jones, 2023). Denoting by $(q_A, q_R) > (0, 0)$ the spillover parameters for brown and green ideas, the demand for biosphere services in the impact inequality, setting $\alpha = 1$ to simplify, can be defined as

$$\frac{(A(t)P(t))^{1+q_A}}{(R(t)P(t))^{1+q_R}}, \tag{2}$$

where $A(t)P(t)$ replaces Py and $R(t)P(t)$ replaces α in the impact inequality definition (1).

In this context, the contribution of this paper is to explore – using the two-factor long-run growth model – whether ideas that drive the accumulation of brown and green materials can provide persistent growth, when the material goods that they help to produce, brown or green, are embedded in a finite Earth. Or, to put it differently, is it possible to have persistent growth supported by ideas without violating the impact inequality? Non-violation could mean that in a utility maximization problem, irrespective of whether the problem is solved by a social planner or by individual agents in the context of a REE, the impact inequality (1) is satisfied as equality so that the biosphere (or the natural capital) S remains constant. In this way, the production of biosphere-harming material goods has a net zero impact on the biosphere. Another way to introduce the impact inequality is to introduce a welfare cost in the optimization if $\frac{(A(t)P(t))^{1+q_A}}{(R(t)P(t))^{1+q_R}} > G(S)$. This cost can be interpreted as the fee for using the services of the biosphere above their natural supply. The deviations between the social planner's solution and the corresponding REE solution could provide the basis for regulation capable of implementing the planner's solution.

In our model, the biosphere includes climate so that exceeding the supply of services by the biosphere can be interpreted in the context of crossing planetary boundaries. A social planner maximizes discounted utility from consumption by allocating time between consumption, and time devoted to accumulating ideas that produce brown and green or biosphere-saving materials, subject to (i) the constraint that (1) is satisfied as equality, or (ii) by accounting for a welfare cost if (1) holds as a strict inequality, or (iii) by solving a dynamic optimization problem by fully considering biosphere dynamics. Furthermore, we consider a similar private optimization problem in which, in a group of countries, each country takes as given the actions of other countries and equilibrium is determined by assuming rational expectations. We develop a policy scheme for implementing the social optimum at a regulated REE.

We study two versions of the two-factor long-run growth model. In the first, the social planner and the agents at the REE do not recognize the contribution of the biosphere in the production of brown or green material goods, which means that the demand for biosphere services (2) is independent of natural capital S. The regulator and the REE agents can infer the aggregate demand for biosphere services but they do not recognize the feedback loop through which human demand affects the rate at which the biosphere can supply services. This assumption could be reasonable when the historic evolution of the growth process is considered. In the second version, natural capital explicitly contributes, along with labor, to the production of brown and green material goods and (2) depends on S.

Finally we extend the model by expanding on the possible time allocation alternatives, and considering the case in which the representative individual can allocate time to child rearing and child educating in addition to time allocated to consumption and production of brown/green ideas. This extension endogenizes population growth and provides further insights into the potential limits to growth.

2 A Model Without Biosphere Feedback on Production

We start with the case in which the planner and the REE agents ignore the contribution of biosphere services in the production process. Consider a representative individual with a unitary time endowment. This endowment is divided between t_C, time devoted to consuming and leisure time; t_A, time that increases the stock of brown ideas and produces brown material goods which are consumed and possibly creates more ideas on how to exploit the environment to create more brown goods; and t_R, time that increases the stock of green ideas R, which is the green or biosphere saving R&D and produces green material goods such as photovoltaics.

Therefore, the flow budget constraint regarding the time allocation of the representative individual is defined as

$$1 = t_C + t_R + t_A. \tag{3}$$

Let $P(t)$ denote the level of population at time t. We assume that the stocks of brown and green ideas starting from some initial levels evolve as

$$(A(t)t_A(t)P(t))^{1+q_A} = \left(\hat{A}_0 e^{\alpha(t)t} t_A(t)P(t)\right)^{1+q_A}, (R(t)t_A(t)P(t))^{1+q_R} = \left(\hat{R}_0 e^{r(t)t} t_R(t)P(t)\right)^{1+q_R} \tag{4}$$

where $\alpha(t), r(t)$ are exogenous time paths of growth rates of the stocks of ideas. We consider zero depreciation a plausible assumption since ideas and techniques once invented never decay. They may become obsolete but new techniques and ideas can use them. The $\hat{A}_0$ and $\hat{R}_0$ could capture time costs of "recalling" old techniques and ideas, at least to some extent.

The biosphere dynamics can be written as

$$\dot{S}(t) = G(S(t)) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_E(t)P(t))^{1+q_R}}, \; S(0) = S_0, \tag{5}$$

where $G(S)$ is the flow of services supplied by Nature, while the demand for Nature's services is given by the impact term

$$I(q_A, q_R, \alpha(t), r(t)) = \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_E(t)P(t))^{1+q_R}}. \tag{6}$$

The multiplication of the stock of ideas at each point in time with populations and their persistent growth is regarded as providing an approximation of the combinatorial exponential growth induced by accumulation of non-rivalrous ideas.

The social planner's utility function using (4) is defined as

$$u(t) = \ln\left[(A(t)t_A(t)P(t))^{1+q_A}(1 - t_R(t) - t_A(t))\right]. \tag{7}$$

The specification of the utility function allows for production spillovers.[4]

[4]It could be argued that the consumption time of the representative agent is augmented by consumption of other agents, so that consumption spillover effects could be written as $t_C^{q_C}$. This type of consumption externalities may reflect, for example, that sports events or concerts, bars, etc, are enjoyed more if there are more people there enjoying them. For $q_C = 0$, these externalities do not emerge. We choose to set $q_C = 0$ in order to simplify the exposition.

2.1 The social optimization management problem

The social planner chooses (t_C, t_A, t_R) to maximize discounted aggregate utility subject to the budget constraint (3), population and biosphere dynamics and the evolution of the stock of ideas, given the exogenous growth rates of ideas and the spillover effects. The planner's objective using (4) can be written as

$$\max_{t_A, t_R} \int_0^\infty e^{-\rho t} \ln \left[(A(t)t_A(t)P(t))^{1+q_A} (1 - t_R(t) - t_A(t)) \right] dt, \tag{8}$$

where ρ is the utility discount rate. We will refer to this optimization as the social optimization management problem (SOMP). Note that in this framework, Nature does not enter the utility function.

In SOMP, the biosphere constraint (5) can be approached in three different ways. First, it can be regarded as an impact inequality which implies a welfare cost if the demand for Nature's services given by (6) exceeds the supply given by $G(S)$. We will refer to this problem as the soft constraint problem (SCP) and define it as

$$\max_{t_A, t_R} \int_0^\infty e^{-\rho t} \left[\ln \left[(A(t)t_A(t)P(t))^{1+q_A} ((1 - t_R - t_A)) \right] \right. \tag{9}$$

$$\left. + C \left[G(S) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}} \right] \right] dt.$$

In this formulation, if the impact inequality is positive, that is, the economy demands fewer services from Nature than the flow of services that Nature can supply, social welfare increases. A negative impact inequality, on the other hand, implies a welfare cost. Thus, C can be interpreted as the planner's implicit valuation of Nature's services given the supply-demand conditions. Since problem (9) has no dynamics, the integrand is maximized term-by-term for each t.

Second, the biosphere constraint can be regarded as a hard constraint such that the net impact is zero at all t. This is the hard constraint problem (HCP) defined as

$$\max_{t_A, t_R} \int_0^\infty e^{-\rho t} \ln \left[(A(t)t_A(t)P(t))^{1+q_A} ((1 - t_R(t) - t_A(t)) \right] dt \tag{10}$$

$$\text{subject to}: \; G(S) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}} = 0. \tag{11}$$

Problem (9)–(11) has no dynamics, therefore the Lagrangian for this problem at each t is defined as

$$\mathcal{L}(t_A, t_R, \mu) = \ln \left[(A(t)t_A(t)P(t))^{1+q_A} \left((1 - t_R(t) - t_A(t)) \right) \right] + \mu \left[G(S) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}} \right].$$

For the HCP to be well defined, it has to be assumed that the constraint is satisfied at initial time. Since the flow of Nature's services $G(S)$ is treated as exogenous, the Lagrangian multiplier associated with constraint (11) can be regarded as the shadow value of Nature's flow of services.

Claim $\mu > 0$.

Proof: Suppose not. Then the value of the objective increases as $t_R(t)$ decreases. But as $t_R(t)$ decreases towards zero, the demand on the biosphere will exceed $G(S)$, which is a contradiction.□

The third approach is to fully take into account biosphere dynamics and study optimal growth, which implies the solution of the optimal control problem (OCP).

$$\max_{t_A, t_R} \int_0^{\infty} e^{-\rho t} \ln \left[(A(t)t_A(t)P(t))^{1+q_A} \left((1 - t_R(t) - t_A(t)) \right) \right] dt \tag{12}$$

$$\text{subject to: } \dot{S}(t) = G(S(t)) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}}, \; S(0) = S_0, \tag{13}$$

with current value Hamiltonian representation

$$\mathcal{H}(S, \lambda, t_A, t_R, t) = \ln \left[(A(t)t_A(t)P(t))^{1+q_A} \left((1 - t_R(t) - t_A(t)) \right) \right] + \lambda(t) \left[G(S(t)) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}} \right], \tag{14}$$

where $\lambda(t)$ is the costate variable which can be interpreted as the dynamics shadow value of the biosphere $S(t)$.

2.1.1 A preliminary result

Before examining the optimality conditions and their implications for growth embedded in a finite biosphere, a preliminary result can be obtained directly from the impact inequality. To attain some notion of sustainability, the demand for Nature's services expressed by the impact term (6) should be bounded above by some bound $\hat{B} < \infty$ for all $t \geq 0$. This bounded impact constraint, assuming fixed population and exogenous growth rates to simplify, implies that

$$\frac{\left[\left(\hat{A}_0 e^{\alpha t} t_A(t)\right) P_0\right]^{1+q_A}}{\left[\left(\hat{R}_0 e^{rt} t_R(t)\right) P_0\right]^{1+q_R}} \leq \hat{B}.$$

Assumption 1: $\alpha\left(1+q_A\right) > r\left(1+q_R\right)$ since more experience is likely to have occurred in accumulating ideas in producing brown A-type goods compared to green R-type goods, given that growing environmental concerns are more recent.

Given that $(t_A(t), t_R(t), t_C(t)) < (1,1,1)$ for all $t \geq 0$, then for any arbitrarily chosen path for $(t_R(t), t_C(t))$, the time $t_A(t)$ must be decreasing exponentially rapidly enough to keep from violating the bounded impact constraint. This result provides a general idea about the properties of solutions of the optimizing models which do not violate the impact constraint.

2.2 The private optimization management problem

For the biosphere management case, the private optimization management problem (POMP) may describe a situation in which a country maximizes own utility and takes as given the actions of other countries. More specifically, we treat each country as small so the actions of others can be regarded as exogenous and each country is a "spillover taker". Then the evolution of brown and green ideas will be given by

$$\begin{aligned} A\left(t_A(t), t_A^e(t), t, P(t)\right) &= \left(\hat{A}_0 e^{\alpha t} t_A(t) P(t)\right)\left(\hat{A}_0 e^{\alpha t} t_A^e(t) P(t)\right)^{q_A} \\ R\left(t_R(t), t_R^e(t), t, P(t)\right. &= \left(\hat{R}_0 e^{rt} t_R(t) P(t)\right)\left(\hat{R}_0 e^{\alpha t} t_R^e(t) P(t)\right)^{q_R}, \end{aligned}$$

where $t_A^e(t), t_R^e(t)$ are fixed in the optimization problem of a specific country. The objective of a specific country will be to

$$\max_{t_A, t_R} \int_0^\infty e^{-\rho t} \ln\left[\left(A\left(t_A(t), t_A^e(t), t, P(t)\right)\right)\left(t_C(t)\right)\right] dt, \tag{15}$$

where $t_C = 1 - t_R - t_A$. Biosphere dynamics are defined in this case as

$$\dot{S}(t) = G(S(t)) - \frac{\left(\hat{A}_0 e^{\alpha t} t_A(t) P(t)\right)\left(\hat{A}_0 e^{\alpha t} t_A^e(t) P(t)\right)^{q_A}}{\left(\hat{R}_0 e^{rt} t_R(t) P(t)\right)\left(\hat{R}_0 e^{\alpha t} t_R^e(t) P(t)\right)^{q_R}}. \tag{16}$$

At an REE,

$$REE: t_C = t_C^e, t_R = t_{R,}^e, t_A = t_{A.}^e \tag{17}$$

The SCF, HCP and OCP defined for the SOMP can also be defined for the POMP given (15), (16) and (17). Deviations between the SOMP and the POMP solutions lead us to the challenge of formulating POMP with incentives to implement the SOMP solution.

3 Optimal time allocation and limits to growth

3.1 The simplified soft constraint problem for the SOMP

Given the complexity of the problem in this section, we present two simplified versions of the SCP in order to obtain insights by providing tractable solutions which are not possible for the full model.

1. $\alpha(t) = 0, r(t) = 0, q_A = 0, q_R = 0$, *constant population.*

In this case, the key part of integrand (9) to be maximized by the choice of t_A, t_R becomes

$$\ln t_A + \ln(1 - t_A - t_R) + C\left(G(S) - \frac{t_A}{t_R}\right).$$

Maximization with respect to t_A, t_R results in the optimal positive choices of

$$t_A^* = \frac{1}{4}\left(2 + C - \sqrt{C\,(4+C)}\right),$$
$$t_R^* = \frac{1}{4}\left(-C + \sqrt{C\,(4+C)}\right),$$

with $t_A^* + t_R^* = \frac{1}{2}$, so the the budget constraint is satisfied. Fig. 1 depicts the optimal choice as functions of C, the planner's implicit valuation of Nature's services.

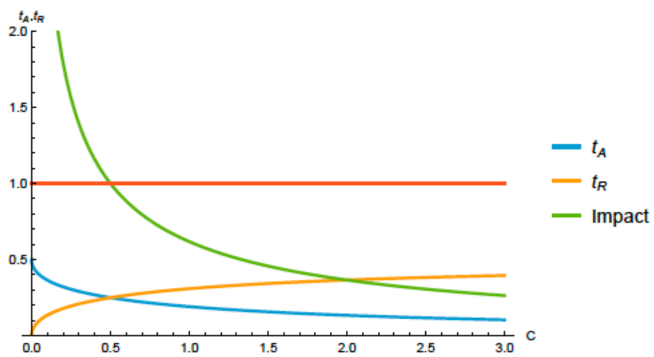


Fig. 1. Optimal time allocations and impact as functions of Nature's valuation.

As can be seen from Fig. 1, when $C = 0$, no time is allocated to green ideas, $t_R = 0$. Half of the available time is allocated to brown ideas, $t_A = 0.5$, and the other half is allocated to leisure and consumption. The impact t_A/t_R is above one for $C < 0.5$. The impact of 1.7 mentioned in the Dasgupta Review is attained for $C = 0.22$. As Nature's valuation increases, t_A is reduced and t_A increases, but their sum is always 1/2.

2. $\alpha = r > 0, q_A = q_R = q$

In this case, the key part of integrand (9) to be maximized by the choice of t_A, t_R becomes

$$(1+q)\left(\alpha t + \ln P(t) + \ln t_A + \ln(1 - t_A - t_R)\right) + C\left(G(S) - \frac{t_A}{t_R}\right).$$

Maximization results in

$$t_A^* = \frac{2(1+q) + C - \sqrt{C}\sqrt{4(1+q)+C}}{4+2q},$$
$$t_R^* = \frac{-C + \sqrt{C}\sqrt{4(1+q)+C}}{4+2q},$$

with $t_A^* + t_R^* = \frac{1+q}{2+q}$, so the the budget constraint is satisfied. Fig. 2 depicts the optimal choice as functions of C, and the spillover parameter q.

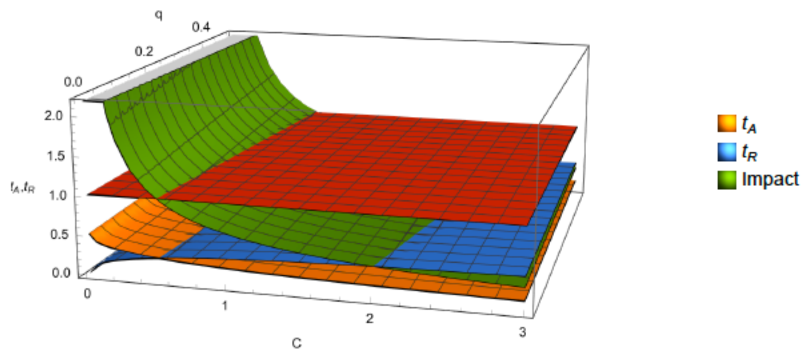


Fig. 2. Optimal time allocations and impact as functions of Nature's valuation and spillover.

The behavior of the time allocations and the impact with respect to Nature's valuation, C, are the same as in case 1. Time allocations increase with the spillover parameter q for any value of C.

The simplified model provides an important insight into the role of Nature's valuation in the regulator's problem. When Nature is valued relatively low, less time is allocated to the production of green ideas and the associated

Nature-preserving material goods. This leads to an impact value greater than 1, indicating that the impact inequality has a negative sign. In the following sections, we consider the general model. Although closed form solutions cannot be derived due to their complexity, the advantage of the general model is that its parameters can be calibrated using real-world data. This allows us to gain insights into the evolution of the pressure exerted on Nature's services by the global economy, as well as the regulatory structures that could help move the REE closer to a socially optimal outcome.

3.2 The soft and hard constraints for the SOMP

We consider the general model described in Section 2. The SCP (9) has no dynamics and the integrand is maximized term-by-term for each t. The optimality conditions are

$$\begin{aligned} t_A^*(t): \frac{1+q_A}{t_A} - \frac{C(1+q_A)\,(t_A(t))^{q_A}\,(A(t)P(t))^{1+q_A}}{(t_R(t)R(t)P(t))^{1+q_R}} &= \frac{1}{1-t_A(t)-t_R(t)} \\ t_R^*(t): \frac{C(1+q_R)\,(t_A(t)A(t))^{1+q_A}\,(P(t))^{q_A}}{t_R\,(t_R(t)R(t))^{1+q_R}\,(P(t))^{q_R}} &= \frac{1}{1-t_A(t)-t_R(t)}. \end{aligned} \tag{18}$$

The HCP (12)–(13) also has no dynamics and the integrand is maximized term-by-term for each t. The optimality conditions are

$$\begin{aligned} t_A^*(t): \frac{1+q_A}{t_A} - \frac{\mu(t)(1+q_A)\,(t_A(t))^{q_A}\,(A(t)P(t))^{1+q_A}}{(t_R(t)R(t)P(t))^{1+q_R}} &= \frac{1}{1-t_A(t)-t_R(t)} \\ t_R^*(t): \frac{\mu(t)(1+q_R)\,(t_A(t)A(t))^{1+q_A}\,(P(t))^{q_A}}{t_R\,(t_R(t)R(t))^{1+q_R}\,(P(t))^{q_R}} &= \frac{1}{1-t_A(t)-t_R(t)} \\ \mu^*(t): G(S(t)) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}} &= 0, \end{aligned} \tag{19}$$

where μ is the Lagrangian multiplier associated with constraint (13).

3.3 The soft and hard constraints for the REE

As in the SOMP case, the SCP and the HCP have no dynamics and the integrands are maximized term-by-term for each t. The optimality conditions are:

For the SCP

$$
\begin{aligned}
t_A^R(t) &: \frac{1}{t_A} - \frac{C\,(t_A^e(t))^{q_A}\,(A(t)P(t))^{1+q_A}}{t_R(t)\,(t_R^e(t))^{q_R}\,(R(t)P(t)^{1+q_R}} = \frac{1}{1-t_A(t)-t_R(t)} \\
t_R^R(t) &: \frac{C t_A(t)\,(t_A^e(t))^{q_A}\,(A(t)P(t))^{1+q_A}}{(t_R(t))^2\,(t_R^e(t))^{q_R}\,(R(t)P(t)^{1+q_R}} = \frac{1}{1-t_A(t)-t_R(t)}
\end{aligned}
\tag{20}
$$

For the HCP

$$
\begin{aligned}
t_A^R(t) &: \frac{1}{t_A} - \frac{\mu(t)\,(t_A^e(t))^{q_A}\,(A(t)P(t))^{1+q_A}}{t_R(t)\,(t_R^e(t))^{q_R}\,(R(t)P(t)^{1+q_R}} = \frac{1}{1-t_A(t)-t_R(t)} \\
t_R^R(t) &: \frac{\mu(t)t_A(t)\,(t_A^e(t))_A^{q_A}\,(A(t)P(t))^{1+q_A}}{(t_R(t))^2\,(t_R^e(t))^{q_R}\,(R(t)P(t)^{1+q_R}} = \frac{1}{1-t_A(t)-t_R(t)} \\
\mu^R(t) &: G(S(t)) - \frac{\left(\hat{A}_0 e^{\alpha t} t_A(t)P(t)\right)\left(\hat{A}_0 e^{\alpha t} t_A^e(t)P(t)\right)^{q_A}}{\left(\hat{R}_0 e^{rt} t_R(t)P(t)\right)\left(\hat{R}_0 e^{\alpha t} t_R^e(t)P(t)\right)^{q_R}} = 0
\end{aligned}
\tag{21}
$$

where μ is the Lagrangian multiplier. To obtain the optimal values for (t_A, t_R), we use the REE condition $t_R = t_{R,}^e, t_A = t_A^e$ in (20) and (21) and solve for each t. By comparing the SOMP optimality conditions with the REE optimality conditions, it is clear that the socially optimal equilibrium time allocation is different relative to the REE time allocation, since at the REE the knowledge spillovers – which represent a positive externality – are not fully internalized.

3.4 Implementing the socially optimal solution

A social planner acting as a world authority can implement the SOMP time allocation by correcting for the spillover distortions of the REE. The policy framework for the implementation of the socially optimal solution can be defined in terms of wages per unit time where wages are in units of dollars or "market baskets". Each human has an endowment of time, e.g. 24 hours/day. Full income per day is $T \cdot w$ where w is wage rate per hour and $T < 24$. The way in which the wage w is fixed can be regarded in the following way. Consider the maximum utility per day that can be attained given T hours/day. If an extra hour of labor time is provided, e.g. $T+1$ hours/day from T hours/day, extra utils are $\Delta U =: \max(U/(T+1)h) - \max(U/Th)$. Extra utils can be converted into consumables by dividing by the marginal utility of consumption. In this way, we get a wage w in units of consumables per day. Once the time budget constraint is converted into income by multiplying the time budget (3) by w, we can define the implementation policy – taxes, subsidies and lump sum transfers – in the usual way.

If, for example, the REE is characterized by more time allocated to brown goods (t_A) and less time allocated to green goods (t_R) relative to the social optimum, an implementation strategy would be to tax t_A and subsidize t_R. For budget balance, the world authority could capture the subsidy expenditures lump sum so SOMP is implemented with a balanced budget. Let (τ, s, t_{tr}) denote tax rate, subsidy rate and transfer respectively. Then the budget constraint would be

$$t_C^R(t) = 1 - (1 - \tau(t))t_A(t) - (1 + s(t))t_R(t) + t_{tr}(t), \tag{22}$$

with

$$\tau(t)t_A(t) + s(t)t_R(t) + tr(t) = 0, \tag{23}$$

so that the budget constraint for the time allocation is satisfied for all t. The objective would be to choose (τ, s, t_{tr}) such that the optimality conditions of the regulated REE coincide with the optimality conditions for the SOMP.

With the budget constraint (22), the objective for the POMP becomes

$$\max_{t_A, t_R} J(t_A, t_R, t; \tau, s, tr) = \max_{t_A, t_R} \int_0^\infty e^{-\rho t} \left\{ \ln \left[\left(\hat{A}_0 e^{\alpha t} t_A(t) P(t) \right) \left(\hat{A}_0 e^{\alpha t} t_A^e(t) P(t) \right)^{q_A} \left(t_C^{RE}(t) \right) \right] \right. \tag{24}$$

$$\left. -C \left[G(S) - \frac{\left(\hat{A}_0 e^{\alpha t} t_A(t) P(t) \right) \left(\hat{A}_0 e^{\alpha t} t_A^e(t) P(t) \right)^{q_A}}{\left(\hat{R}_0 e^{rt} t_R(t) P(t) \right) \left(\hat{R}_0 e^{\alpha t} t_R^e(t) P(t) \right)^{q_R}} \right] \right\} dt,$$

subject to (22) and (23), so that the budget constraint is satisfied for all t.

Given the nonlinearity of the problem, the optimal instruments can be determined in two stages.

1. First, consider the SCP for the SOMP and let $(t_A^*(t), t_B^*(t))$ be the solutions of optimality conditions (18). Consider the optimality conditions for the SCP-POMP problem with objective function (24). The optimality conditions will be similar to (20) but will also be functions of the instruments $(\tau(t), s(t), tr(t))$ for all t. Let the optimality conditions be of the general form:

$$F_A \left(t_A(t), t_R(t), t; \tau(t), s(t), tr(t), C \right) = 0 \tag{25}$$
$$F_R \left(t_A(t), t_R(t), t; \tau(t), s(t), tr(t), C \right) = 0 \,.$$

In order to attain the SOMP solutions for $(t_A(t), t_R(t))$ from the POMP optimality conditions of the regulated POMP, the system of (25) and (23) evaluated at the SOMP solutions is solved for the unknowns $(\tau(t), s(t), tr(t))$, for all t, or

$$\begin{aligned} F_A\left(t_A^*(t), t_R^*(t), t; \tau(t), s(t), tr(t), C\right) = 0 \\ F_R\left(t_A^*(t), t_R^*(t), t; \tau(t), s(t), tr(t), C\right) = 0 \\ \tau t_A^*(t) + s t_R^*(t) + tr = 0. \end{aligned} \tag{26}$$

2. Next, to verify that the solution $(\tau^*(t), s^*(t), tr^*(t))$ of system (23) actually generates the SOMP solution for all t, this solution is substituted into (24) and the regulated POMP problem is solved. Verification requires that the $(t_A(t), t_R(t))$ solution reproduce the SOMP in terms of time allocation but also, and most importantly, in terms of the ratio $I/G(S)$. If, in the SCP for the SOMP, Nature's valuation is such that $I/G(S) \leq 1$, then the regulated POMP should reproduce this result.

The implementation stages are presented in a more detailed way in the simulations part of the paper. A similar approach can be used for the HCP.

3.5 Optimal growth and biosphere dynamics

In the associated optimal control problem, $S(t)$ is a state variable while the objective is to choose paths for $t_A(t), t_R(t)$. From the current value Hamiltonian (14), the necessary conditions for optimality – noting from (14) that the derivatives of the Hamiltonian with respect to the controls are independent of the state S – can be stated as

$$t_A^*(t): \frac{\partial \mathcal{H}(S, \lambda, t_A, t_R, t)}{\partial t_A} = 0 \Rightarrow g_A\left(t_{A.}, t_R, \lambda, t\right) = 0 \tag{27}$$

$$t_R^*(t): \frac{\partial \mathcal{H}(S, \lambda, t_A, t_R, t)}{\partial t_R} = 0 \Rightarrow g_R\left(t_{A.}, t_R, \lambda, t\right) = 0\,. \tag{28}$$

Assuming that the conditions for the application of the implicit function theorem applies, then (27) and (28) can be solved as functions of the costate variable λ, or

$$t_A^*(t) = h_A(\lambda, t) \tag{29}$$

$$t_R^*(t) = h_R(\lambda, t)\,. \tag{30}$$

Note that the control functions are independent of the state variable S. Given these control functions, the Hamiltonian system can be written as

$$\dot{\lambda} = \lambda\left(\rho - G'(S)\right) \quad (31)$$

$$\dot{S} = G(S) - I(\lambda, t)\,, \quad (32)$$

where $I(\lambda, t)$ is the impact function with $t_A(t)$ and $t_R(t)$ substituted by the optimal controls (29) and (30).

To study the solution of this problem, consider for simplicity the case in which the population remains constant and there is no exogenous technical change, so that the problem is autonomous. Consider a steady state for (31) where $\dot{\lambda} = 0$. Then a steady state for Nature can be defined as $S^* : \rho = G'(S^*)$. This is a well-known result from the theory of optimal management of renewable resources. If Nature's dynamics follow the standard logistic growth law, then the steady state for Nature, S^*, will be less than the maximum sustainable yield level. When S is at a steady state, $\dot{S} = 0$, then (32) implies

$$G(S^*) = I(\lambda). \quad (33)$$

If (33) has a solution, then $\lambda^* = v\left(S^*\right)$.

For the non-autonomous problem, (31) can be solved, given the function $G(S)$, to obtain $\lambda(t) = \lambda_0 e^{\int_0^t \left(\rho - G'(S(u))\right) du}$. This solution can be substituted into (32) to solve for the optimal path for $S^*(t)$. The optimal path for the costate is then determined as

$$\lambda^*(t) = \lambda_0 e^{\int_0^t \left(\rho - G'(S^*(u)\right) du},$$

and the optimal paths for the controls (t_A, t_R) are obtained by substituting $\lambda^*(t)$ into (29) and (30).

The purpose of the models presented in this section is to explore the way in which optimal time allocations between activities that produce brown goods and Nature-saving goods – given exogenous technical change and spillover effects – will affect the evolution of the impact inequality. The SCP provides insights about the relationship between the valuation of Nature reflected in C and the sign of the impact inequality, while the HCP provides insights about the implicit shadow value of Nature's flow of services, μ, that equates Nature's supply of services with the economy's demand. The SCP can, therefore, provide insights regarding the optimal time allocation and the level of valuation C that keeps the demand for Nature's services below the supply and determines the associated growth of per capita output. The HCP provides similar insights when demand and supply are always in balance. The solution of the optimal control model provides the socially optimal path of the impact inequality and the associated path of output.

Although the statement of the optimality conditions which were presented in this section for all the models is relatively straightforward, the nonlinearities involved prevent the derivation of closed form solutions. Thus, in order to obtain some of the insight described in the previous paragraph, we resort to numerical simulations.

4 Numerical results

4.1 Calibration

The exogenous growth rates for brown and green ideas and the associated production of brown and green material goods are defined by the functions $\hat{A}_0 e^{\alpha(t)t}, \hat{R}_0 e^{r(t)t}$ respectively. We approximate $\alpha(t)$ by TFP which can be regarded as the main driver of output growth in the 20th century and has been associated, to a large extent, with brown goods. We set $\alpha(t) = 1\%$ as a long-term approximation of the estimated TFP in advanced economies for the period 1890–2015 (Bergeaud et al., 2017). We approximate the rate of growth for Nature-saving ideas by the declines in the levelized cost of energy (LCOE) for utility-scale renewable energy generation technologies.

More specifically, we consider the paths of LCOE for solar photovoltaics (S-PV) at the utility level and wind on-shore for the period 2009–2023 (Lazard, 2023). For S-PV there is a 7.5% average annual reduction between 2009–2023 and a 5.3% reduction between 2013–2023, with the reduction being much slower after 2018. For on-shore wind, there is a 4.5% average annual reduction in LCOE between 2009–2023 and a 3% reduction between 2013–2023, with the LCOE being almost constant after 2018. Furthermore, the ratio of wind-to-solar in renewable energy in 2023 was 62% to 48%. Looking forward, it should not be expected that the LCOE for renewable energy will keep decreasing. It seems that the LCOE should be stabilized at a lower bound like the fossil fuel LCOE. In view of these points, we considered a time dependent growth $r(t) = \bar{r} + r_0 e^{-r_1 t}$, which by 2100 will converge to the overall TFP rate and remain constant. Thus, $\bar{r} = 0.01$, with r_0 being chosen such that at $t = 0$, which is regarded as the present period, the LCOE decline rate is equal to the weighted average decline rates for S-PV and wind on-shore for the period 2013–2023 which is 0.044. Finally, r_1 is calibrated such that $r(t) \to \bar{r}$ after 75 periods. The parameters $\hat{A}_0, \hat{R}_0$ were set equal to one.

For the growth spillover parameter q_A, we follow Yilmazkuday (2024, section 6.2, Table 1) which indicates that when all statistically-significant country pairs are considered (3,732 out of 7,140), the statistically-significant median growth spillover is about 0.241. That is to say, for each 1% positive growth shock in the source country, the receiver gets on average a 0.241% increase in growth. Using this information, the spillover parameter was calculated as $q_A = \ln(1.00241)/\ln(1.01) = .009926 \approx 0.01$. For q_R, we could not find a similar result. Assuming that since green technologies are more recent the spillovers might be stronger, we considered a slightly higher value of $q_R = 0.012$.

To model population growth, we use information from a report by the United Nations Department of Economic and Social Affairs, Population Division (2024, page 8) which states that "The world's population is expected to continue growing over the coming fifty or sixty years, reaching a peak of around 10.3 billion people in the mid-2080s, up from 8.2 billion in 2024. After peaking, the global population is projected to start declining gradually, returning to 10.2 billion people by the end of the century." In view of this, population growth was modeled by a logistic function and the growth rate was calibrated such that, with an initial value of 8.2, the upper bound of the associated sigmoid path after 75 periods is 10.3.

For biosphere dynamics, we assume logistic growth of the general form

$$G(S(t)) = rS(t)\left(1 - \frac{S(t)}{K}\right). \tag{34}$$

The Dasgupta Review (2021, Box 4.6) states that currently the global ecological footprint, which is the current demand for Nature's services, exceeds the sustainable supply by Nature by a factor of 1.7. In order to make our results compatible, we calibrated the logistic growth parameters such that the maximum sustainable yield (MSY) corresponding to (34) is one.[5] This captures the idea that the biosphere's services are used in the neighborhood of the MSY, an assumption which could be supported by the recent results about approaching or even crossing planetary boundaries. Using the value of one as the MSY and the rest of the parameters in (34) should be regarded as a normalization that allows comparisons with the factor of 1.7.

4.2 Growth under soft and hard Nature constraints

4.2.1 The socially optimal solution

The soft constraint In order to obtain a benchmark point for our calculations that is consistent with the result that the demand for Nature's services exceeds the corresponding supply by a factor of 1.7, we numerically solve the optimality conditions (18) of the SCP using the calibration described in the previous section with cost C chosen such that the ratio $I/G(S)$ is 1.7 at

[5] In this calibration, $K = 200$ and $r = 0.02$.

$t = 0$.[6] Keeping C constant, we solve the SCP problem for $t = \{0, 1, 2, ..., 1,000\}$. The optimal time allocations for the first 100 periods are shown in Fig. 3 as $(t_A(SOMP), t_R(SOMP))$.[7] The ratio $I/G(S)$ ranges from 1.7 at $t = 0$ to 1.57 at $t = 1,000$. Per capita output grows at an average rate of 1.13% for the first 100 periods. If we interpret the cost C as the implicit valuation of Nature's services when the demand for and supply of these services are not in equilibrium, this result suggests that with such a valuation the demand for Nature's services will always exceed supply. This implies a continuous decumulation of Nature's "stock". Therefore, although per capita output grows persistently, this growth is not sustainable since the biosphere's stock will eventually be depleted.

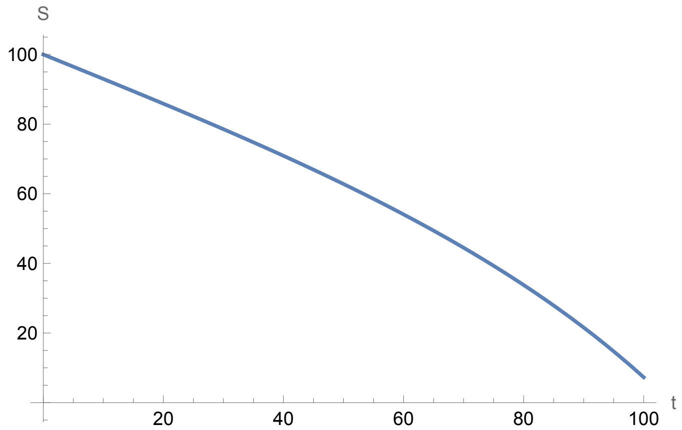


Fig. 3. Biosphere depletion.

To find a sustainable solution, the implicit valuation of Nature's services was increased up to a point at which the current $I/G(S)$ was below one. This value of C exceeded by 125% the value that resulted in the 1.7 ratio at $t = 0$ and implies that, in order to turn the impact inequality into an equality, Nature's valuation should be substantially increased. The optimal time allocations for the first 100 periods are shown in Fig. 4 as $(t_A(SOMP), t_R(SOMP))$. The ratio $I/G(S)$ ranges from 0.989 at $t = 0$ to 0.926 at $t = 1,000$. Per capita output grows at an average rate of 1.13% for the first 100 periods and the level of output per capita in this case is approximately 20% less relative to the case in which the $I/G(S)$ ratio ranged from 1.7 to 1.57. Thus sustainability – given the current state regarding technical change, growth spillovers and population growth – requires appropriate valuation of Nature's services, which in turn has negative effects on the level of per capita output relative to the unsustainable case.

[6] The supply of Nature's services was normalized to 1 and treated as exogenous.

[7] The Hessian matrix at the optimal (t_A, t_R) which was numerically calculated for all time periods is negative definite; therefore, second-order conditions are satisfied. The second-order conditions are satisfied for all the solutions that follow.

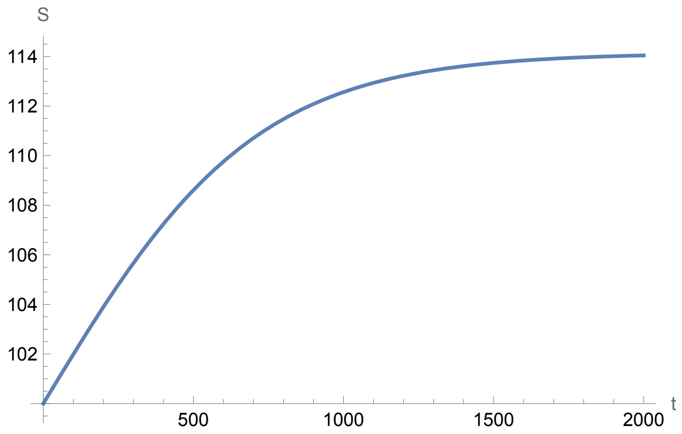


Fig. 4. Biosphere preservation in the long run with impact at the MSY.

The biosphere with dynamics described by (5), the growth function given by (34) and the impact kept constant at 1.7 will be depleted after 104 periods as shown in Fig. 3. On the other hand, with the impact kept constant at 0.98, the biosphere will converge in the long run to a steady state which is approximately the stock that corresponds to MSY demand as shown in Fig. 4.

The insights from this exercise suggest that, with realistic estimates regarding the evolution of technical change and population, if the current demand for Nature's services exceeds the supply by a factor of 1.7 and this excess demand remains persistent, then the situation is not sustainable in the sense of Nature's sustainability and the biosphere tends to depletion. If the valuation for ecosystem services substantially increases, this will increase the time allocated to develop Nature-saving technologies and reduce the time allocated to brown technologies. This will provide a sustainable path for the biosphere and will reduce per capita output relative to the unsustainable case.

Sensitivity analysis To explore the sensitivity of this "unsustainability" result to technical change and spillovers effects, we conducted sensitivity tests to the benchmark solution in which Nature's valuation is such that the ratio $I/G(S)$ is equal to 1.7 at $t = 0$. First we solve the SCP with Nature's valuation that resulted in $I/G(S) = 1.7$ at $t = 0$ with the assumption that the path of Nature-saving technical change will converge to a level which is twice the current long-run TFP rate of $\bar{r} = 0.01$, that is, $\bar{r} = 0.02$. This seeks to capture recent results (Way et al., 2022) indicating persistent reduction in the LCOE of renewable energy driven by Wright's law that these costs drop as a power law of cumulative production. Under this assumption, the $I/G(S)$ ratio starts at 1.7 for $t = 0$, but drops to 1.1 after 100 periods and to 0.67 after 200 periods, which a persistent declining time path. This suggests biosphere sustainability.

It should be noted that the increase in the exogenous rate of growth of green technology allows relatively higher allocation of time to brown goods/ideas and lower time allocation to green goods/ideas since the rate of green technical change compensates for the reduction in time allocated to green goods/ideas. The outcome is an increase in the output per capita level of 24.7% at $t = 100$ relative to case of $\bar{r} = 0.01$.

To test the sensitivity to an increase in the brown TFP rate, we increased α from 1% to 2% while the rate of growth of the exogenous Nature-saving technical change remains at the benchmark level. This increase results in a relative decrease in the time allocated to brown goods/ideas and a relative increase in the time allocated to green goods/ideas. The sustainability conditions worsen relative to the benchmark case with the $I/G(S)$ ratio increasing from 1.7 at $t = 0$ to 2.4 after 100 periods.

Finally, we increased the green spillover from 0.2 to 0.5 by keeping $\bar{r} = 0.01$, and the brown spillover fixed at 1.1. In this case, the $I/G(S)$ ratio declines at a much slower rate relative to the case of increasing the growth rate of Nature-saving technology. The ratio approaches one after 200 periods. After 100 periods, the level of output per capita increases by 15.3% relative to the case of $q_R = 0.2$.

The hard constraint In the hard constraint case, demand and supply are always in equilibrium. The optimal time allocation for (t_A, t_R) obtained by numerically solving the optimality conditions (19) is very similar to the paths for $(t_A HC, t_R HC)$ in Fig. 3, while the Lagrangian multiplier μ varies between 0.48 and 0.467. The $I/G(S)$ is equal to one for the 1,000 periods for which the ratio was calculated. The average per capita growth for the first 100 periods is 1.16%, while the per capita output exceeds that of the sustainable case under the soft constraint presented in Section 4.2.1, by 2%–4.5% during the first 100 periods. This result follows from the fact that the Lagrangian multiplier μ that reflects the shadow value of Nature's supply varies at each period of time, as shown in Fig. 4, in response to changes in the technology and population levels. This increases the efficiency of the whole system and allows it to attain relatively higher growth rates and levels of per capita output.

4.2.2 Rational expectations equilibrium

We solve for the REE first by assuming that Nature's valuation parameter C for the soft constraint is: (i) the same as the one that produced a current ratio of $I/G(S) = 1.7$ at the SOMP, and (ii) the same as the one that produced an $I/G(S)$ ratio in the neighborhood of 0.98 at the SOMP. We also solve for the REE under the hard constraint that $I/G(S) = 1$ for all t.

The soft constraint We numerically solve the optimality conditions (20). For the parametrization (i) at $t = 0$, the $I/G(S)$ ratio is 1.675 but after 100 periods it increases to 2.43. The growth rate of output per capita during the same period is 0.78% compared to 1.13% for the SOMP, while output per capita at the REE is lower relative to the SOMP case by 4% at $t = 0$ and by 25% at $t = 100$.

For the parametrization (ii) at $t = 0$, the $I/G(S)$ ratio is 0.98 but after 100 periods it increases to 1.34, which suggests that the REE is unsustainable in terms of the biosphere, while for the same valuation for Nature the SOMP is sustainable. The growth rate of output per capita during the same period is 0.65% compared to 1.13% for the SOMP, while output per capita at the REE is lower relative to the SOMP case by 3.5% at $t = 0$ and by 40% at $t = 100$.

The hard constraint We numerically solve the optimality conditions (21). The time allocations at the REE satisfy the hard constraint for the 100 periods. The growth rate of output per capita during the same period is 0.43% relative to 1.15% for the SOMP, while output per capita at the REE is lower relative to the SOMP case by 5% at $t = 0$ and by 53% at $t = 100$.

The SOMP solution outperforms the REE in terms of levels and growth of per capita output and in the ability to attain a sustainable solution with the soft constraint. This is because the SOMP, in contrast to the REE, fully internalizes the positive externality of spillovers in the production of brown and green goods.

The time allocations for the SOMP and the POMP for the two cases of the soft constraint and the case of the hard constraint are presented in Figs. 5, 6 and 7.

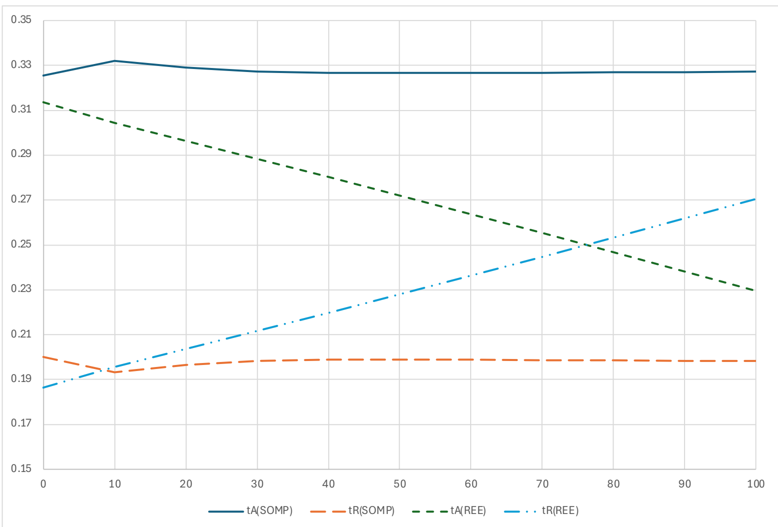


Fig. 5. Soft constraint and Nature unsustainability for SOMP-REE.

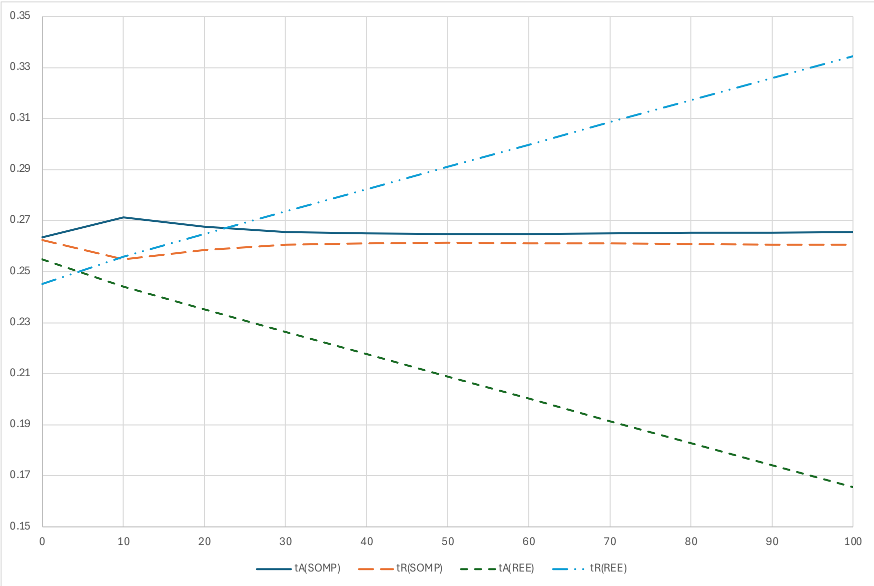


Fig. 6. Soft constraint and Nature sustainability for SOMP, and Nature unsustainability for REE.

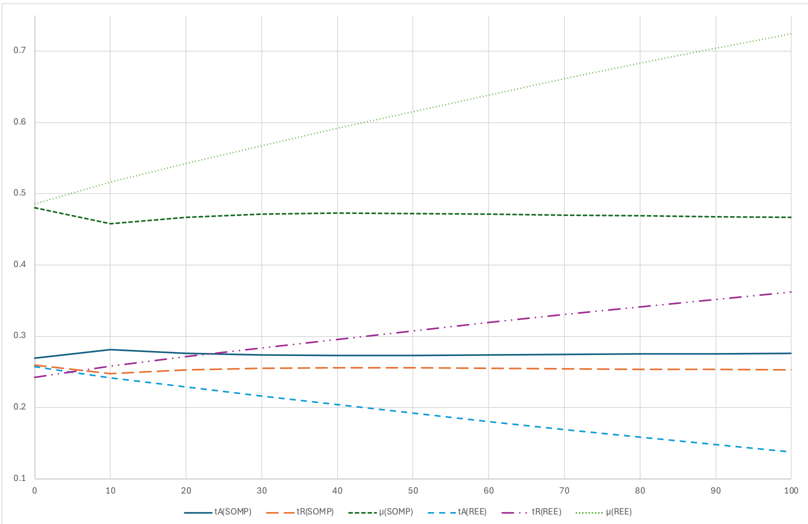


Fig. 7. Hard constraint for SOMP and POMP.

Moving from the unsustainable (Fig. 5) to the sustainable at the SOMP (Fig. 6), the path for time allocated to brown goods (t_A) shifts downwards while the the path for time allocated to green goods (t_R) shifts upwards. A similar behavior is exhibited by the REE but the situation is not sustainable. In Fig. 6, both solutions satisfy the hard constraint. The REE solution is characterized by a higher shadow value for the biosphere. This is because spillovers are not fully internalized at the REE and therefore, in order to satisfy the constraint, the shadow value of the biosphere increases at the solution.

Fig. 8 presents the implementation of the SOMP at a regulated POMP. The regulatory instruments were calculated following the two-stage procedure described in Section 3.3, which is presented analytically in Appendix 1. The target SOMP solution was the one shown in Fig. 4 in which Nature's value is chosen so that the system is sustainable. As shown, the regulated REE solution provides biosphere sustainability similar to the SOMP solution.

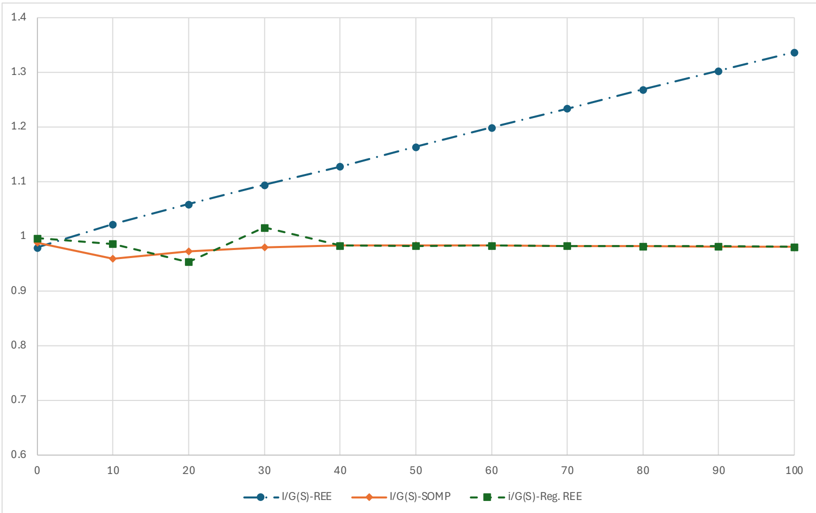


Fig. 8. Implementation of the SOMP solution at the regulated REE.

4.3 Optimal growth

With the functional specifications of this paper, the full optimal growth problem can be written as

$$J = \max_{t_A(t), t_R(t)} \int_0^\infty e^{-\rho t} \ln \left[\left(\left(\hat{A}_0 e^{\alpha t} t_A(t) P(t) \right)^{1+q_A} (1 - t_R - t_A) \right) \right] dt \qquad (35)$$

$$\text{subject to: } \dot{S}(t) = G(S(t)) - \frac{\left(\hat{A}_0 e^{\alpha(t)t} t_A(t) P(t) \right)^{1+q_A}}{\left(\hat{R}_0 e^{r(t)t} t_R(t) P(t) \right)^{1+q_R}}, \qquad (36)$$

$$S(0) = S_0,$$

where

$$G(S(t)) = rS(t)\left(1 - \frac{S(t)}{K}\right) \quad (37)$$

$$P(t) = \frac{P_0}{(1 + P_1 e^{-\pi t})} \quad (38)$$

$$r(t) = r_0 + r_1 e^{r_2 t}, \quad (39)$$

with current value Hamiltonian function

$$H(t_A, t_R, S, \lambda) = \ln\left[\left(\left(\hat{A}_0 e^{\alpha t} t_A(t) P(t)\right)^{1+q_A} (1 - t_R - t_A)\right)\right] + \lambda(t)\left[G(S(t)) - \frac{\left(\hat{A}_0 e^{\alpha(t)t} t_A(t) P(t)\right)^{1+q_A}}{\left(\hat{R}_0 e^{r(t)t} t_R(t) P(t)\right)^{1+q_R}}\right]$$

where $\lambda(t) = \frac{\partial J}{\partial S}$ is the costate variable expressing the shadow value of the biosphere.

4.3.1 The time autonomous model

To provide a closed form solution, we consider the autonomous optimal control problem without spillover effects. The current value Hamiltonian, omitting t to simplify, is

$$\mathcal{H}(t_A, t_R, S, \lambda) = \ln\left[P_0 t_A (1 - t_A - t_R)\right] + \lambda\left[rS\left(1 - \frac{S}{K}\right) - \frac{t_A}{t_R}\right]. \quad (40)$$

The optimality conditions for the controls imply, for interior solutions,

$$t_A^* : \frac{1}{t_A} - \frac{\lambda}{t_R} = \frac{1}{1 - t_A - t_R} \quad (41)$$
$$t_R^* : \frac{\lambda t_A}{t_R^2} = \frac{1}{1 - t_A - t_R}.$$

Solving conditions (41), we obtain

$$t_A^*(\lambda) = 0.125\left(4 + 2\lambda \pm 2\sqrt{\lambda}\sqrt{4 + \lambda}\right) \quad (42)$$

$$t_R^*(\lambda) = -0.25\lambda \pm \sqrt{\lambda}\sqrt{4 + \lambda}. \quad (43)$$

Using the positive solutions for the controls, the dynamic Hamiltonian system is

$$\dot{\lambda} = \lambda \left(\rho - r \left(1 - \frac{2S}{K} \right) \right) \tag{44}$$
$$\dot{S} = rS \left(1 - \frac{S}{K} \right) - \frac{t_A^*(\lambda)}{t_R^*(\lambda)} \,.$$

We use the calibration values for the parameters to solve for the steady state. We chose the combination of roots from the solutions (42),(43) that provide a steady state with the saddle point property. The optimal steady state is $(S^*, \lambda^*) = (50, 0.762)$ and it has the saddle point property with associated eigenvalues $(e_1, e_2) = (0.0164, -0.0064)$. The steady-state time allocation ratio is $\frac{t_A^*}{t_R^*} = \frac{0.214}{0.286} \approx 0.75$, while the steady-state growth of the biosphere $G(S^*) = G(50) = 0.75$. Thus, at the steady state, the demand for and supply of Nature's services are equal and the biosphere is in a state of equilibrium. Although this version of the model is highly simplified, it points out the optimality of keeping the biosphere in equilibrium and stresses the role of Nature-saving technical change which, in the simplified model, is reflected in t_R.

4.3.2 Spillover effects

To obtain insights into the impact of the spillover effects on the optimal steady state, we introduce spillover effects into (40) and obtain a new Hamiltonian that incorporates the spillover effects:

$$\mathcal{H}(t_A, t_R, S, \lambda; q_A, q_R) = \ln \left[P_0 (t_A)^{1+q_A} (1 - t_A - t_R) \right] + \lambda \left[rS \left(1 - \frac{S}{K} \right) - \frac{(P_0 t_A)^{1+q_A}}{(P_0 t_R)^{1+q_R}} \right].$$

To solve the resulting nonlinear control problem, we take a first-order expansion around $q_A = q_R = 0$. Solving the optimal control model for positive values of (q_A, q_R) in the neighborhood of $(0, 0)$ will provide information about the impact of positive spillovers on the socially optimal steady state. The expanded Hamiltonian is

$$\mathcal{H}^{\mathrm{EXP}}(t_A, t_R, S, \lambda; q_A, q_R) = \ln \left[P_0 (t_A)(1 - t_A - t_R) \right] + \lambda \left[rS \left(1 - \frac{S}{K} \right) - \frac{t_A}{t_R} \right]$$
$$q_A \ln (P_0 t_A) + \lambda \left[\frac{q_R t_A \ln (P_0 t_R)}{t_R} - \frac{q_A t_A \ln (P_0 t_A)}{t_R} + \frac{q_A q_R t_A \ln (P_0 t_A) \ln (P_0 t_R)}{t_R} \right]. \tag{45}$$

Approximating $\ln (P_0 t_R), \ln (P_0 t_A)$ at the steady-state values for (t_A, t_R) when $q_A = q_R = 0$, the new steady states resulting from small increases in the spillover parameters in the neighborhood of $(0,0)$ are shown in Table 1.

Table 1. The impact of spillovers at the steady state

q_A	q_R	t_A^*	t_R^*	S^*	λ^*	Utility
0	0	0.214	0.286	50	0.762	0.232
0.01	0.01	0.221	0.281	50	0.747	0.247
0	0.01	0.200	0.301	50	0.798	0.212
0.01	0	0.235	0.266	50	0.710	0.258
0.02	0.02	0.232	0.271	50	0.716	0.265
0	0.02	0.188	0.315	50	0.831	0.188
0.02	0	0.260	0.241	50	0.645	0.271

In all cases, the impact inequality holds as equality at the steady state. A uniform increase in the spillover effects increases steady-state utility and increases the time allocated to the production of material goods, while the time allocated to Nature-saving technologies decreases because these reductions are compensated by the spillover effects. This result is in line with the results of the sensitivity analysis of the SCP in Section 4.2.1. An increase in q_A with $q_R = 0$ increases welfare while the opposite happens when q_R increases and $q_A = 0$. This can be explained by the fact that q_R does not affect utility and therefore the role of the optimal control problem is to provide a signal for the optimal time allocation to green goods/ideas that will sustain the impact equality at the steady state.

4.3.3 The time dependent model

To provide results regarding the impact of the exogenous technical change from the time dependent model, we simplify (12)–(36) by setting $r(t) = \bar{r}, q_A = q_R = 0$, and assume constant population. We impose a terminal time T, and a terminal condition $S(T) = S(0) = S_0$. This condition reflects a sustainability constraint indicating that at the end of the planning horizon the biosphere should be the same as at the beginning. The current value Hamiltonian for the problem is

$$H(t_A, t_R, \lambda, S, t, \alpha, \bar{r}) = \ln\left[\left(\left(\hat{A}_0 e^{\alpha t} t_A(t)\bar{P}\right)(1 - t_R - t_A)\right)\right]$$
$$\lambda(t)\left[G(S(t)) - \frac{\left(\hat{A}_0 e^{\alpha t} t_A(t)\right)}{\left(\hat{R}_0 e^{\bar{r}t} t_R(t)\right)}\right].$$

We take a first-order expansion of this Hamiltonian around values $(\alpha_0, \bar{r}_0)$ and apply the maximum principle on the expanded Hamiltonian which is of the general form

$$\hat{H}(t_A, t_R, \lambda, S, t; \alpha, \bar{r}) = H(t_A, t_R, \lambda, S, t, \alpha_0, \bar{r}_0) + \frac{\partial H(\centerdot, \alpha_0, \bar{r}_0)}{\partial \alpha}(\alpha - \alpha_0) + \frac{\partial H(\centerdot, \alpha_0, \bar{r}_0)}{\partial \bar{r}}(\bar{r} - \bar{r}_0).$$

The optimal control problem can be solved for $(\alpha, \bar{r})$ values in the neighborhood of $(\alpha_0, \bar{r}_0)$. Let $t_A^*(\lambda, S, t; \alpha, \bar{r}, \alpha_0, \bar{r}_0), t_R^*(\lambda, S, t; \alpha, \bar{r}, \alpha_0, \bar{r}_0)$ be the controls that maximize the expanded Hamiltonian evaluated at some $(\alpha, \bar{r})$ in the neighborhood of $(\alpha_0, \bar{r}_0)$. Then the time dependent Hamiltonian system is

$$\dot{S}(t) = \frac{\partial \hat{H}}{\partial \lambda}$$
$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial \hat{H}}{\partial S}$$
$$S(0) = S_0 = S(T),$$

with all derivatives evaluated at $t_A^*(\lambda, S, t; \alpha, \bar{r}, \alpha_0, \bar{r}_0), t_R^*(\lambda, S, t; \alpha, \bar{r}, \alpha_0, \bar{r}_0)$. Because of the terminal condition on S, the transversality conditions imply that $\lambda(T)$ is free. We solve the problem at $(\alpha_0, \bar{r}_0) = (0.009, 0.009)$ and $(\alpha, \bar{r}) = (0.1, 0.1)$ which are the long-run TFP rates for brown and green technical progress, with $S(0) = 100 = S(T)$, so that the initial and terminal biosphere growth rate is at the MSY. Results are presented in Fig. 9 for $T = 100$.

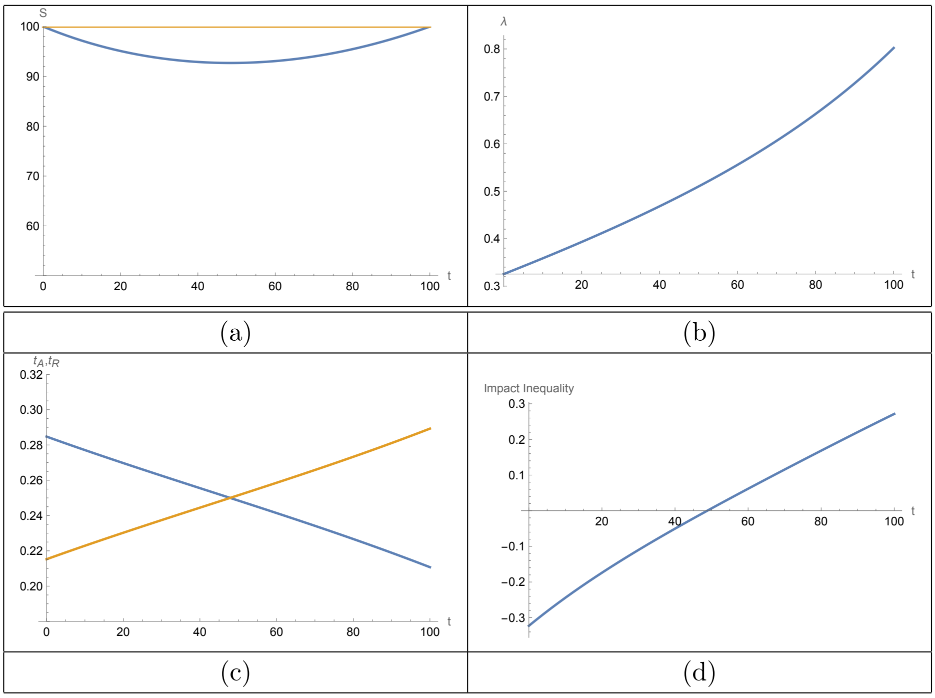


Fig. 9. (a) The optimal time path of the stock of biosphere $S(t)$. (b) The time path of the shadow value of the biosphere $\lambda(t)$. (c). Optimal time

allocation with $t_A(t)$ decreasing, $t_R(t)$, increasing (d). The time path of the impact inequality.

The results in Fig. 9 indicate that optimal control methods can provide solutions that conserve the biosphere (9(a)) and can turn the impact inequality from negative to positive for half of the planning horizon (9(d)). The shadow value of the biosphere (9(b)) is critical since it should be used for policy design based on Nature's valuation. The optimal control framework can be used to study more ambitious targets that involve increasing the biosphere during the planning horizon. Fig. 10 presents the case in which the terminal value of the biosphere is set at 150 with an initial value of 100.

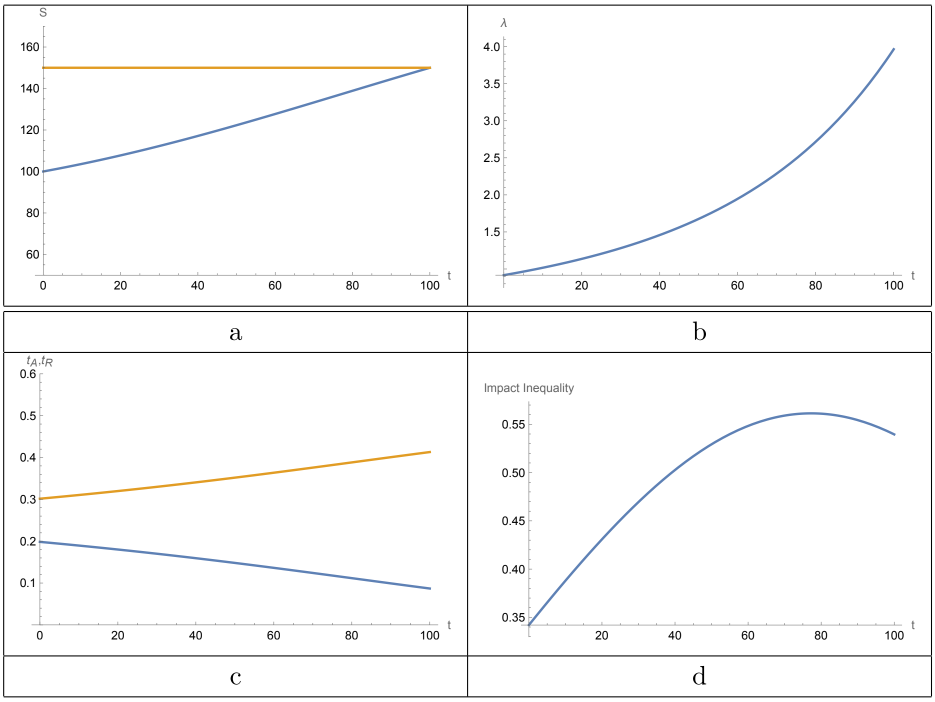


Fig. 10. (a) The optimal time path of the stock of biosphere $S(t)$. (b) The time path of the shadow value of the biosphere $\lambda(t)$. (c). Optimal time allocation with $t_A(t)$ decreasing, $t_R(t)$, increasing (d). The time path of the impact inequality.

To attain the more ambitious target, Nature's valuation increases, time allocated to green goods/ideas increases, and time allocated to brown goods/ideas decreases relative to the case illustrated in Fig. 9.

5 A forward-looking biosphere model

The social planner and the REE agents take into account the fact that the biosphere contributes explicitly to the evolution of the stocks of brown and green ideas, which can be written as

$$A\left(t_A(t), S(t), t, P(t)\right) = \left(\hat{A}_0 e^{\alpha(t)t} S(t)^{b_A} t_A(t) P(t)\right)^{1+q_A} \tag{46}$$
$$R\left(t_R(t), S(t), t, P(t)\right) = \left(\hat{R}_0 e^{r(t)t} S(t)^{b_R} t_R(t) P(t)\right)^{1+q_R}.$$

Then the demand side of the impact inequality, using (6), is

$$\hat{I}(t; q_A, q_R, \alpha, r, b_A, b_R) = \left(S(t)^{\hat{b}}\right) I(t; q_A, q_R, \alpha, r)$$
$$\hat{b} = b_A(1+q_A) - b_R(1+q_R).$$

Since the biosphere stock is endogenous, the problem should be analyzed in the context of optimal control. The optimal control problem for the social planner, following (35) and omitting (t) when possible to ease notation, is

$$J = \max_{t_A(t), t_R(t)} \int_0^\infty e^{-\rho t} \ln\left[\left(\left(\hat{A}_0 e^{\alpha t} S^{b_A} t_A P(t)\right)^{1+q_A} (1 - t_R - t_A)\right)\right] dt \tag{47}$$

$$\text{subject to}: \ \dot{S}(t) = G(S(t)) - S^{\hat{b}} \frac{\left(\hat{A}_0 e^{\alpha(t)t} t_A P(t)\right)^{1+q_A}}{\left(\hat{R}_0 e^{r(t)t} t_R(t) P(t)\right)^{1+q_R}}, \tag{48}$$
$$S(0) = S_0,$$

with current value Hamiltonian function

$$H(t_A, t_R, S, \lambda) = \ln\left[\left(\left(\hat{A}_0 e^{\alpha t} S^{b_A} t_A P\right)^{1+q_A} (1 - t_R - t_A)\right)\right] +$$
$$\lambda(t)\left[G(S) - S^{\hat{b}} \frac{\left(\hat{A}_0 e^{\alpha(t)t} t_A P\right)^{1+q_A}}{\left(\hat{R}_0 e^{r(t)t} t_R P\right)^{1+q_R}}\right],$$

where $\lambda(t) = \frac{\partial J}{\partial S}$ is the costate variable expressing the shadow value of the biosphere.

5.1 The time autonomous model

To provide a closed form solution, we consider the autonomous optimal control problem without spillover effects. The current value Hamiltonian, omitting t to simplify, is

$$H(t_A, t_R, S, \lambda) = \ln\left[P_0 S^{b_A} t_A(1 - t_A - t_R)\right] + \lambda\left[rS\left(1 - \frac{S}{K}\right) - \frac{S^{(b_A - b_R)} t_A}{t_R}\right]. \tag{49}$$

The optimality conditions for the controls imply, for interior solutions,

$$\begin{aligned} t_A^* &: \frac{1}{t_A} - \frac{\lambda S^{(b_A - b_R)}}{t_R} = \frac{1}{1 - t_A - t_R} \\ t_R^* &: \frac{\lambda t_A S^{(b_A - b_R)}}{t_R^2} = \frac{1}{1 - t_A - t_R}. \end{aligned} \tag{50}$$

Solving conditions (50), we obtain

$$t_A^*(\lambda) = 0.125\left(4 + 2\lambda S^{(b_A - b_R)} \pm \sqrt{\lambda S^{(b_A - b_R)}\left(16 + 4\lambda S^{(b_A - b_R)}\right)}\right) \tag{51}$$

$$t_R^*(\lambda) = -0.25\lambda S^{(b_A - b_R)} \pm 0.125\sqrt{\lambda S^{(b_A - b_R)}\left(16 + 4\lambda S^{(b_A - b_R)}\right)}. \tag{52}$$

Using the positive solutions for the controls, the dynamic Hamiltonian system is

$$\begin{aligned} \dot{\lambda} &= \lambda\left(\rho - r\left(1 - \frac{2S}{K}\right)\right) \\ \dot{S} &= rS\left(1 - \frac{S}{K}\right) - \frac{t_A^*(\lambda)}{t_R^*(\lambda)}. \end{aligned} \tag{53}$$

We use the calibration values for the parameters to solve for the steady state. We chose the combination of roots from the solutions (51),(52) that provide a steady state with the saddle point property. The optimal steady state is $(S^*, \lambda^*) = (50, 0.762)$ and it has the saddle point property with associated eigenvalues $(e_1, e_2) = (0.0164, -0.0064)$. The steady-state time allocation ratio is $\frac{t_A^*}{t_R^*} = \frac{0.214}{0.286} \approx 0.75$, while the steady-state growth of the biosphere $G(S^*) = G(50) = 0.75$. Thus, at the steady state, the demand for and supply of Nature's services are equal and the biosphere is in a state of equilibrium. Although this version of the model is highly simplified, it points out the optimality of keeping the biosphere in equilibrium and stresses the role of Nature-saving technical change which, in the simplified model, is reflected in t_R.

6 Sustainable development

The well-known definition of sustainable development from the Brundtland Report (Brundtland, 1987) describes it as "development that meets the needs of the present without compromising the ability of future generations to meet their own needs." Among the various definitions that follow the spirit of Brundtland's definition, a central concept – particularly in definitions that are rigorous enough to support theoretical and applied research – is the concept of *non-declining social welfare* or *non-declining intergenerational well-being over time* (Dasgupta and Mäler, 2001; Arrow et al., 2003, Arrow et al., 2012). In the context of the model developed in Section 2 in which the biosphere does not affect production, the non-declining social welfare definition can be stated, using the definition of the utility flow (7), as

$$\frac{dW(t)}{dt} \geq 0 \quad for\, all\, t \geq 0$$

$$W(t) = \int_t^\infty e^{-\rho(s-t)} u(s|t) + C\left[\overline{G(S)} - \frac{\left(A(z)t_A^*(s|t)P(z)\right)^{1+q_A}}{\left(R(z)t_R^*(s|t)P(z)\right)^{1+q_R}}\right] ds, z = s - t, \tag{54}$$

where the supply of Nature's services is considered as exogenous $\overline{G(S)}$, and $t_R^*(z), t_A^*(z)$ are the optimal time allocation obtained from the solution of the SCP. We focus on this problem because the evolution of the biosphere is not accounted for by the regulator, which can be regarded as an assumption not far from current practices in the real world. In such a case, if the impact inequality remains persistently negative, the services provided by Nature will be depleted. In equation (34), the evolution of Nature is modeled using logistic growth. Under this framework, exceeding the carrying capacity can be interpreted as transgressing planetary boundaries, leading to negative growth in ecosystem services. This, in turn, implies additional costs to the current valuation of Nature.

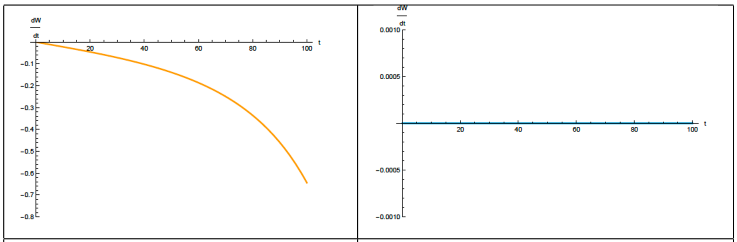
Our analysis builds on the simplified model presented in Section 3.1. As illustrated in Fig. 1, when the planner's implicit valuation of Nature's services is sufficiently low, the optimal time allocation (t_A^*, t_R^*) generates an environmental impact that exceeds Nature's supply at the MSY of the logistic growth function, normalized to 1 under our parametrization. In this regime, the impact inequality is negative. Moreover, the optimal allocation (t_A^*, t_R^*) is invariant with respect to both the level of Nature's services and the passage of time. In the absence of technical change and under a fixed population, if the social planner treats the supply of Nature's services as constant, the social welfare indicator can be written as

$$W(t) = \int_t^\infty e^{-\rho(s-t)} \left[\ln \left[t_A^* \left((1 - t_R^* - t_A^*) \right) \right] + C \left[\overline{G(S)} - \frac{t_A^*}{t_R^*} \right] \right] ds = \tag{55}$$

$$\frac{\ln \left[t_A^* \left((1 - t_R^* - t_A^*) \right) \right] + C \left[\overline{G(S)} - \frac{t_A^*}{t_R^*} \right]}{\rho}. \tag{56}$$

It is clear that $\frac{dW(t)}{dt} = 0$. Thus, in this case, development may appear to be sustainable even when the impact inequality is persistently negative. However, this conclusion is misleading, as it implicitly assumes that the supply of Nature's services remains constant, despite a persistently negative impact inequality. In reality, a negative impact inequality indicates that the demand for Nature's services exceeds the supply, which is determined by Nature's regenerative capacity, $G(S)$. This means that Nature's services will be declining. Assuming that at $t = 0$, $G(S) = 1$ – the MSY in (34) – and the time allocations (t_A^*, t_R^*) are such that the impact is maintained at 1.7, then $G(S)$ declines over time. As a result, the time derivative of the social welfare indicator becomes negative for all $t > 0$, as shown in Fig. 11a, indicating that development is, in fact, not sustainable. This result makes clear the impact of the low valuation of Nature's services, reflected in C, on sustainable development. If C is increased such that the new optimal time allocation (t_A^*, t_R^*) implies that the demand for Nature's services is equal to its supply, and that the impact inequality is persistently always zero, then $\frac{dW(t)}{dt} = 0$ as shown in Fig. 11b, and development is sustainable even when the planner does not take into account the impact of Nature's services in production. This simple model provides an important insight into conditions for sustainable development. Low valuation of Nature's services leading to excess demand for these services and a persistent negative impact inequality leads to non-sustainable development. A higher valuation resulting in no excess demand leads to sustainability.

This simple model highlights an important condition for sustainable development: a low valuation of Nature's services leads to excess demand, a persistently negative impact inequality, and ultimately, non-sustainability. In contrast, a higher valuation that eliminates excess demand ensures a balance between demand and supply, making sustainable development possible.



(a)	(b)

Fig. 11. (a) Low valuation of Nature's services and non-sustainable development. (b) High valuation of Nature's services and sustainable development.

The results are robust to the inclusion of technical change and population growth. Let $\alpha = 0.01$ and $r = 0.012$ denote the growth rates of brown and green ideas, respectively, with population growing at 1%. At $t = 0$, Nature's valuation and time allocation are set so that the environmental impact equals 1.7 and the impact inequality is -0.7. With time allocation held constant, the impact inequality initially falls due to technical change and population growth, but later rises as natural capital is depleted. The welfare flow declines monotonically, indicating unsustainable development. Sustainability is restored if Nature's valuation rises enough for the resulting time allocation to yield zero impact inequality.

6.1 Sustainability and optimality

In the previous section, the social planner incorporated potential welfare costs from a negative impact inequality, yet disregarded the dynamic evolution of natural capital. Once this evolution is explicitly incorporated, the planner's decision problem becomes an optimal control problem, represented by the corresponding Hamiltonian formulation,

$$\mathcal{H}(S,\lambda,t_A,t_R,t) = \ln\left[(A(t)t_A(t)P(t))^{1+q_A}\left((1-t_R(t)-t_A(t))\right)\right] \tag{57}$$

$$(C+\lambda(t))\left[G(S(t)) - \frac{(A(t)t_A(t)P(t))^{1+q_A}}{(R(t)t_R(t)P(t))^{1+q_R}}\right],$$

where C is the planner's arbitrary valuation of Nature's services and $\lambda(t)$ is the shadow valuation emerging from the optimization. Considering the simplified time autonomous model that can provide tractable results, the positive optimal time allocation is

$$t_A = 0.5 + 0.25(C + \lambda(t)) - 0.125\sqrt{-16 + (4 + 2C + 2\lambda(t))^2}$$
$$t_R = -0.25(C + \lambda(t)) + 0.25\sqrt{C^2 + \lambda(t)(4 + \lambda(t)) + C(4 + 2\lambda(t))}\,.$$

Using (34) for Nature's evolution, Table 2 shows the steady-state welfare along with the optimal steady-state values for λ and S corresponding to different values of Nature's valuation C. Note that all optimal steady states have the saddle point property.

Table 2. Optimal steady states

C	Social Welfare	λ	S
0	0.0985	0.76	50.0
0.1	0.1478	0.57	57.5
0.2	0.1927	0/39	66.8
0.3	0.2272	0.24	77.8
0.4	0.2469	0.11	89.3
0.5	**0.2527**	**0**	**100.0**
0.6	0.2484	-0.09	109.2
0.7	0.2379	-0.18	117.0
0.8	0.2339	-0.25	123.5
0.9	0.2079	-0.33	128.9
1.0	0.1908	-0.40	133.7

At the valuation that maximizes steady-state social welfare, time is allocated equally between activities ($t_A = t_R = 0.25$), and the impact inequality is zero, reflecting equilibrium between the demand for and supply of Nature's services. The saddle-point stability of this steady state ensures that the maximized welfare is sustainable. This confirms that sustainability requires the demand for and supply of Nature's services to be balanced, a condition attainable through appropriate valuation of Nature's services.

7 Endogenous fertility

In the previous sections, population growth was assumed to be exogenous. This section extends the model by endogenizing population dynamics, allowing it to depend on the representative individual's fertility choices. The unitary time endowment of the representative individual is allocated to raising and educating children, $n\tau$. This time is in addition to consuming and leisure time t_C, time that increases the stock of brown ideas t_A, and time that increases the stock of green ideas t_R.

The time devoted to raising and educating children is defined as $\tau = \tau^q + \tau^e e$, where τ^q, τ^e is time devoted to raising and educating children, respectively, and e is the education level. Following Galor (2012), the utility associated with the number of children and their education is represented by the term $\gamma\left[\ln n + \beta h\left(g, e\right)\right]$. We simplify this term to $\gamma \ln\left(yn\tau\right)$ by using the simplifying assumption $\beta h(e, g) = \beta \min\left\{e, g\right\} = \beta g$, where β is a parameter between 0 and 1. The underlying assumption is that by interpreting g as the level of technical progress, we let g "pull up" the demand for education e via a Leontief function. Then, by setting $\beta = 1$, $\tau = \tau^q + e\tau^e = \tau^q + g\tau^e$. Thus $n\tau = n\left(\tau^q + g\tau^e\right)$ is the total time spent in raising and educating children with y being its opportunity cost in dollars per unit of time. Therefore $yn\tau$ is dollars of consumption spent in the form of raising and educating children. This assumption tries to capture the role of g in pulling up the demand for education, i.e., human capital, in each child, thus increasing τ, which, in turn, raises the "price" of each child. When time for raising and educating children is introduced, the flow budget constraint regarding the time allocation of the representative individual is defined as

$$1 = t_C + t_R + t_A + n\tau. \tag{58}$$

The evolution of population is described by the sigmoid function,

$$P(t) = \frac{\bar{P}}{\left(1 + Ze^{(n-\delta)t}\right)}, \; Z = \frac{\left(\bar{P} - P_0\right)}{P_0}, \tag{59}$$

where δ is global mortality rate, $\bar{P}$ is the upper bound of the associated sigmoid path regarded as carrying capacity, and P_0 is the initial population level. Thus the intrinsic population rate of growth is the difference between the fertility rate n, which is a choice of the representative individual, and the mortality rate. Note that we are using the logistic formulation for the population growth instead of something such as $dP/dt = P(n - \delta)$ in order to produce a population growth trajectory that looks more like observed data and what we expect to happen as Earth's population and consumption gets closer to Earth's carrying capacity.

Assuming that the representative individual gives utility weights γ and $1-\gamma$ to utility from children and utility from consumption respectively, and that the population path is given by (59), the utility function can be written as

$$\begin{aligned} u(t_A, t_R, n) = (1-\gamma)\ln\left[\left(A(t)t_A(t)P(t,n)\right)^{1+q_A}\left(1 - t_R - t_A - n\tau\right)\right] \\ + \gamma \ln\left[\left(\left(A(t)t_A(t)P(t,n)\right)^{1+q_A}\right) n\tau\right]. \end{aligned}$$

Then the SCP in which the social planner chooses (t_A, t_R, n) can be written as

$$\max_{t_A, t_R, n} \int_0^{\infty} e^{-\rho t} \left[u(t_A, t_R, n) + C \left[G(S) - \frac{(A(t) t_A(t) P(t,n))^{1+q_A}}{(R(t) t_R(t), P(t,n))^{1+q_R}} \right] \right] dt. \quad (60)$$

The optimality conditions, which hold for every t, are

$$t_A^*(t): \frac{1+q_A}{t_A} - \frac{C(1+q_A)A(t)P(t,n)\left(A(t)t_A P(t,n)\right)^{q_A}}{\left(t_R R(t) P(t,n)\right)^{1+q_R}} = \frac{1-\gamma}{1-t_A-t_R-\tau n} \quad (61)$$

$$t_R^*(t): \frac{C(1+q_R)A(t)P(t,n)\left(A(t)t_A P(t,n)\right)^{q_A}}{t_R^2 R(t)\left(P(t,n)t_R R(t)\right)^{q_R}} = \frac{1-\gamma}{1-t_A-t_R-\tau n} \quad (62)$$

$$n^*(t): \frac{\gamma}{n} + \frac{C(q_A-q_R)\left(t_A A(t)\right)\left(t_A A(t) P(t,n)\right)^{q_A}\left(\frac{\partial P(t,n)}{\partial n}\right)}{\left(t_R R(t) P(t)\right)^{1+q_R}} + \frac{(1+q_A)\left(\frac{\partial P(t,n)}{\partial n}\right)}{P(t,n)} = \quad (63)$$

$$\frac{\tau(1-\gamma)}{1-t_A(t)-t_R(t)-\tau n(t)}.$$

Using the calibration for the SCP analyzed in Section 4.2.1, the initial and terminal population values of Section 4.1, a global average mortality rate of 7.76 deaths per one thousand individuals,[8] and initial values for $n\tau$ such that the population level after 75 years without any optimization is 10.3 billion (that is, equal to the UN predictions), the solution of problem (60) for $\gamma = \{0.5, 0.25, 0.75\}$ is shown in Fig. 12. The paths $(t_R, t_A, \tau n)$, $(t'_A, t'_R, \tau n')$, $(t''_A, t''_R, \tau n'')$ in Fig. 12 represent the time allocated to brown and green ideas production and child-rearing, corresponding to $\gamma = \{0.5, 0.25, 0.75\}$ respectively. For comparison, the paths t_A^* and t_R^* are those derived in Section 4.2.1 for the SCP which yields an persistent impact inequality of 1.7.

[8]See Macrotrends, World Death Rate 1950-2025, at https://www.macrotrends.net/global-metrics/countries/wld/world/death-rate (accessed May 28, 2025).

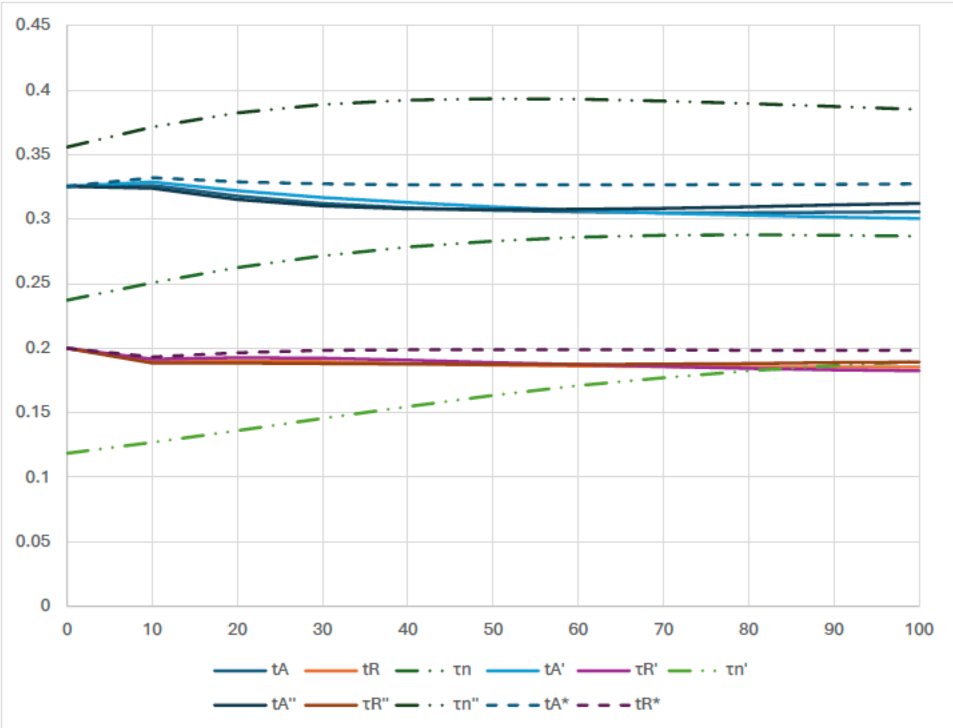


Fig. 12. Time allocation paths for $\gamma = \{0.5, 0.25, 0.75\}$.

When the representative individual incorporates utility from having children, the time allocated to producing brown and green ideas decreases relative to the case of $\gamma = 0$, as shown by comparison of paths $\{(t_A, t_R), (t'_A, t'_R), (t''_A, t''_R)\}$ with paths $\{t^*_A, t^*_R\}$. Changes in the utility weight γ associated with children do not significantly affect the time allocated to material goods production, but they do influence the time devoted to child-rearing, as shown by the trajectories $(\tau n, \tau n', \tau n'')$.

Fig. 13 displays the corresponding fertility paths $n(t)$ which can be interpreted as the chosen fertility rate by the representative individual at each time t. At the initial time $t = 0$, the fertility rate is 17.3 which coincides with the observed crude reproduction rate for 2025.[9] An increase in the utility weight for children, as expected, increases the fertility rate.

[9]See Macrotrends, World Birth Rate 1950-2025, at https://www.macrotrends.net/global-metrics/countries/wld/world/birth-rate (accessed May 28, 2025).

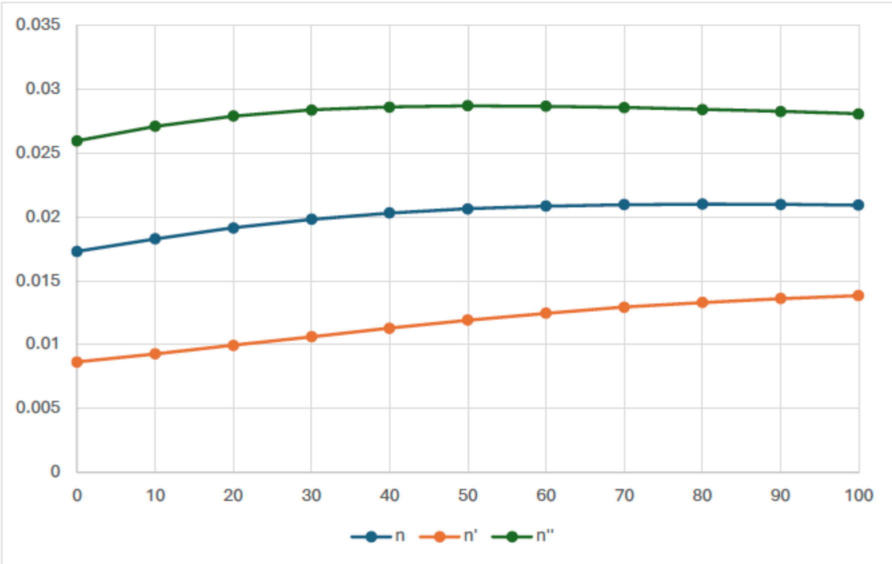


Fig. 13. Fertility paths for $\gamma = \{0.5, 0.25, 0.75\}$.

In Section 4.2.1, the implicit social value of Nature in the SCP model was calibrated so that the initial impact was 1.7, in line with the Dasgupta Review. It was shown that, under the currently projected trajectories of technological change and population growth, this impact level does not substantially decline over the next 100 years, indicating that the growth path is unsustainable in terms of both Nature and development. Maintaining the same social value of Nature as in Section 4.2.1, the endogenous fertility model yields a similar result regarding unsustainability. For $\gamma = 0.5$, the impact value in the impact inequality ranges from 1.69 at $t = 0$ to 1.68 at $t = 100$. The result remains consistent – with only minor numerical deviations for $\gamma = 0.25$ and $\gamma = 0.75$.

Sustainability can be achieved by sufficiently increasing the implicit value of Nature in the SCP. In Fig. 14, the paths t_A^* and t_R^* correspond to the SCP model with endogenous fertility for $\gamma = 0.5$, maintaining the valuation for Nature that corresponds to the 1.7 impact for the exogenous population model. In this case, the impact remains between 1.69 to 1.68 over the time horizon. In contrast, the paths t_A, t_R illustrate a scenario with identical assumptions, except that the valuation of Nature is increased by 125%. Under this adjustment, the impact value decreases, ranging from 0.99 ($t = 0$) to 0.98 ($t = 100$), and sustainability is attainable.

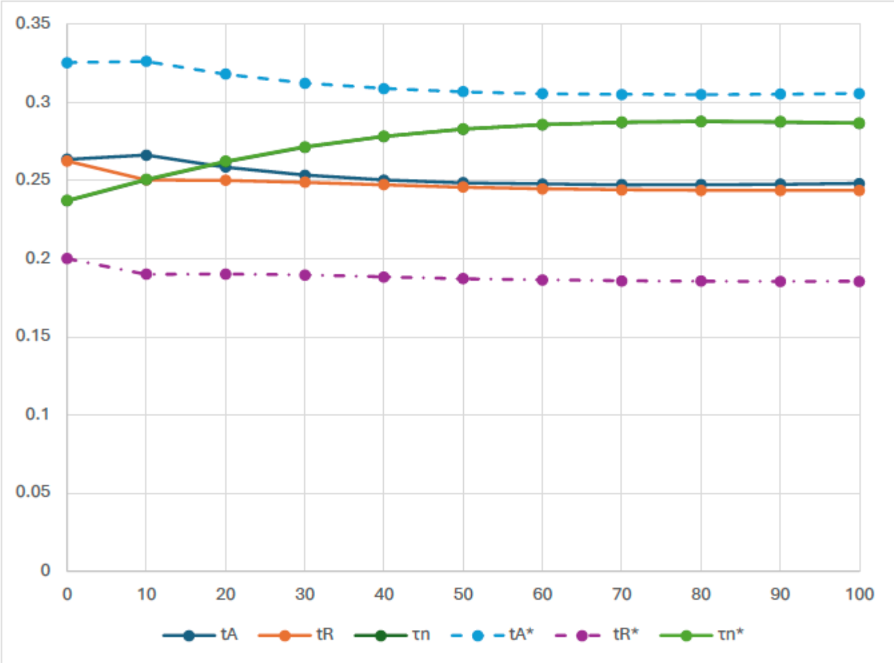


Fig. 14. Unsustainable and sustainable paths with endogenous fertility.

It is interesting to note that the change in impact, in this case, arises solely from the reallocation of time between production of brown and green ideas, as illustrated by the shift from the t_A^* and t_R^* paths to the t_A, t_R paths. The time allocated to child-rearing remains virtually unchanged as the valuation of Nature increases, as shown by the overlapping τn and τn^* paths.

8 Conclusions

The purpose of this paper is to examine whether the accumulation of brown and green ideas can support persistent economic growth when such growth is embedded within the constraints of a finite Earth. The analysis is based on the concept of impact inequality, which captures the gap between Nature's supply of services and the economy's demand for them along a growth trajectory.

Using a long-run growth model where labor is the sole generator of value, and growth is driven by idea generation and knowledge spillovers stemming from the non-rival nature of ideas, we investigate whether a persistent excess demand for Nature's services (i.e., a negative impact inequality) can sustain continuous GDP growth.

We develop several versions of an idea-based growth model. In a setting where Nature's role in material production is ignored, we demonstrate that both a short-sighted social planner and a rational expectations equilibrium that undervalue Nature's services can lead to biosphere depletion and unsustainable development – measured by a declining social welfare indicator. Sustainability becomes feasible when a social planner assigns a high value to Nature's services, thereby internalizing the welfare costs of excess demand, or when a hard constraint enforces a balance between supply and demand for these services. We further show how a policy framework can replicate the planner's solution under rational expectations.

Our results hold in a calibrated model that reflects current data and incorporates technological progress in brown and green technologies, active positive spillovers, and population growth (either exogenous or endogenously determined through fertility decisions). This setting allows for combinatorial growth. Nevertheless, we find that undervaluing Nature's services eventually results in biosphere depletion and unsustainable development – even in the presence of ongoing technological progress.

A key insight is that if the evolution of the biosphere is ignored, growth paths may appear sustainable and even outperform those with high valuation of Nature's services. However, this perception is misleading: eventual biosphere depletion halts growth despite continued technical innovation. This outcome can be reversed by assigning a sufficiently high value to Nature's services. Under this condition, supply-demand equilibrium is achieved, and growth becomes both persistent and sustainable.

Finally, when Nature's services are explicitly included in the production function and the biosphere's dynamics are incorporated into the regulator's optimization problem, sustained growth without Nature's depletion is achievable.

Our main conclusion is that under current rates of technological change and population growth, low valuation of Nature's services leads to unsustainable growth paths. Sustainable growth is feasible either by assigning high value to Nature's contributions or by embedding biosphere dynamics into the structure of the optimizing models.

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