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# Statistical Properties of Two Asymmetric Stochastic Volatility in Power Mean Models

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Here we investigate the statistical properties of two autoregressive normal asymmetric SV models with possibly time varying risk premia. These, although they seem very similar, it turns out, that they possess quite different statistical properties. The derived properties can be employed to develop tests or to check for up to forth order stationarity, something important for the asymptotic properties of various estimators.

## 1 Introduction

Empirical investigations in economics and finance have uncovered several robust statistical regularities, commonly referred to as stylized facts. Among the most prominent of these is volatility clustering, wherein periods of elevated (or subdued) volatility tend to be succeeded by similar periods. This phenomenon, first documented in economic time series, has also been observed in the physical sciences, notably in turbulence data—where it is referred to as indeterminacy (Barndorff-Nielsen 1997 [11]).

The recognition of volatility clustering has led to substantial developments in time series modeling, beginning with the introduction of the Autoregressive Conditional Heteroskedasticity (ARCH) model by Engle (1982) [25] and its generalization by Bollerslev (1986) [16], the Generalized ARCH (GARCH) model. These foundational models have since evolved into a wide range of dynamic heteroskedasticity frameworks (see e.g., Bollerslev, Chou, and Kroner, 1992 [17], Bera and Higgins, 1993 [12], and Francq and Zakoian, 2010 [31]).

Financial theory frequently posits explicit relationships between the conditional first and second moments of asset returns. For example, in the context of equity markets, the conditional mean of excess returns is often modeled as a function of conditional variance (Merton, 1980 [55], and Glosten, Jagannathan and Runkle, 1993 [35]). Rational, risk-averse investors are hypothesized to demand higher expected returns in periods of increased volatility, implying a positive risk-return trade-off, an association empirically supported by French et al. (1987) [32], Campbell and Hentschel (1992) [18], and Poon and Taylor (1992) [61]. To formally capture this relationship, Engle, Lilien and Robins (1987) [29] proposed the ARCH-in-Mean (ARCH-M) model.

Subsequent empirical findings (e.g., Glosten, Jagannathan and Runkle (1993) [35]; Nelson (1991) [58]) also highlight a negative association between the unanticipated component of returns and future volatility. French et al. (1987) [32] interpret this as indirect evidence for a positive correlation between expected risk premia and ex-ante volatility. According to the volatility feedback hypothesis, a large unanticipated shock increases expected future volatility, which, if positively related to returns, leads to an immediate drop in asset prices (Campbell and Hentschel (1992) [18]).

Another well-documented asymmetry in financial markets is that volatility tends to increase more following negative unexpected returns than positive ones of equivalent magnitude. This leverage effect, originally discussed by Black (1976) [14], can be attributed to increases in financial leverage following declines in firm value. It is typically modeled within asymmetric GARCH frameworks such as the Exponential GARCH (Nelson (1991) [58]), Quadratic GARCH (Sentana (1995) [65]), or the GJR-GARCH model (Glosten, Jagannathan and Runkle (1993) [35]).

These empirical regularities have spurred extensive research, particularly within empirical finance, focusing on estimating and quantifying these dynamics through symmetric and asymmetric GARCH-M specifications, which offer computational tractability for inference (e.g., Gonzales-Rivera 1996 [36], Arvanitis and Demos 2004 [6] and 2004a [7], and Bali and Peng 2006 [10]).

The statistical properties of various GARCH-type models have been explored extensively in the literature (Milhoj (1985) [57]; Karanasos (1999) [49]; Rodriguez and Ruiz (2012) [63]; He C. and T. Terasvirta [43]; Demos (2002) [23]; He, Terasvirta and Malmsten (2002) [44]; Tsiotas (2007) [69]). For GARCH-M models, relevant properties are presented in Anyfantaki and Demos ((2011) [4], (2016) [5]) and Arvanitis and Demos (2004a,b) [6, 7].

While GARCH models are characterized by the influence of past mean shocks on future conditional variance, an alternative class of models—Stochastic Volatility (SV) models—introduces an additional innovation term that drives the conditional variance, which may be correlated with the return innovation (Andersen, 1996 [1]). The archetypal SV model defines volatility as a log-linear first-order autoregressive process, commonly referred to as the SV(1) model. Despite their conceptual appeal, SV models have seen limited practical adoption, largely due to the unobservability of volatility, which necessitates both filtering and smoothing for inference (Andersen and Benzoni, 2009 [2]).

Fundamental differences exist between GARCH-M and SV-M models not only in their structural specification—pertaining to the number of innovations affecting the return and volatility processes—but also in their statistical properties and capacity to accommodate stylized facts. Notably, while GARCH-M models estimate the contemporaneous relationship between conditional mean and variance, SV-M models are equipped to jointly model ex-ante return-volatility relationships and the volatility feedback mechanism (Koopman and Uspensky, 2002 [51]).

Classical estimation techniques for SV models are presented in Taylor (1986) [67] and Harvey, Ruiz, and Shephard (1994) [41]. Bayesian approaches are found in Jacquier, Polson, and Rossi (1994 [47], 2004 [48]), Kim, Shephard, and Chib (1998) [50], and Koopman and Uspensky (2002) [51]. For approaches to approximate likelihood estimation, see Bermudez, Marin, and Veiga (2020) [13] and Romero and Ropero-Moriones (2023) [62], who use data cloning, as well as Marín, Romero, and Veiga (2024) [52], who apply the Laplace transformation. Methods based on indirect inference are discussed in Gallant and Tauchen (1996) [33], Smith (1993) [66], Gourieroux, Monfort, and Renault (1993) [37], and for the simulated and generalized method of moments see Duffie and Singleton (1993) [24], Melino and Turnbull (1990) [54], and Andersen and Sorensen (1996) [3].

In the present study, we examine the statistical properties of asymmetric SV models with potentially time-varying risk premia, i.e., where the conditional variance, raised to a power, enters the mean equation (SV-PM specification). The standardized innovations are assumed to be normally distributed, and correlation between the return and volatility innovations is permitted. We focus on two autoregressive SV specifications that, while seemingly similar, exhibit markedly different statistical characteristics.

For foundational statistical properties of SV models, see Taylor (1994)

[68], Harvey and Shephard (1996) [42], Jacquier, Polson, and Rossi (1994) [47], Danielsson (1998) [21], and Harvey, Ruiz, and Shephard (1994) [41]. For asymmetric SV models, see Tsiotas (2012) [70], and for long-memory SV models, consult Ghysels, Harvey, and Renault [34], Harvey (2007) [39], and Perez and Ruiz (2003) [59]. Mao et al. (2020) [53] present a Generalized Asymmetric SV-M model in which asymmetry arises via a transformation of the mean innovation in the volatility equation. Here, we adopt the classical approach of modeling asymmetry through a correlation between return and volatility innovations, as proposed by Jacquier, Polson, and Rossi (2004) [48] (hereafter, JPR). To our knowledge, the statistical properties of such an asymmetric SV-PM model are presented here for the first time.

The properties derived herein provide a basis for constructing statistical tests analogous to those proposed by Horvath, Kokoszka, and Zitikis (2006) [46], and for checking stationarity of various orders, which is essential for establishing the asymptotic behavior of estimators. These results can also inform GMM-based estimation procedures.

In the next section we present the two SV models and derive their static and dynamic moments. In the final section we compare the properties of the two models and conclude.

# 2 The Two SV-PM Models

We consider the following normal Autoregressive Stochastic Volatility in Power Mean class of models:

$$y_t = c + \lambda \sigma_t^{2\alpha} + \varepsilon_t^* = c + \lambda \sigma_t^{2\alpha} + \varepsilon_t \sigma_t \quad where, \tag{2.1}$$

$$\ln \sigma_t^2 = \omega + \psi \ln \sigma_{t-1}^2 + \eta_{t-1} \; (SV1) \; and \tag{2.2}$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \overset{iid}{\sim} N\left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{pmatrix} \right).$$

We call this model SV1-M. The estimation of the model above, with $\alpha = 0.5$, can be found in Arvanitis and Demos (2024) [8], whereas restricted version of it, with $c = \lambda = 0$, has been estimated by quasi maximum likelihood in Harvey and Shephard (1996) [42], and by MCMC in Meyer and Yu (2000) [56]. Further, Asai and McAleer (2011) [9] present some properties of the restricted model concerning mainly the asymmetry and the leverage effect.[1]

However, a second model has been considered in applied work. Specifically, instead of the above conditional variance specification 2.2 the following one is employed :

$$\ln \sigma_t^2 = \omega + \psi \ln \sigma_{t-1}^2 + \eta_t \ (SV2). \tag{2.3}$$

We name this one the SV2-M model. A similar model, with $\rho = 0$ and $\alpha = 0.5$, has been estimated in Koopman and Uspensky (2002) [51] by simulated maximum likelihood, but they add an autoregressive term in the mean. Further, a model with $c = \lambda = 0$ and $\alpha = 1$, but with non-normal error distribution, has been employed by Jacquier, Polson and Ross (2004) [48] (JPR).

Although the two models look very similar, there are important differences between the statistical properties that they can accommodate. In fact, notice that for the SV1-M model the mean error and the conditional variance are contemporaneously uncorrelated, which is not the case for the SV2-M one. Nevertheless, Yu (2005) [72] proved that the partial derivative of future volatility with respect to the error is not necessarily negative when $\rho < 0$, i.e. it could be the case that even if $\rho < 0$ future volatility could decrease with a negative error, claiming the the variance specification in 2.2 is a more "natural" one (see details in Yu 2005 [72]).

Now, form equation 2.1, and for $\alpha = 1$, we get

$$y_t = c + \lambda E_{t-1}\left(\sigma_t^2\right) + \lambda\left(\sigma_t^2 - E_{t-1}\left(\sigma_t^2\right)\right) + \varepsilon_t \sigma_t \quad where,$$

$E_{t-1}\left(\sigma_t^2\right)$ is the expected volatility given information at time $t-1$. Hence $\lambda$ could also represent the volatility feedback coefficient, as the term $\sigma_t^2 - E_{t-1}\left(\sigma_t^2\right)$ is the unexpected part for volatility. In fact, this is a restricted version of the model considered in Campbell and Hentschel (1992) [18], where the risk premium and the feedback coefficients are different. In a GARCH-M type model, this parameterization is not possible as $E_{t-1}\left(\sigma_t^2\right) = \sigma_t^2$, and consequently this constitutes a comparative advantage of the SV-PM model (see Koopman and Uspensky 2002 [51] for more details).

Let us now explore the properties of these models.

[1] Notice that the findings of Carnero, Pena and Ruiz (2004) [19] suggest that the assumption of normality is reasonably appropriate for financial time series.

## 2.1 Properties of the SV1-PM Model

Let us now investigate the statistical static and dynamic properties of the $SV1-M$ model.

### 2.1.1 Static Properties

First, it is easy to prove that the variance of $\varepsilon_t^* = \varepsilon_t \sigma_t$ is given by

$$V\left(\varepsilon_t^*\right) = E\left(\varepsilon_t^2 \sigma_t^2\right) = E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right],$$

and its kurtosis coefficient is given by

$$\kappa_{\varepsilon_t^*} = \frac{3E\left(\sigma_t^4\right)}{\left(V\left(\varepsilon_t \sigma_t\right)\right)^2} = 3\exp\left(\frac{\sigma_\eta^2}{1-\psi^2}\right) > 3.$$

Notice that the kurtosis coefficient is bigger than 3, i.e. the stochastic volatility increases the kurtosis of the errors, a well known fact of the SV models. Further the square coefficient of variation of the conditional variance is given by

$$CV^2 = \frac{Var\left(\sigma_t^2\right)}{E^2\left(\sigma_t^2\right)} = \exp\left(\frac{\sigma_\eta^2}{1-\psi^2}\right) - 1.$$

Now

$$E\left(y_t\right) = c + \lambda \exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]. \tag{2.4}$$

Notice that for $c = 0$ the unconditional risk has the sign of $\lambda$, positive, as $\lambda$ represents the price of risk, in Financial Economics, where as for $c = \lambda = 0$, as in Harvey and Shephard (1996) [42], the risk premium is zero.

Now

$$V\left(y_t\right) = \lambda^2 \left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right] - 1\right\} \exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right] + \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]. \tag{2.5}$$

It is worth mentioning that the price of risk parameter $\lambda$ increases the variance of the observed process, independent of the sign of the price of risk parameter $\lambda$ (see Appendix A).

The skewness coefficient of observed process $y_t$ is given by

$$sk(y_t) = \lambda \frac{\lambda^2 \left\{\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 3\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + 2\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]}{[V(y_t)]^{3/2}}$$
$$+3\lambda \frac{\left\{\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)} + \frac{(\alpha^2+1)\sigma_\eta^2}{2(1-\psi^2)}\right]}{[V(y_t)]^{3/2}}, \tag{2.6}$$

where $V(y_t)$ is given in 2.5. Notice that the skewness coefficient has the sign of $\lambda$, i.e. positive under the assumption of risk premium positivity, as it is highly unlikely the first part of the expression to be negative and bigger in absolute value to the second one. Further, for $\lambda = 0$ the skewness is zero.

Now the kurtosis coefficient of $y_t$ is:

$$\kappa(y_t) = \frac{\lambda^4 \left\{\exp\left[\frac{6\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 4\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + 6\exp\left[\frac{2\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] - 3\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]}{[V(y_t)]^2}$$
$$+\frac{6\lambda^2 \left\{\left\{\exp\left[\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right] - 2\right\}\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] + 1\right\}\exp\left[\frac{(2\alpha+1)\omega}{(1-\psi)} + \frac{(2\alpha^2+1)\sigma_\eta^2}{2(1-\psi^2)}\right]}{[V(y_t)]^2}$$
$$+3\frac{\exp\left[\frac{2\omega}{(1-\psi)} + \frac{4\sigma_\eta^2}{2(1-\psi^2)}\right]}{[V(y_t)]^2}, \tag{2.7}$$

where $V(y_t)$ is, again, given in 2.5. Notice that for $\alpha = 0.5$ it is possible that $\kappa(y_t) - \kappa_{\varepsilon_t^*}$ can positive or negative, i.e. the presence of the price of risk, $\lambda$, could either increase or decrease the kurtosis coefficient of the data (see Appendix A).

### 2.1.2 Dynamic Properties

For the autocovariances of the observed process, $y_t$, we have that:

$$Cov(y_t, y_{t-k}) = \lambda^2 \left[\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right]\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$+\lambda\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(2\alpha+1)\omega}{2(1-\psi)} + \frac{\sigma_\eta^2(4\alpha^2 + 4\alpha\psi^k + 1)}{8(1-\psi^2)}\right]$$

and the correlations are given by dividing the above expression by the variance in 2.5. It is worth noticing that, provided that $\lambda$ is positive then the correlations can be either positive or negative, depending on the relative values of of the parameters. Notice that for $\rho = 0$ the $Corr\left(y_t, y_{t-k}\right)$ is proportional to $\lambda^2$ (see Appendix A), whereas for $\lambda = 0$ the autocorrelations are zero.

In terms of leverage effect we have that

$$Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right) = \rho\sigma_\eta\psi^{k-1}\frac{\exp\left[\frac{4\psi^k-1}{8(1-\psi^2)}\sigma_\eta^2\right]}{\sqrt{\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)}} \tag{2.8}$$

and has the sign of $\rho$, i.e. the leverage effect can be satisfied by the model if and only if $\rho < 0$, provided that $\psi > 0$ something which is very plausible due to volatility clustering (see below).

In terms of dynamic asymmetry, as we call $Corr\left(y_t^2, y_{t-k}\right)$,

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}\right) = {} & \lambda^3\left(\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{3\alpha\omega}{(1-\psi)}+\frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] \\
& +2\lambda\alpha\rho\sigma_\eta\psi^{k-1}\left\{\lambda\exp\left[\frac{\alpha\omega}{(1-\psi)}+\frac{\alpha\left(\psi^k+3\alpha\right)\sigma_\eta^2}{2(1-\psi^2)}\right]+c\right\}\exp\left[\frac{(2\alpha+1)\,\omega}{2(1-\psi)}+\frac{\sigma_\eta^2\left(4\alpha\psi^k+1+4\alpha^2\right)}{8(1-\psi^2)}\right] \\
& +\lambda\left(\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)}+\frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right] \\
& +2c\lambda^2\left(\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
& +\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8(1-\psi^2)}\right].
\end{aligned}$$

Now for $\lambda = 0$ we get that $Corr\left(y_t^2, y_{t-k}\right)$ is proportional to $\rho\sigma_\eta\psi^{k-1}$ and consequently has the sign of $\rho$, and, of course, it is independent of $\alpha$ (see Appendix A). Notice that in this case and for $\rho < 0$ we have that $Corr\left(y_t^2, y_{t-k}\right) > Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right)$, i.e. the leverage effect is stronger than the dynamic asymmetry. In Appendix A we also present the dynamic asymmetry for $\rho = 0$.

Now the volatility clustering is given by (see Appendix A for a proof)

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2 \psi^k}{(1-\psi^2)}\right] - 1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1}, \tag{2.9}$$

and it is the same with the SV2-PM specification (see below).

For the dynamic kurtosis, as we call here $Cov\left(y_t^2, y_{t-k}^2\right)$, the formula is very complicated and presented in Appendix A. However, under the Efficient Market hypothesis, i.e. for $c = 0$, the autocovariances are given by:

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) &= \lambda^4 \left\{\exp\left[\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
&+2\lambda^2 \left\{\left(1 + 2\alpha^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{(4\alpha^2+1)\,\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
&+4\lambda^3\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(6\alpha+1)\,\omega}{2\,(1-\psi)} + \frac{\sigma_\eta^2\left(8\alpha\,(2\alpha+1)\,\psi^k + (2\alpha+1)^2 + 16\alpha^2\right)}{8\,(1-\psi^2)}\right] \\
&+ \left\{\left(1 + \rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right] \\
&+2\rho\sigma_\eta\psi^{k-1}\left\{\lambda\exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha\left(2\psi^k + \alpha + 1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right] + c\right\}\exp\left[\frac{3\omega}{2\,(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k + 5\right)}{8\,(1-\psi^2)}\right].
\end{aligned}$$

and

$$\begin{aligned}
V\left(y_t^2\right) &= \lambda^4 \left\{\exp\left[\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
&+2\lambda^2\left\{3\exp\left[\frac{2\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{(4\alpha^2+1)\,\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
&+\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right].
\end{aligned} \tag{2.10}$$

Further, for $\lambda = c = 0$ we get

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(1 + \rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right] - 1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1},$$

which is the same as equation 18 Perez, Ruiz and Veiga (2009) [60] and equation 5 in Ruiz and Veiga (2008) [64], and again it is positive for any values of $\rho$. Further, it is easy to see that $Corr\left(y_t^2, y_{t-k}^2\right) < Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$, in most plausible cases.

In Appendix A we present the autocovariances for $\lambda = 0$, and $\rho = 0$.

## 2.2 Properties of the SV2-PM Model

### 2.2.1 Static Moments

Now, in Appendix B we prove that if instead the conditional variance specification we employ the specification in JPR, equation 2.3, we get

$$E\left(y_t\right) = c + \lambda \exp\left(\frac{\alpha\omega}{1-\psi} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) + \frac{\rho\sigma_\eta}{2}\exp\left(\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and

$$\begin{aligned} Var\left(y_t\right) &= \lambda^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - 1\right]\exp\left(\frac{2\alpha\omega}{\left(1-\psi\right)} + \frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right) \\ &+ \lambda\rho\sigma_\eta\left[\left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right]\exp\left(\frac{\left(2\alpha+1\right)\omega}{2\left(1-\psi\right)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\ &+ \left\{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - \frac{\left(\rho\sigma_\eta\right)^2}{4}\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right). \end{aligned}$$

Notice that if $\rho = 0$ we get that the expected value and variance of $y_t$ for the SV2 model are the same as those for the SV1 one (see equations 2.4 and 2.5). Further, if $c = \lambda = 0$, as in JPR, $E\left(y_t\right)$ is negative, provided that $\rho < 0$ (leverage effect).

The skewness coefficient of the mean error $\varepsilon_t^* = \sigma_t\varepsilon_t$ is given by

$$sk\left(\varepsilon_t^*\right) = \frac{3}{2}\rho\sigma_\eta\frac{\left\{\left(3+\left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - \left(1+\left(\rho\sigma_\eta\right)^2\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + \frac{1}{6}\left(\rho\sigma_\eta\right)^2}{\left(\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right)^{3/2}}. \quad (2.11)$$

It is worth noticing that although the distribution of the standardized errors is normal the skewness of the mean error is non-zero, and if fact it is negative, provided that $\rho < 0$ due to the leverage effect.

Now the kurtosis of the mean error is given by

$$\kappa\left(\varepsilon_t^*\right)=\frac{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}$$
$$+3\left(\frac{\rho\sigma_\eta}{2}\right)^2\frac{\left\{2\left(1+\left(\rho\sigma_\eta\right)^2\right)-\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2},$$

and for $\rho=0$, as in Koopman and Uspensky (2002) [51], $\kappa\left(\varepsilon_t^*\right)$ is the same as for the SV1-M model.

The asymmetry of the observed process, $y_t$, is very complicated for the full model and it is presented in Appendix B. Now for $\lambda=0$ the skewness coefficient of $y_t$, $sk\left(y_t\right)$, is, of course, the same as this one of $\varepsilon_t^*$, see equation 2.11, above, whereas for $\rho=0$ we get that $sk\left(y_t\right)$ for the SV2-M model is the same as for the SV1-M one (see equation 2.6).

Further, the kurtosis coefficient for $y_t$, $\kappa\left(y_t\right)$ , is also very complicated and is provided in Appendix B. However, for $\lambda=0$ $\kappa\left(y_t\right)$ is the same as the kurtosis coefficient of $\varepsilon_t^*$, above, whereas for $\rho=0$ we have that $\kappa\left(y_t\right)$ is the same as the one of SV1-M model (see equation 2.7).

### 2.2.2 Dynamic Moments

The dynamic moments for the SV2-M specification are more complicated than the SV1-M one. This due to the fact the for the SV2-M model we have that $Cov\left(g\left(\varepsilon_t\sigma_t\right),f\left(\varepsilon_{t-k},\sigma_{t-k}\right)\right)\neq 0$for various functions $g\left(.\right)$ and $f\left(.\right)$.

Now, the autocovariances the observed process, $y_t$, are given by

$$Cov\left(y_t,y_{t-k}\right)=\lambda^2\left\{\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{2\alpha\omega}{\left(1-\psi\right)}+\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right)$$
$$+\lambda\rho\sigma_\eta\left\{\left(\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\left(2\alpha+1\right)\omega}{2\left(1-\psi\right)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$
$$+\frac{1}{4}\left(\rho\sigma_\eta\right)^2\left\{\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).$$

It is possible that, depending on the parameter values, the $k^{th}$ order autocovariance can be either positive or negative.

$$Var\left(y_t\right) = \lambda^2 \left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right] \exp\left(\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$+\lambda\rho\sigma_\eta \left[(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right] \exp\left(\frac{(2\alpha+1)\,\omega}{2\left(1-\psi\right)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$
$$+\left\{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - \frac{\left(\rho\sigma_\eta\right)^2}{4}\right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

Now in the model of Koopman and Uspensky (2002) [51] we have $\rho = 0$ and it follows that that the autocorrelation functions of the SV2-M model is the same as for the SV1-M one (see equation 3.8) and consequently, $Corr\left(y_t, y_{t-k}\right)$ can be only positive. The same is true in the case of JPR, where we have that $\lambda = 0$, i.e.

$$Corr\left(y_t, y_{t-k}\right) = \frac{1}{4}\left(\rho\sigma_\eta\right)^2 \frac{\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right) - 1}{\left(1+\rho^2\sigma_\eta^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) - \frac{(\rho\sigma_\eta)^2}{4}},$$

and $Corr\left(y_t, y_{t-k}\right)$ can be only positive. However, in this case, i.e. if $\lambda = 0$, the autocorrelations of the observed process for the SV1-M model is zero.

For the leverage effect we get

$$Corr\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) = \frac{1}{2}\rho\sigma_\eta \frac{\left(2\psi^k+1\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right] - 1}{\sqrt{\left[\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right]\left[\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) - \left(\frac{1}{2}\rho\sigma_\eta\right)^2\right]}},$$

which is negative provided that $\rho < 0$, something which also is true for the SV1-M model (see equation 2.8).

For the SV2-M model the dynamic asymmetry is very complicated for the full model and it is presented in Appendix B (equation 3.15). Now for the Koopman and Uspensky (2002) [51] model, i.e. for $\rho = 0$ and $c \neq 0$, we get that $Cov\left(y_t^2, y_{t-k}\right)$ for the model are the same as for the SV1-M model.

For $\lambda = 0$ but $\rho \neq 0$ and $c \neq 0$ we get

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-k}\right) = \frac{\rho\sigma_\eta}{2}\left(1+(\rho\sigma_\eta)^2\right)\left[\left(2\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right] \\
\times\exp\left(\frac{3\omega}{2\left(1-\psi\right)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
+c\frac{(\rho\sigma_\eta)^2}{2}\left[\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
+\lambda\left(\rho\sigma_\eta\right)^2\frac{(2\alpha+1)}{2}\left\{\left[(2\alpha+1)\,\psi^k+1\right]\exp\left(\frac{(2\alpha+1)\,\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}. \\
\times\exp\left(\frac{(\alpha+1)\,\omega}{(1-\psi)}+\frac{\left((2\alpha+1)^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)
\end{aligned}
$$

and $Cov\left(y_t^2, y_{t-k}\right)$ has probably the sign of $\rho$.

The stochastic volatility process is the same for the two model. Consequently the volatility clustering is given by equation 2.9 for the SV2-M model, as well.

Again, the dynamic kurtosis is very complicated. In Appendix B the autocovariances, $Corr\left(y_t^2, y_{t-k}^2\right)$, and the variance $V\left(y_t^2\right)$ are presented. Further if $\lambda = c = 0$, as in JPR, we get

$$
Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(1+(\rho\sigma_\eta)^2\right)\left\{\left(1+\left(\left(\psi^k+1\right)\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)\right\}}{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)^2},
$$

which is positive for any $k$.

For $c \neq 0$ and $\rho = 0$, as in Koopman and Uspensky (2002) [51], the $Covv\left(y_t^2, y_{t-k}^2\right)$ and $V\left(y_t^2\right)$ are the same as for the SV1 process and are presented in Appendix A.. The same applies for the case of $c = \rho = 0$.

# 3 Comparisons and Conclusions

Overall, the SV2-PM model exhibits more intricate static and dynamic moment structures compared to the SV1-PM model. This added complexity stems from the contemporaneous correlation between the in-mean error and the conditional variance in the SV2-PM specification when $\rho \neq 0$. When $\rho = 0$, however, the two models produce identical static and dynamic moments, making them observationally indistinguishable.

For the remainder of this section, we assume $\lambda > 0$ to represent a positive price of risk, $\rho < 0$ to capture the leverage effect, and $c = 0$ to ensure market efficiency. Under these assumptions, the expected return $E(y_t)$ is positive in the SV1-M model, while it may be either positive or negative in the SV2-PM model. Additionally, in the SV2-PM model, the variance of the observed process is influenced by both $\lambda$ and $\rho$.

When $\lambda = 0$ but $\rho \neq 0$, the skewness of the observed process is zero in the SV1-M model, yet negative in the SV2-M model. More generally, when both $\lambda \neq 0$ and $\rho \neq 0$, the skewness of the observed series can take on either sign, though the skewness of the mean error remains definitively negative. For example, using parameter estimates from JBR (Table 4, weakly equal-weighted NYSE returns), the skewness of the data is estimated at –0.54. Notably, JBR assumes a fat-tailed distribution for the mean error, giving the SV2-PM model a comparative advantage: it can replicate negative skewness in returns without imposing asymmetry on the standardized error distribution (see, e.g., Harvey and Palumbo 2023 [40]; Blasques, Francq, and Laurent 2023 [15]).

Regarding kurtosis, when $\lambda = 0$, the kurtosis of the observed process equals that of the mean error in the SV1-PM model. By contrast, in the SV2-PM model, the kurtosis of the observed process is lower than that of the mean error.

Autocorrelations, $Corr\left(y_t, y_{t-k}\right)$, can be positive or negative in both models depending on the relative magnitudes of $\rho$ and $\lambda$. When $\lambda = 0$, autocorrelations vanish in the SV1-M model but remain strictly positive in the SV2-PM model.

The leverage effect, measured by $Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right)$, is negative in both models and disappears when $\rho = 0$. Dynamic asymmetries can be either positive or negative, depending on the interaction between $\rho$ and $\lambda$. However, in the SV1-PM model, if the autocorrelations of the observed process are negative, dynamic asymmetries are also negative. When $\lambda = 0$ and $\rho \neq 0$, both models exhibit negative dynamic asymmetries. Importantly, the two models share identical expressions for volatility clustering, i.e., $Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$.

Table 1 reports estimated moments for the SV1-PM model based on parameter values treated as true. The first three columns use restricted estimates with $c = \lambda = 0$, taken from Yu (2005) [72] (Table 1) and Asai and

McAleer (2011) [9] (Table 2). As predicted by the theory in Sections 2.1.1 and 2.1.2, both skewness and autocorrelations are zero in this setup. Columns 4 and 5 use full-model estimates from Arvanitis and Demos (2024) [8], yielding non-zero skewness and small positive autocorrelations. Interestingly, both full-model estimates report positive skewness coefficients.

Table 1. Estimated Moments for SV1-PM Models

| | Yu | AM | | AD | |
|---|---|---|---|---|---|
| | S&P | S&P | Topix | S&P | DAX |
| $sk(y_t)$ | 0 | 0 | 0 | 0.027 | 0.037 |
| $\kappa(y_t)$ | 4.497 | 3.536 | 4.025 | 5.178 | 5.478 |
| $Cor(y_t, y_{t-1})$ | 0 | 0 | 0 | -0.004 | -0.003 |
| $Cor(y_t, y_{t-2})$ | 0 | 0 | 0 | -0.003 | -0.003 |
| $Cor\left(\sigma_t^2, \varepsilon_{t-1}^*\right)$ | -0.078 | -0.154 | -0.176 | -0.213 | -0.140 |
| $Cor\left(\sigma_t^2, \varepsilon_{t-2}^*\right)$ | -0.075 | -0.152 | -0.166 | -0.197 | -0.139 |
| $Cor(y_t^2, y_{t-1})$ | -0.029 | -0.060 | -0.059 | -0.082 | -0.050 |
| $Cor(y_t^2, y_{t-1})$ | -0.002 | -0.040 | -0.055 | -0.075 | -0.046 |
| $Cor\left(y_t^2, y_{t-1}^2\right)$ | 0.139 | 0.071 | 0.110 | 0.163 | 0.169 |
| $Cor\left(y_t^2, y_{t-2}^2\right)$ | 0.134 | 0.070 | 0.104 | 0.150 | 0.159 |

Table 2 presents moment estimates for several SV2-PM model variants. In Columns 1 and 2, based on Koopman and Uspensky (2002) [51], where $\rho = 0$, the leverage effect is absent and autocorrelations, though small, are positive. Koopman and Uspensky originally incorporated an additional AR(1) component to capture autocorrelation, which is excluded here. Columns 3 and 4 use restricted estimates from Yu (2005) [72] and Jacquier, Polson, and Rossi (2005) [48], both with $c = \lambda = 0$. Consistent with Section 2.2.2, the observed process shows positive autocorrelations and a negative leverage effect. In all four SV2-M cases, the estimated skewness is negative.

Table 2. Estimated Moments for SV2-PM Models

| | KU | | Yu | JPR |
|---|---|---|---|---|
| | FT | S&P | S&P | VW |
| $sk\left(y_t\right)$ | -0.0234 | -0.040 | -0.131 | -0.502 |
| $\kappa\left(y_t\right)$ | 5.425 | 4.975 | 4.487 | 5.789 |
| $Cor\left(y_t, y_{t-1}\right)$ | 0.000 | 0.000 | 0.000 | 0.004 |
| $Cor\left(y_t, y_{t-2}\right)$ | 0.000 | 0.000 | 0.000 | 0.003 |
| $Cor\left(\sigma_t^2, \varepsilon_{t-1}^*\right)$ | 0 | 0 | -0.061 | -0.172 |
| $Cor\left(\sigma_t^2, \varepsilon_{t-2}^*\right)$ | 0 | 0 | -0.059 | -0.160 |
| $Cor\left(y_t^2, y_{t-1}\right)$ | -0.003 | -0.007 | -0.023 | -0.063 |
| $Cor\left(y_t^2, y_{t-2}\right)$ | -0.001 | -0.006 | -0.011 | -0.058 |
| $Cor\left(y_t^2, y_{t-1}^2\right)$ | 0.173 | 0.158 | 0.148 | 0.166 |
| $Cor\left(y_t^2, y_{t-2}^2\right)$ | 0.169 | 0.153 | 0.138 | 0.153 |

To summarize, both models can replicate key empirical features of financial return series, depending on parameter values. However, the SV2-PM model—particularly in its unrestricted form—exhibits greater moment complexity. A distinct advantage of SV2-PM is its ability to capture negative return skewness without resorting to asymmetric standardized error distributions.

A potential extension for both models involves introducing autocorrelation in the standardized mean error, such as through an ARMA specification. While this would further complicate the moment structure, it may enhance model fit.

# Appendix A

## Static Moments

From Demos 2002 [23] we get:

$$A_{k,i}^{(s,d)} = \rho\phi_\eta \left(s\psi_{i+k} + d\psi_i\right) = B_{k,i}^{(s,d)}$$
$$\mathit{\Gamma}_{k,i}^{(s,d)} = \frac{\phi_\eta^2}{2}\left(s\psi_{i+k} + d\psi_i\right)^2 = \Delta_{k,i}^{(s,d)}$$

$$\varpi_k^{(s,d)} = \prod_{i=0}^{\infty}\left[\Phi\left(A_{k,i}^{(s,d)}\right)\exp\left(\mathit{\Gamma}_{k,i}^{(s,d)}\right) + \exp\left(\Delta_{k,i}^{(s,d)}\right)\Phi\left(-B_{k,i}^{(s,d)}\right)\right] \tag{3.1}$$
$$= \prod_{i=0}^{\infty}\left\{\exp\left(\mathit{\Gamma}_{k,i}^{(s,d)}\right)\left[\Phi\left(A_{k,i}^{(s,d)}\right) + \Phi\left(-B_{k,i}^{(s,d)}\right)\right]\right\} = \exp\left(\sum_{i=0}^{\infty}\mathit{\Gamma}_{k,i}^{(s,d)}\right)$$
$$= \exp\left(\frac{\phi_\eta^2}{2}\sum_{i=0}^{\infty}\left(s\psi^{i+k} + d\psi^i\right)^2\right) = \exp\left(\frac{\phi_\eta^2\left(s\psi^k + d\right)^2}{2}\sum_{i=0}^{\infty}\psi^{2i}\right) = \exp\left(\frac{\phi_\eta^2\left(s\psi^k + d\right)^2}{2\left(1-\psi^2\right)}\right)$$

$$\mathit{\Lambda}_{(k-1)}^{s} = \prod_{i=0}^{k-1}\left[\Phi\left(A_{0,i}^{(0,s)}\right) exp\left(\mathit{\Gamma}_{0,i}^{(0,s)}\right) + exp\left(\Delta_{0,i}^{(0,s)}\right)\Phi\left(-B_{0,i}^{(0,s)}\right)\right]$$
$$= \prod_{i=0}^{k-1}\exp\left(\mathit{\Gamma}_{0,i}^{(0,s)}\right) = \exp\sum_{i=0}^{k-1}\left(\frac{\phi_\eta^2}{2}s^2\psi_i^2\right)$$

By employing the above formulae we get:

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\right) = \exp\left[\left(s+d\right)\frac{\omega}{1-\psi}\right]\varpi_k^{(s,d)}\mathit{\Lambda}_{(k-1)}^{s} \tag{3.2}$$
$$= \exp\left[\frac{(s+d)\,\omega}{1-\psi} + \frac{\sigma_\eta^2\left(s\psi^k + d\right)^2}{2\left(1-\psi^2\right)} + \frac{\sigma_\eta^2 s^2}{2}\sum_{i=0}^{k-1}\psi^{2i}\right]$$
$$= \exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k + d^2 + s^2\right)}{2\left(1-\psi^2\right)}\right]$$

Now from Demos 2002 [23], again,

$$\begin{aligned} D^s_{k-1} &= A^{(0,s)}_{0,k-1}\Phi\left(A^{(0,s)}_{0,k-1}\right)\exp\left(\Gamma^{(0,s)}_{0,k-1}\right) + B^{(0,s)}_{0,k-1}\exp\left(\Delta^{(0,s)}_{0,k-1}\right)\Phi\left(-B^{(0,s)}_{0,k-1}\right) \\ &= A^{(0,s)}_{0,k-1}\exp\left(\Gamma^{(0,s)}_{0,k-1}\right) = s\rho\phi_\eta\psi^{k-1}\exp\left(\frac{\phi^2_\eta}{2}s^2\psi^{2k-2}\right) \end{aligned}$$

$$\begin{aligned} Y^s_{k-1} &= \Phi\left(A^{(0,s)}_{0,k-1}\right)\exp\left(\Gamma^{(0,s)}_{0,k-1}\right) + \exp\left(\Delta^{(0,s)}_{0,k-1}\right)\Phi\left(-B^{(0,s)}_{0,k-1}\right) \\ &= \exp\left(\Gamma^{(0,s)}_{0,k-1}\right) = \exp\left(\frac{\phi^2_\eta}{2}s^2\psi^{2k-2}\right) \end{aligned}$$

$$\begin{aligned} F^s_{k-1} &= \left(A^{(0,s)}_{0,k-1}\right)^2\Phi\left(A^{(0,s)}_{0,k-1}\right)\exp\left(\Gamma^{(0,s)}_{0,k-1}\right) + \left(B^{(0,s)}_{0,k-1}\right)^2\exp\left(\Delta^{(0,s)}_{0,k-1}\right)\Phi\left(-B^{(0,s)}_{0,k-1}\right) \\ &= \left(A^{(0,s)}_{0,k-1}\right)^2\exp\left(\Gamma^{(0,s)}_{0,k-1}\right) = s^2\rho^2\phi^2_\eta\psi^{2k-2}\exp\left(\frac{\phi^2_\eta}{2}s^2\psi^{2k-2}\right) \end{aligned}$$

$$\begin{aligned} E\left(\sigma^{2s}_t\sigma^{2d}_{t-k}\varepsilon_{t-k}\right) &= E\left(\sigma^{2s}_t\sigma^{2d}_{t-k}\right)D^s_{k-1}\left(Y^s_{k-1}\right)^{-1} \quad \textit{and} \\ E\left(\sigma^{2s}_t\sigma^{2d}_{t-k}\varepsilon^2_{t-k}\right) &= E\left(\sigma^{2s}_t\sigma^{2d}_{t-k}\right)\left[1 + F^s_{k-1}\left(Y^s_{k-1}\right)^{-1}\right] \end{aligned}$$

Hence, employing the above formulae we get

$$\begin{aligned} E\left(\sigma_t\sigma_{t-k}\varepsilon_{t-k}\right) &= \frac{1}{2}\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\omega}{1-\psi} + \frac{\sigma^2_\eta\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right] \\ E\left(\sigma^2_t\sigma_{t-k}\varepsilon_{t-k}\right) &= \rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{1-\psi} + \frac{\sigma^2_\eta\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right] \\ E\left(\sigma_t\sigma^2_{t-k}\varepsilon_{t-k}\right) &= \frac{1}{2}\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{1-\psi} + \frac{\sigma^2_\eta\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right] = \frac{1}{2}E\left(\sigma^2_t\sigma_{t-k}\varepsilon_{t-k}\right) \\ E\left(\sigma^2_t\sigma^2_{t-k}\varepsilon_{t-k}\right) &= \rho\sigma_\eta\psi^{k-1}\exp\left[2\frac{\omega}{1-\psi} + \frac{\sigma^2_\eta\left(\psi^k+1\right)}{\left(1-\psi^2\right)}\right] \end{aligned}$$

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}\right) = \exp\left(\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\phi_\eta^2\left(s\psi^k+d\right)^2}{2\left(1-\psi^2\right)}\right)\exp\left(\sum_{i=0}^{k-1}\frac{\phi_\eta^2}{2}s^2\psi_i^2\right)$$
$$\times s\rho\phi_\eta\psi^{k-1}\frac{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}$$
$$= s\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k+d^2+s^2\right)}{2\left(1-\psi^2\right)}\right] \tag{3.3}$$

and

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}^2\right) = \exp\left(\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(s\psi^k+d\right)^2}{2\left(1-\psi^2\right)}\right)\exp\left(\sum_{i=0}^{k-1}\frac{\sigma_\eta^2}{2}s^2\psi_i^2\right)$$
$$\times\left[1+s^2\rho^2\sigma_\eta^2\psi^{2k-2}\frac{\exp\left(\frac{\sigma_\eta^2}{2}s^2\psi^{2k-2}\right)}{\exp\left(\frac{\sigma_\eta^2}{2}s^2\psi^{2k-2}\right)}\right]$$
$$= \left(1+s^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k+d^2+s^2\right)}{2\left(1-\psi^2\right)}\right] \tag{3.4}$$

Now

$$E\left(\sigma_t\right) = \exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],\quad E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$
$$E\left(\sigma_t^3\right) = \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{9\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],\quad \textit{and}\quad E\left(\sigma_t^4\right) = \exp\left[2\frac{\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{\left(1-\psi^2\right)}\right].$$

$$E\left(\sigma_t^{2s}\right) = \exp\left[\frac{s\omega}{(1-\psi)} + \frac{s^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] \tag{3.5}$$

Hence it follows that

$$V\left(\varepsilon_t^*\right) = E\left(\varepsilon_t^2\sigma_t^2\right) = E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right],$$
$$sk\left(\varepsilon_t^*\right) = 0$$

and

$$\kappa\left(\varepsilon_t^*\right) = \frac{3E\left(\sigma_t^4\right)}{\left(V\left(\varepsilon_t\sigma_t\right)\right)^2} = 3\exp\left[\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right] > 3.$$

Further, employing equation 3.5, we get

$$E\left(y_t\right) = c + \lambda_1 E\left(\sigma_t^{2\alpha}\right) = c + \lambda_1 \exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

and

$$V\left(y_t\right) = E\left(y_t - E\left(y_t\right)\right)^2 = E\left[\ \lambda_1\left(\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right) + \varepsilon_t\sigma_t\ \right]^2 =$$
$$= \lambda_1^2 E\left(\sigma_t^{4\alpha}\right) - \lambda_1^2 E^2\left(\sigma_t^{2\alpha}\right) + E\left(\sigma_t^2\right)$$
$$= \lambda_1^2\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right].$$

N0otice that

$$\frac{V\left(y_t\right)}{V\left(\varepsilon_t^*\right)} = 1 + \frac{\lambda^2\left\{\exp\left[\frac{2\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]}{\exp\left[\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]} > 1.$$

Now notice that

$$E\left(y_t^2\right) = E\left(c + \lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t\right)^2 =$$
$$= c^2 + \lambda^2 E\left(\sigma_t^{4\alpha}\right) + E\left(\sigma_t^2\right) + 2c\lambda E\left(\sigma_t^{2\alpha}\right)$$
$$= c^2 + \lambda\left\{\lambda\exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{3\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] + 2c\right\}\exp\left[\frac{\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$
$$+\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

Further,

$$V\left(y_t^2\right) = E\left(y_t^4 - E^2\left(y_t^2\right)\right) =$$
$$= E\left(\left(c + \lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t\right)^4\right) - \left(c^2 + \lambda^2 E\left(\sigma_t^{4\alpha}\right) + E\left(\sigma_t^2\right) + 2c\lambda E\left(\sigma_t^{2\alpha}\right)\right)^2 =$$
$$= \lambda^4\left[E\left(\sigma_t^{8\alpha}\right) - E^2\left(\sigma_t^{4\alpha}\right)\right] + 4c^2\lambda^2\left[E\left(\sigma_t^{4\alpha}\right) - E^2\left(\sigma_t^{2\alpha}\right)\right]$$
$$+4c\lambda^3\left[E\left(\sigma_t^{6\alpha}\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha}\right)\right] + 4c^2 E\left(\sigma_t^2\right)$$
$$+4c\lambda\left[3E\left(\sigma_t^{2\alpha+2}\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^2\right)\right] + 3E\left(\sigma_t^{2\alpha+2}\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^2\right)$$
$$+2\lambda^2\left[3E\left(\sigma_t^{4\alpha+2}\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^2\right)\right] + 3E\left(\sigma_t^4\right) - E^2\left(\sigma_t^2\right).$$

Hence

$$V\left(y_t^2\right)=\lambda^4\left\{\exp\left[\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \tag{3.6}$$
$$+4c\lambda^3\left\{\exp\left[\frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)}+\frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]$$
$$+4c\lambda\left\{3\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)}+\frac{(\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right]$$
$$+4c^2\lambda^2\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$+2\lambda^2\left\{3\exp\left[\frac{2\alpha\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)}+\frac{(4\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right]$$
$$+4c^2\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{2(1-\psi^2)}\right]+\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right].$$

Further

$$E\left(y_t-E\left(y_t\right)\right)^3=E\left[\ \lambda_1\left(\sigma_t^{2\alpha}-E\left(\sigma_t^{2\alpha}\right)\right)+\lambda_2\left(\varepsilon_t^2-1\right)\sigma_t^2+\varepsilon_t\sigma_t\ \right]^3$$
$$=\lambda_1^3E\left[\ \left[\left(\sigma_t^{2\alpha}-E\left(\sigma_t^{2\alpha}\right)\right)\right]^3\ \right]+3\lambda_1\lambda_2E\left[\ \left[\left(\sigma_t^{2\alpha}-E\left(\sigma_t^{2\alpha}\right)\right)\right]^2\left(\varepsilon_t^2-1\right)\sigma_t^2\ \right]$$
$$+3E\left[\ \left[\lambda_1\left(\sigma_t^{2\alpha}-E\left(\sigma_t^{2\alpha}\right)\right)\right]^2\varepsilon_t\sigma_t\ \right]$$
$$+E\left[3\left[\lambda_1\left(\sigma_t^{2\alpha}-E\left(\sigma_t^{2\alpha}\right)\right)\right]\left[\lambda_2\left(\varepsilon_t^2-1\right)\sigma_t^2+\varepsilon_t\sigma_t\right]^2+\left[\lambda_2\left(\varepsilon_t^2-1\right)\sigma_t^2+\varepsilon_t\sigma_t\right]^3\right]$$
$$=\lambda^3\left[E\left(\sigma_t^{6\alpha}\right)-3E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha}\right)+2E^3\left(\sigma_t^{2\alpha}\right)\right]+3\lambda\left[E\left(\sigma_t^{2\alpha+2}\right)-E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^{2}\right)\right]$$

$$sk\left(y_t\right)=\lambda\frac{\lambda^2\left\{\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-3\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]+2\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)}+\frac{3\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]}{A^{3/2}}.$$
$$+\lambda\frac{3\left\{\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)}+\frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right]}{A^{3/2}},$$

where

$$A=\lambda^2\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]+\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{2(1-\psi^2)}\right].$$

Notice that for $\alpha=0.5$ or $\alpha=1$ we have that the numerator is positive provided that $\exp\left[\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]-3\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]+2>0$, which is, in general, true.

Hence, in this case, the skewness coefficient $sk(y_t)$ has the sign of $\lambda$, i.e. positive under the assumption of positive price of risk.

Now

$$E(y_t - E(y_t))^4 = E\left(\lambda^4 \left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right]^4 + 4\lambda^3 \left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right]^3 \varepsilon_t \sigma_t\right)$$
$$+E\left(6\lambda^2 \left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right]^2 \varepsilon_t^2 \sigma_t^2 + 4\left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right] \varepsilon_t^3 \sigma_t^3 + \varepsilon_t^4 \sigma_t^4\right)$$
$$= \lambda^4 \left[E\left(\sigma_t^{8\alpha}\right) - 4E\left(\sigma_t^{6\alpha}\right) E\left(\sigma_t^{2\alpha}\right) + 6E\left(\sigma_t^{4\alpha}\right) E^2\left(\sigma_t^{2\alpha}\right) - 3E^4\left(\sigma_t^{2\alpha}\right)\right]$$
$$+6\lambda^2 \left[E\left(\sigma_t^{4\alpha+2}\right) - 2E\left(\sigma_t^{2\alpha+2}\right) E\left(\sigma_t^{2\alpha}\right) + E\left(\sigma_t^2\right) E^2\left(\sigma_t^{2\alpha}\right)\right] + 3E\left(\sigma_t^4\right),$$

and it follows

$$\kappa(y_t) = \frac{\lambda^4 \left\{\exp\left[\frac{6\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 4\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + 6\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 3\right\} \exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]}{A^2}$$
$$+\frac{6\lambda^2 \left\{\left\{\exp\left[\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right] - 2\right\} \exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] + 1\right\} \exp\left[\frac{(2\alpha+1)\omega}{(1-\psi)} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{2(1-\psi^2)}\right]}{A^2},$$
$$+3\frac{\exp\left[\frac{2\omega}{(1-\psi)} + \frac{4\sigma_\eta^2}{2(1-\psi^2)}\right]}{A^2}$$

where now

$$A = \lambda^2 \left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right].$$

Now for $\alpha = 0.5$we get

$$\kappa(y_t) = \frac{\left(\lambda^4 + 6\lambda^2 + 3\right) \exp\left[\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left\{\left(\lambda^2+1\right) \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \lambda^2\right\}^2}$$
$$+\lambda^2 \frac{6\left(\lambda^2+1\right) \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 4\left(\lambda^2+3\right) \exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\lambda^2}{\left\{\left(\lambda^2+1\right) \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \lambda^2\right\}^2}. \qquad (3.7)$$

Notice that

$$\kappa(y_t) - \kappa_{\varepsilon_t^*} = \lambda^2 \left(A^2 - 1\right)^2 \frac{\left(-2A^3 + 3A + 3\right)(A-1)\lambda^2 + 6A}{\left[\left(\lambda^2+1\right)A - \lambda^2\right]^2}$$
$$where\ A = \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right].$$

It is possible that $\kappa(y_t) - \kappa_{\varepsilon_t^*}$ can positive or negative, i.e. the presence of the price of risk, $\lambda$, could either increase or decrease the kurtosis coefficient of the data. However, for plausible values of the parameters, it is clear that the price of risk increases the kurtosis of the data.

## Dynamic Moments

For the dynamic properties of $y_t$ we get that:

$$
\begin{aligned}
Cov(y_t, y_{t-k}) &= E\left(\left(\lambda\left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right] + \varepsilon_t\sigma_t\right)\left(\lambda\left[\sigma_{t-k}^{2\alpha} - E\left(\sigma_{t-k}^{2\alpha}\right)\right] + \varepsilon_{t-k}\sigma_{t-k}\right)\right) \\
&= E\left(\lambda^2\left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right]\left[\sigma_{t-k}^{2\alpha} - E\left(\sigma_{t-k}^{2\alpha}\right)\right] + \lambda\left[\sigma_t^{2\alpha} - E\left(\sigma_t^{2\alpha}\right)\right]\varepsilon_{t-k}\sigma_{t-k}\right) \\
&\quad + E\left(\lambda\left[\sigma_{t-k}^{2\alpha} - E\left(\sigma_{t-k}^{2\alpha}\right)\right]\varepsilon_t\sigma_t + \varepsilon_t\sigma_t\varepsilon_{t-k}\sigma_{t-k}\right) \\
&= \lambda^2\left[E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha}\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_{t-k}^{2\alpha}\right)\right] + \lambda E\left(\sigma_t^{2\alpha}\sigma_{t-k}\varepsilon_{t-k}\right) \\
&= \lambda^2\left[\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right]\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
&\quad + \lambda\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(2\alpha+1)\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\alpha^2 + 4\alpha\psi^k + 1\right)}{8(1-\psi^2)}\right].
\end{aligned}
$$

Notice that for $\rho = 0$we have

$$
Corr(y_t, y_{t-k}) = \frac{\lambda^2\left[\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right]\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]}{\lambda^2\left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] + \exp\left[\frac{\omega}{(1-\psi)}(3.8) + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]}.
$$

Now, given that

$$
\begin{aligned}
Cov\left(\sigma_t^2, \varepsilon_{t-k}^*\right) &= E\left(\sigma_t^2\varepsilon_{t-k}^*\right) = E\left(\sigma_t^2\sigma_{t-k}\varepsilon_{t-k}\right) \\
&= \rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k + 5\right)}{8(1-\psi^2)}\right],
\end{aligned}
$$

the leverage is given by

$$Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right) = \frac{\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8(1-\psi^2)}\right]}{\sqrt{V\left(\sigma_t^2\right)V\left(\varepsilon_{t-k}^*\right)}}$$

$$= \frac{\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{4\psi^k-1}{8(1-\psi^2)}\sigma_\eta^2\right]}{\sqrt{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}}.$$

**Dynamic Asymmetry**

First notice that

$$\begin{gathered} Cov\left(y_t^2, y_{t-k}\right) = E\left(\left[y_t^2 - E\left(y_t^2\right)\right]\left[y_{t-k} - E\left(y_{t-k}\right)\right]\right) = \\ = \lambda^3\left(E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{2\alpha}\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_{t-k}^{2\alpha}\right)\right) + \lambda\left(E\left(\sigma_t^2\sigma_{t-k}^{2\alpha}\right) - E\left(\sigma_t^2\right)E\left(\sigma_{t-k}^{2\alpha}\right)\right) \\ +2c\lambda^2\left(E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha}\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_{t-k}^{2\alpha}\right)\right) + \lambda^2 E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}\sigma_{t-k}\right) \\ +E\left(\sigma_t^2\varepsilon_{t-k}\sigma_{t-k}\right) + 2c\lambda E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}\sigma_{t-k}\right), \end{gathered}$$

and employing equations 3.2, 3.3 and 3.4 we get

$$\begin{aligned} Cov\left(y_t^2, y_{t-k}\right) = {} & \lambda^3\left(\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] \\ & +2\lambda^2\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(4\alpha+1)\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(8\alpha\psi^k+1+16\alpha^2\right)}{8(1-\psi^2)}\right] \\ & +2c\lambda\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(2\alpha+1)\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\alpha\psi^k+1+4\alpha^2\right)}{8(1-\psi^2)}\right] \\ & +\lambda\left(\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)} + \frac{(\alpha^2+1)\sigma_\eta^2}{2(1-\psi^2)}\right] \\ & +2c\lambda^2\left(\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\ & +\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8(1-\psi^2)}\right]. \end{aligned}$$

For $\lambda = 0$ we get

$$Corr\left(y_t^2, y_{t-k}\right) = \rho\sigma_\eta\psi^{k-1}\frac{\exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+1\right)}{8(1-\psi^2)}\right]}{\sqrt{4c^2 + \left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]}}.$$

For $\rho = 0$ we get

$$\begin{aligned} Cov\left(y_t^2, y_{t-k}\right) &= \lambda^3\left(\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{5\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] \\ &+\lambda\left(\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)\exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)} + \frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] \\ &+2c\lambda^2\left(\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)\exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]. \end{aligned} \tag{3.9}$$

For $c = 0$

$$\begin{aligned} Cov\left(y_t^2, y_{t-k}\right) &= \lambda^3\left(\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)\exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{5\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] \\ &+\lambda\left(\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)\exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)} + \frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] \\ &+2\lambda^2\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(4\alpha+1)\,\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2\left(5\alpha\psi^k + 7\alpha^2 + 1\right)}{8\left(1-\psi^2\right)}\right] \\ &+\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right]. \end{aligned}$$

Notice that it is possible to have $Cov\left(y_t^2, y_{t-k}\right) > 0$ for $\rho < 0$, i.e. to have the leverage effect, but positive dynamic asymmetry.

**Volatility Clustering**

As

$$Cov\left(\sigma_t,\sigma_{t-k}\right)=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right],$$

$$Cov\left(\sigma_t^2,\sigma_{t-k}\right)=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{3\omega}{2\left(1-\psi\right)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]=Cov\left(\sigma_t,\sigma_{t-k}^2\right)$$

and

$$Cov\left(\sigma_t^2,\sigma_{t-k}^2\right)=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right],$$

and it follows that

$$Corr\left(\sigma_t^2,\sigma_{t-k}^2\right)=\frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}.$$

Further

$$\begin{aligned}Cov\left(\varepsilon_t^{*2},{\varepsilon_{t-k}^{*}}^2\right)&=Cov\left(\varepsilon_t^2\sigma_t^2,\varepsilon_{t-k}^2\sigma_{t-k}^2\right)=E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-E^2\left(\varepsilon_t^2\sigma_t^2\right)\\&=E\left(\sigma_t^2\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-E^2\left(\sigma_t^2\right)\\&=\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right]\end{aligned}$$

and

$$\begin{aligned}Corr\left(\varepsilon_t^{*2},{\varepsilon_{t-k}^{*}}^2\right)&=\frac{\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right]}{\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{1-\psi}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right]}\\&=\frac{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}\end{aligned}$$

**Dynamic Kurtosis**

Now as

$$Cov\left(\varepsilon_t\sigma_t, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = 0,\ Cov\left(\varepsilon_t\sigma_t^{2\alpha}, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = 0\ \ and$$
$$Cov\left(\varepsilon_t^2\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = Cov\left(\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right),$$

we get that

$$\begin{gathered}Cov\left(y_t^2, y_{t-k}^2\right) = \lambda^4 Cov\left(\sigma_t^{4\alpha}, \sigma_{t-k}^{4\alpha}\right) + \lambda^2 Cov\left(\sigma_t^{4\alpha}, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + 2c\lambda^3 Cov\left(\sigma_t^{4\alpha}, \sigma_{t-k}^{2\alpha}\right) \\ +2c\lambda^2 E\left(\sigma_t^{4\alpha}\sigma_{t-k}\varepsilon_{t-k}\right) + 2\lambda^3 E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right) + \lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}^{4\alpha}\right) + Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) \\ +2c\lambda Cov\left(\sigma_t^2, \sigma_{t-k}^{2\alpha}\right) + 2cE\left(\sigma_t^2\sigma_{t-k}\varepsilon_{t-k}\right) + 2\lambda E\left(\sigma_t^2\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right) + 2c\lambda^3 Cov\left(\sigma_t^{2\alpha}, \sigma_{t-k}^{4\alpha}\right) \\ +2c\lambda Cov\left(\sigma_t^{2\alpha}, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + 4c^2\lambda^2 Cov\left(\sigma_t^{2\alpha}, \sigma_{t-k}^{2\alpha}\right) + 4c^2\lambda E\left(\sigma_t^{2\alpha}\sigma_{t-k}\varepsilon_{t-k}\right) \\ +4c\lambda^2 E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right),\end{gathered}$$

and employing equations 3.2, 3.5, 3.3 and 3.4 we get

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) = \lambda^4 & \left\{\exp\left[\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
+2\lambda^2 & \left\{\left(1+2\alpha^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)}+\frac{(4\alpha^2+1)\,\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
& +4c\lambda^3\left\{\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)}+\frac{5\alpha^2\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
& +4\lambda^3\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(6\alpha+1)\,\omega}{2\,(1-\psi)}+\frac{\sigma_\eta^2\left(8\alpha\,(2\alpha+1)\,\psi^k+(2\alpha+1)^2+16\alpha^2\right)}{8\,(1-\psi^2)}\right] \\
& +4c\lambda^2\alpha\rho\sigma_\eta\psi^{k-1}\mathcal{A}\exp\left[\frac{(4\alpha+1)\,\omega}{2\,(1-\psi)}+\frac{\sigma_\eta^2\left(4\alpha\psi^k+1+8\alpha^2\right)}{8\,(1-\psi^2)}\right] \\
& +\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right] \\
+2c\lambda & \left\{\left(2+\alpha^2\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-2\right\}\exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)}+\frac{(\alpha^2+1)\,\sigma_\eta^2}{2\,(1-\psi^2)}\right] \\
+2\rho\sigma_\eta\psi^{k-1} & \left\{\lambda\exp\left[\frac{\alpha\omega}{(1-\psi)}+\frac{\alpha\left(2\psi^k+\alpha+1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right]+c\right\}\exp\left[\frac{3\omega}{2\,(1-\psi)}+\frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\,(1-\psi^2)}\right] \\
& +4c^2\lambda^2\left\{\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\
& +4c^2\lambda\alpha\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(2\alpha+1)\,\omega}{2\,(1-\psi)}+\frac{\sigma_\eta^2\left(4\alpha\psi^k+1+4\alpha^2\right)}{8\,(1-\psi^2)}\right],
\end{aligned}
$$

where

$$
\mathcal{A} = \exp\left[\frac{\alpha\left(\psi^k+2\alpha\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right]+\exp\left[\frac{\alpha\left(2\alpha\psi^k+1\right)\sigma_\eta^2}{2\,(1-\psi^2)}\right].
$$

For $\lambda = 0$ we get

$$Cov\left(y_t^2, y_{t-k}^2\right) = \left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$+2c\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8(1-\psi^2)}\right]$$

and

$$V\left(y_t^2\right) = 4c^2\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{2(1-\psi^2)}\right]+\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right].$$

Now for $\rho = 0$, we get:

$$\begin{aligned} Cov\left(y_t^2, y_{t-k}^2\right) = {} & \lambda^4\left\{\exp\left[\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{4\alpha\omega}{(1-\psi)}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \\ & +2\lambda^2\left\{\exp\left[\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(2\alpha+1)\omega}{(1-\psi)}+\frac{(4\alpha^2+1)\sigma_\eta^2}{2(1-\psi^2)}\right] \\ & +4c\lambda^3\left\{\exp\left[\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{3\alpha\omega}{(1-\psi)}+\frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right] \\ & +\left\{\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right] \\ & +4c\lambda\left\{\exp\left[\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{(\alpha+1)\omega}{(1-\psi)}+\frac{(\alpha^2+1)\sigma_\eta^2}{2(1-\psi^2)}\right] \\ & +4c^2\lambda^2\left\{\exp\left[\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\alpha\omega}{(1-\psi)}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right], \end{aligned} \tag{3.10}$$

and

$$V\left(y_t^2\right) = \lambda^4 \left\{\exp\left[\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] \tag{3.11}$$

$$+4c\lambda^3 \left\{\exp\left[\frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{3\alpha\omega}{(1-\psi)} + \frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right]$$

$$+4c\lambda \left\{3\exp\left[\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{(\alpha+1)\,\omega}{(1-\psi)} + \frac{(\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right]$$

$$+4c^2\lambda^2 \left\{\exp\left[\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{2\alpha\omega}{(1-\psi)} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$

$$+2\lambda^2 \left\{3\exp\left[\frac{2\alpha\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{(4\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right]$$

$$+4c^2 \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right] + \left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right].$$

Further, if additionally $\rho = 0$ as well, i.e. $\rho = c = 0$, we get:

$$Cov\left(y_t^2, y_{t-k}^2\right) = \lambda^4 \left\{\exp\left[\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{4\alpha\omega}{(1-\psi)} + \frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right]$$

$$+2\lambda^2 \left\{\exp\left[\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{(4\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right]$$

$$+\left\{\exp\left[\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right\} \exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right].$$

# Appendix B

The SV2-M is given by

$$y_t = c + \lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t,$$
$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_t = \frac{\omega}{1-\psi} + \sum_{i=0}^{\infty}\psi^i\eta_{t-i}.$$

First, form Perez, Ruiz and Veiga (2009) [60] we have that

$$E\left[\varepsilon_t\exp\left(A\eta_t\right)\right] = \rho\sigma_\eta A\exp\left[\frac{A^2\sigma_\eta^2}{2}\right], \tag{3.12}$$

and as $\Phi\left(\frac{3}{2},\frac{1}{2};z\right)=2\left(z+\frac{1}{2}\right)\Phi\left(\frac{1}{2},\frac{1}{2};z\right) = 2\left(z+\frac{1}{2}\right)\exp\left(z\right)$ see Gradshteyn and Ryzhik (1994) [38], formula 9.212.4, where $\Phi\left(.,.;z\right)$ is the confluent hypergeometric function, and from Perez, Ruiz and Veiga (2009) [60], equation 19, we get

$$E\left(\varepsilon_t^2\exp\left(A\eta_t\right)\right) = \left(1+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right].$$

Additionally,

$$E\left(\varepsilon_t^3\exp\left(A\eta_t\right)\right) = E\left(\varepsilon_t^3 E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]E\left(\varepsilon_t^3\exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right)$$
$$= \exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\frac{1}{\sqrt{2\pi}}\int\left(x-\left(\rho\sigma_\eta A\right)\right)^3\exp\left[-\frac{\left(x-\rho\sigma_\eta A\right)^2}{2}\right]dx$$
$$+\exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\frac{1}{\sqrt{2\pi}}\int\left[3\left(\rho\sigma_\eta A\right)x^2 - 3\left(\rho\sigma_\eta A\right)^2 x + \left(\rho\sigma_\eta A\right)^3\right]\exp\left[-\frac{\left(x-\rho\sigma_\eta A\right)^2}{2}\right]dx$$
$$= \left(\rho\sigma_\eta A\right)\left(3+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right].$$

Now notice that

$$\ln\sigma_t^2 = \frac{\omega}{1-\psi} + \sum_{i=0}^{\infty}\psi^i\eta_{t-i}$$

then form equation 3.2 we have

$$E\left(\sigma_t^{2C}\right)=E\left[\exp\left(C\ln\sigma_t^2\right)\right]=\exp\left(\frac{C\omega}{1-\psi}+\frac{C^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right). \qquad (3.13)$$

It follows that $E\left(\sigma_t\right)$, $E\left(\sigma_t^2\right)$, $E\left(\sigma_t^3\right)$ and $E\left(\sigma_t^4\right)$ are the same as for the SV1-M model.

## Static Moments

First,

$$E\left(\sigma_t^B\varepsilon_t^A\right)=E\left(\exp\left(\frac{B\omega}{2}+\frac{B}{2}\psi\ln\sigma_{t-1}^2+\frac{B}{2}\eta_t\right)\varepsilon_t^A\right)$$

$$=E\left(\exp\left(\frac{B}{2}\eta_t\right)\varepsilon_t^A\right)\exp\left(\frac{B\omega}{2}\frac{1}{(1-\psi)}+\frac{(B\psi)^2\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \qquad (3.14)$$

and employing equation 3.12 we get

$$E\left(\sigma_t\varepsilon_t\right)=\frac{\rho\sigma_\eta}{2}\exp\left(\frac{1}{2}\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).$$

It follows that

$$E\left(y_t\right)=c+\lambda E\left(\sigma_t^{2\alpha}\right)+E\left(\varepsilon_t\sigma_t\right)=c$$
$$+\lambda\exp\left(\frac{\alpha\omega}{1-\psi}+\frac{\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)+\frac{\rho\sigma_\eta}{2}\exp\left(\frac{\omega}{2(1-\psi)}+\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

Notice that depending on the parameter values the unconditional risk premium can be either positive or negative.

Now to find $Var\left(y_t\right)$ we need $E\left(\sigma_t^2\varepsilon_t\right)$ and $E\left(\varepsilon_t^2\sigma_t^2\right)$. Hence employing equations 3.14 and 3.12 we get

$$V\left(\varepsilon_t^*\right)=V\left(\varepsilon_t\sigma_t\right)=\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}$$
$$\times\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).$$

Hence,

$$Var\left(y_t\right) = \lambda^2 V\left(\sigma_t^{2\alpha}\right) + V\left(\varepsilon_t\sigma_t\right) + 2Cov\left(\sigma_t^{2\alpha}, \varepsilon_t\sigma_t\right)$$

and employing the formulae above we get the result of the main text.

From formula 9.212.4 in Gradshteyn and Ryzhik (1994) [38] we get

$$\Phi\left(\frac{5}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right) = \frac{2}{3}\left[\left(\frac{A^2\rho^2\sigma_\eta^2}{2} + \frac{5}{2}\right)\Phi\left(\frac{3}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right) - \Phi\left(\frac{1}{2}, \frac{1}{2}, \frac{A^2\rho^2\sigma_\eta^2}{2}\right)\right],$$

and from Perez, Ruiz and Veiga (2009) [60], equation 19, we get

$$E\left(\varepsilon_t^4 \exp\left(A\eta_t\right)\right) = \exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\left[3 + 6\left(\rho\sigma_\eta A\right)^2 + \left(\rho\sigma_\eta A\right)^4\right].$$

Now, employing equation 3.14, we have that the skewness and kurtosis coefficients, of the mean error $\varepsilon_t^* = \varepsilon_t\sigma_t$, are as given in the main text.

The asymmetry of the observed process, $y_t$, is given by

$$\begin{aligned} E\left(y_t - E\left(y_t\right)\right)^3 &= E\left(\lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t - \left[\lambda E\left(\sigma_t^{2\alpha}\right) + E\left(\varepsilon_t\sigma_t\right)\right]\right)^3 \\ &= \lambda^3 E\left(\sigma_t^{6\alpha}\right) + 3\lambda^2 E\left(\sigma_t^{4\alpha+1}\varepsilon_t\right) + 3\lambda E\left(\sigma_t^{2\alpha+2}\varepsilon_t^2\right) + E\left(\varepsilon_t^3\sigma_t^3\right) \\ &-3\lambda^3 E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^{4\alpha}\right) - 6\lambda^2 E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - 3\lambda E\left(\sigma_t^{2\alpha}\right)E\left(\varepsilon_t^2\sigma_t^2\right) \\ &-3\lambda^2 E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_t\sigma_t\right) - 6\lambda E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)E\left(\varepsilon_t\sigma_t\right) - 3E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\varepsilon_t\sigma_t\right) \\ &+2\lambda^3 E^3\left(\sigma_t^{2\alpha}\right) + 6\lambda^2 E^2\left(\sigma_t^{2\alpha}\right)E\left(\varepsilon_t\sigma_t\right) + 6\lambda E\left(\sigma_t^{2\alpha}\right)E^2\left(\varepsilon_t\sigma_t\right) + 2E^3\left(\varepsilon_t\sigma_t\right). \end{aligned}$$

It follows that

$$\begin{aligned} E\left(y_t - E\left(y_t\right)\right)^3 &= \lambda^3\left[\exp\left(\frac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 2\right]\exp\left(\frac{3\alpha\omega}{1-\psi} + \frac{3\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\ &+\frac{3}{2}\lambda^2\rho\sigma_\eta\left[\begin{array}{c} (4\alpha+1)\exp\left(\frac{\alpha(\alpha+1)\sigma_\eta^2}{(1-\psi^2)}\right) \\ -2\left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right) \\ -\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 2 \end{array}\right]\exp\left(\frac{\left(4\alpha+1\right)\omega}{2\left(1-\psi\right)} + \frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\ &+3\lambda\left[\left(1 + \left(\rho\sigma_\eta\left(\alpha+1\right)\right)^2\right)\exp\left(\frac{\alpha\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1 + \left(\rho\sigma_\eta\right)^2\right)\right]\exp\left(\frac{\left(\alpha+1\right)\omega}{(1-\psi)} + \frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\ &+\frac{3}{2}\lambda\left(\rho\sigma_\eta\right)^2\left[1 - \left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)\right]\exp\left(\frac{\left(\alpha+1\right)\omega}{(1-\psi)} + \frac{\left(2\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\ &+\frac{3}{2}\rho\sigma_\eta\left[\begin{array}{c} \left(3 + \left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right) \\ -\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + \frac{1}{6}\left(\rho\sigma_\eta\right)^2 \end{array}\right]\exp\left(\frac{3\omega}{2\left(1-\psi\right)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right). \end{aligned}$$

Now for $\rho = 0$, as in Koopman and Uspensky (2002), the asymmetry can be only positive, whereas for $\lambda = 0$, as in JPR, we get that $sk\left(y_t\right) = sk\left(\varepsilon_t^*\right)$ and the asymmetry of the observed process can be only negative.

Now

$$
\begin{aligned}
&E\left(y_t - E\left(y_t\right)\right)^4 = \lambda^4 E\left(\sigma_t^{8\alpha}\right) - 4\lambda^4 E\left(\sigma_t^{2\alpha}\right) E\left(\sigma_t^{6\alpha}\right) + 6\lambda^4 E^2\left(\sigma_t^{2\alpha}\right) E\left(\sigma_t^{4\alpha}\right) \\
&-3\lambda^4 E^4\left(\sigma_t^{2\alpha}\right) + 4\lambda^3 E\left(\sigma_t^{6\alpha+1}\varepsilon_t\right) - 12\lambda^3 E\left(\sigma_t^{2\alpha}\right) E\left(\sigma_t^{4\alpha+1}\varepsilon_t\right) - 4\lambda^3 E\left(\sigma_t^{6\alpha}\right) E\left(\varepsilon_t\sigma_t\right) \\
&+12\lambda^3 E^2\left(\sigma_t^{2\alpha}\right) E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - 12\lambda^3 E^3\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t\sigma_t\right) + 12\lambda^3 E\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t\sigma_t\right) E\left(\sigma_t^{4\alpha}\right) \\
&+6\lambda^2 E\left(\sigma_t^{4\alpha+2}\varepsilon_t^2\right) - 12\lambda^2 E\left(\sigma_t^{2\alpha}\right) E\left(\sigma_t^{2\alpha+2}\varepsilon_t^2\right) - 12\lambda^2 E\left(\sigma_t^{4\alpha+1}\varepsilon_t\right) E\left(\varepsilon_t\sigma_t\right) \\
&+6\lambda^2 E^2\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t^2\sigma_t^2\right) + 24\lambda^2 E\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t\sigma_t\right) E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) + 6\lambda^2 E\left(\sigma_t^{4\alpha}\right) E^2\left(\varepsilon_t\sigma_t\right) \\
&-18\lambda^2 E^2\left(\sigma_t^{2\alpha}\right) E^2\left(\varepsilon_t\sigma_t\right) + 4\lambda E\left(\sigma_t^{2\alpha+3}\varepsilon_t^3\right) - 12\lambda E\left(\sigma_t^{2\alpha+2}\varepsilon_t^2\right) E\left(\varepsilon_t\sigma_t\right) - 4\lambda E\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t^3\sigma_t^3\right) \\
&+12\lambda E\left(\sigma_t^{2\alpha}\right) E\left(\varepsilon_t\sigma_t\right) E\left(\varepsilon_t^2\sigma_t^2\right) + 12\lambda E^2\left(\varepsilon_t\sigma_t\right) E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - 12\lambda E\left(\sigma_t^{2\alpha}\right) E^3\left(\varepsilon_t\sigma_t\right) \\
&+E\left(\varepsilon_t^4\sigma_t^4\right) - 4E\left(\varepsilon_t^3\sigma_t^3\right) E\left(\varepsilon_t\sigma_t\right) + 6E^2\left(\varepsilon_t\sigma_t\right) E\left(\varepsilon_t^2\sigma_t^2\right) - 3E^4\left(\varepsilon_t\sigma_t\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&E\left(y_t - E\left(y_t\right)\right)^4 = \lambda^4 \boldsymbol{A} \exp\left(\frac{4\alpha\omega}{(1-\psi)} + \frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) \\
&+2\lambda^3\rho\sigma_\eta \boldsymbol{B} \exp\left(\frac{(6\alpha+1)\,\omega}{2(1-\psi)} + \frac{\left(12\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
&+3\lambda^2 \boldsymbol{C} \exp\left(\frac{(2\alpha+1)\,\omega}{(1-\psi)} + \frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) + 2\lambda\rho\sigma_\eta \boldsymbol{D} \exp\left(\frac{(2\alpha+3)\,\omega}{2(1-\psi)} + \frac{\left(4\alpha^2+3\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
&+\left\{\left[3 + 24\left(\rho\sigma_\eta\right)^2 + 16\left(\rho\sigma_\eta\right)^4\right] \exp\left(\frac{3\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - \frac{3}{16}\left(\rho\sigma_\eta\right)^4\right\} \exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\
&-3\left(\rho\sigma_\eta\right)^2 \left\{\begin{array}{c} \left(3 + \left(\rho\sigma_\eta\frac{3}{2}\right)^2\right) \exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) \\ -\frac{1}{2}\left(1 + \left(\rho\sigma_\eta\right)^2\right) \end{array}\right\} \exp\left(\frac{2\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{A} = &\ \exp\left(\tfrac{6\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 4\exp\left(\tfrac{3\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) + 6\exp\left(\tfrac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) - 3, \\
\boldsymbol{B} = &\ (6\alpha+1)\exp\left(\frac{3\alpha\,(2\alpha+1)\,\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 3\,(4\alpha+1)\exp\left(\frac{\alpha\,(\alpha+1)\,\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - \exp\left(\frac{3\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right) \\
&+3\,(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) + 3\exp\left(\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - 3,
\end{aligned}
$$

and

$$\boldsymbol{C} = 2\left(1+(\rho\sigma_\eta(2\alpha+1))^2\right)\exp\left(\frac{[4\alpha(\alpha+2)+1]\,\sigma_\eta^2}{4(1-\psi^2)}\right) + 2\left[1+(\rho\sigma_\eta)^2\right]\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$-4\left(1+(\rho\sigma_\eta(\alpha+1))^2\right)\exp\left(\frac{(4\alpha+1)\,\sigma_\eta^2}{4(1-\psi^2)}\right) + 2(\rho\sigma_\eta)^2(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right)$$
$$-(\rho\sigma_\eta)^2(4\alpha+1)\exp\left(\frac{\alpha(\alpha+1)\,\sigma_\eta^2}{(1-\psi^2)}\right) + \frac{1}{2}(\rho\sigma_\eta)^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)-3\right],$$
$$\boldsymbol{D} = (2\alpha+3)\left(3+\left(\rho\sigma_\eta\frac{(2\alpha+3)}{2}\right)^2\right)\exp\left(\frac{3(2\alpha+1)\,\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$-3\left(1+(\rho\sigma_\eta(\alpha+1))^2\right)\exp\left(\frac{(4\alpha+1)\,\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+3\left\{\left(1+(\rho\sigma_\eta)^2\right)-\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+\frac{3}{4}(\rho\sigma_\eta)^2\left\{(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right)-1\right\}.$$

## Autocorrelation

First, notice that

$$E\left(\sigma_t^B\sigma_{t-k}^A\right) = E\left(\exp\left(\frac{B}{2}\eta_t\right)\varepsilon_t^A\right)\exp\left(\frac{B\omega}{2}\frac{1}{(1-\psi)}+\frac{(B\psi)^2\,\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$= \exp\left((B+A)\frac{\omega}{2(1-\psi)}+\frac{\sigma_\eta^2}{8(1-\psi^2)}\left(2AB\psi^k+A^2+B^2\right)\right)$$

$$E\left(\varepsilon_t^A\sigma_t^B\sigma_{t-k}^D\right) = E\left(\varepsilon_t^A\exp\left(\frac{B}{2}\eta_t\right)\right)\exp\left(\frac{B}{2}\omega\right)$$
$$\times\exp\left((B\psi+D)\frac{\omega}{2(1-\psi)}+\frac{\sigma_\eta^2}{8(1-\psi^2)}\left(2DB\psi^k+D^2+(B\psi)^2\right)\right)$$
$$= E\left(\varepsilon_t^A\exp\left(\frac{B}{2}\eta_t\right)\right)\exp\left((B+D)\frac{\omega}{2(1-\psi)}+\frac{\sigma_\eta^2}{8(1-\psi^2)}\left(2DB\psi^k+D^2+(B\psi)^2\right)\right)$$

$$E\left(\sigma_t^B\sigma_{t-k}^D\varepsilon_{t-k}^C\right) = E\left(\exp\left(\frac{B}{2}\omega + \frac{B}{2}\psi\ln\sigma_{t-1}^2 + \frac{B}{2}\eta_t\right)\varepsilon_{t-k}^C\sigma_{t-k}^D\right)$$
$$= \exp\left[\frac{B}{2}\omega + \frac{B}{2}\psi\omega + ... + \frac{B}{2}\psi^{k-1}\omega + \frac{B^2\sigma_\eta^2}{8} + \frac{B^2\psi^2\sigma_\eta^2}{8} + ... + \frac{B^2\psi^{2k-2}\sigma_\eta^2}{8}\right]$$
$$\times E\left(\exp\left(\frac{B}{2}\psi^k\ln\sigma_{t-k}^2\right)\varepsilon_{t-k}^C\sigma_{t-k}^D\right)$$
$$= \exp\left[\frac{B}{2}\omega\frac{1-\psi^k}{1-\psi} + \frac{B^2\sigma_\eta^2}{8}\frac{1-\psi^{2k}}{1-\psi^2}\right]E\left(\varepsilon_{t-k}^C\sigma_{t-k}^{B\psi^k+D}\right)$$
$$= E\left(\exp\left(\frac{B\psi^k+D}{2}\eta_{t-k}\right)\varepsilon_{t-k}^C\right)$$
$$\times\exp\left((B+D)\frac{\omega}{2(1-\psi)} + \left(B^2 + \left(\psi^2-1\right)B^2\psi^{2k} + 2BD\psi^{k+2} + D^2\psi^2\right)\frac{\sigma_\eta^2}{8(1-\psi^2)}\right)$$

$$E\left(\varepsilon_t^A\sigma_t^B\varepsilon_{t-k}^C\sigma_{t-k}^D\right) = E\left(\varepsilon_t^A\exp\left(\frac{B}{2}\eta_t\right)\right)\exp\left(\frac{\omega B}{2}\right)E\left(\exp\left(\frac{B}{2}\psi\ln\sigma_{t-1}^2\right)\varepsilon_{t-k}^C\sigma_{t-k}^D\right)$$
$$= E\left(\varepsilon_t^A\exp\left(\frac{B}{2}\eta_t\right)\right)E\left(\exp\left(\frac{B\psi^k+D}{2}\eta_{t-k}\right)\varepsilon_{t-k}^C\right)$$
$$\times\exp\left((B+D)\frac{\omega}{2(1-\psi)} + \left((B\psi)^2 + \left(\psi^2-1\right)B^2\psi^{2k} + 2BD\psi^{k+2} + D^2\psi^2\right)\frac{\sigma_\eta^2}{8(1-\psi^2)}\right)$$

It follows that

$$Cov\left(y_t, y_{t-k}\right) = Cov\left(c + \lambda\sigma_t^{2\alpha} + \varepsilon_t\sigma_t, c + \lambda\sigma_{t-k}^{2\alpha} + \varepsilon_{t-k}\sigma_{t-k}\right)$$
$$= \lambda^2 Cov\left(\sigma_t^{2\alpha}, \sigma_{t-k}^{2\alpha}\right) + \lambda Cov\left(\sigma_t^{2\alpha}, \varepsilon_{t-k}\sigma_{t-k}\right) + \lambda Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}^{2\alpha}\right) + Cov\left(\varepsilon_t\sigma_t, \varepsilon_{t-k}\sigma_{t-k}\right)$$

and the result of in the main text follows.

## Leverage $Cov\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right)$

Now

$$Cov\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) = E\left(\sigma_t^2\sigma_{t-k}\varepsilon_{t-k}\right) - E\left(\sigma_t^2\right)E\left(\sigma_{t-k}\varepsilon_{t-k}\right)$$
$$= \rho\sigma_\eta\frac{2\psi^k+1}{2}\exp\left[3\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\left(4\psi^k+5\right)\right] - \frac{1}{2}\rho\sigma_\eta\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8(1-\psi^2)}\right)$$

and

$$Corr\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) = \frac{1}{2}\rho\sigma_\eta \frac{\left(2\psi^k+1\right)\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]-1}{\sqrt{\left[\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\left[\left(1+(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\frac{1}{2}\rho\sigma_\eta\right)^2\right]}},$$

which has the sign of $\rho$, for any $k$, provide that $\psi > 0$ due to volatility clustering.

## Dynamic Asymmetry $Cov\left(y_t^2, y_{t-k}\right)$

First notice that

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}\right) &= E\left(y_t^2, y_{t-k}\right) - E\left(y_t^2\right)E\left(y_{t-k}\right) = \\
&= \lambda^3\left[E\left(\sigma_{t-k}^{2\alpha}\sigma_t^{4\alpha}\right) - E\left(\sigma_{t-k}^{2\alpha}\right)E\left(\sigma_t^{4\alpha}\right)\right] + \lambda\left[E\left(\sigma_{t-k}^{2\alpha}\varepsilon_t^2\sigma_t^2\right) - E\left(\sigma_{t-k}^{2\alpha}\right)E\left(\varepsilon_t^2\sigma_t^2\right)\right] \\
&\quad +2c\lambda^2\left[E\left(\sigma_{t-k}^{2\alpha}\sigma_t^{2\alpha}\right) - E^2\left(\sigma_t^{2\alpha}\right)\right] + 2c\lambda\left[E\left(\sigma_{t-k}^{2\alpha}\varepsilon_t\sigma_t\right) - E\left(\sigma_{t-k}^{2\alpha}\right)E\left(\varepsilon_t\sigma_t\right)\right] \\
&\quad +2\lambda^2\left[E\left(\sigma_{t-k}^{2\alpha}\sigma_t^{2\alpha+1}\varepsilon_t\right) - E\left(\sigma_{t-k}^{2\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right] \\
&\quad +\lambda^2\left[E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}\sigma_{t-k}\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_{t-k}\sigma_{t-k}\right)\right] + E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}\sigma_{t-k}\right) - E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\varepsilon_{t-k}\sigma_{t-k}\right) \\
&\quad +2c\lambda\left[E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}\sigma_{t-k}\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\varepsilon_{t-k}\sigma_{t-k}\right)\right] + 2c\left[E\left(\varepsilon_t\sigma_t\varepsilon_{t-k}\sigma_{t-k}\right) - E\left(\varepsilon_t\sigma_t\right)E\left(\varepsilon_{t-k}\sigma_{t-k}\right)\right] \\
&\quad +2\lambda\left[E\left(\sigma_t^{2\alpha+1}\varepsilon_t\varepsilon_{t-k}\sigma_{t-k}\right) - E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)E\left(\varepsilon_{t-k}\sigma_{t-k}\right)\right]
\end{aligned}$$

It follows that

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-k}\right) = {} & \lambda^3\left[\exp\left(\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{3\alpha\omega}{1-\psi}+\frac{5\alpha^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\
& +\lambda\left(1+(\rho\sigma_\eta)^2\right)\left[\exp\left(\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{(\alpha+1)\,\omega}{1-\psi}+\frac{\left(\alpha^2+1\right)\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\
& +2c\lambda^2\left[\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{2\alpha\omega}{1-\psi}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right) \\
& +2c\lambda\rho\sigma_\eta\left\{\left(\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{(2\alpha+1)\,\omega}{2\left(1-\psi\right)}+\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +\lambda^2\rho\sigma_\eta\boldsymbol{F}\exp\left(\frac{(4\alpha+1)\,\omega}{2\left(1-\psi\right)}+\frac{\left(8\alpha^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +\frac{\rho\sigma_\eta}{2}\left(1+(\rho\sigma_\eta)^2\right)\left[\left(2\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{3\omega}{2\left(1-\psi\right)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +c\frac{(\rho\sigma_\eta)^2}{2}\left[\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
& +\lambda\left(\rho\sigma_\eta\right)^2\frac{(2\alpha+1)}{2}\boldsymbol{G}\exp\left(\frac{(\alpha+1)\,\omega}{(1-\psi)}+\frac{\left((2\alpha+1)^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
\boldsymbol{F} = {} & (2\alpha+1)\left[\exp\left(\frac{\alpha\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right]\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\
& +\frac{1}{2}\left[\left(4\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right), \\
\boldsymbol{G} = {} & \left[(2\alpha+1)\,\psi^k+1\right]\exp\left(\frac{(2\alpha+1)\,\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1.
\end{aligned}
$$

## Volatility Clustering

It is easy to prove that $Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$ is the same as for the SV1-M model.

First, notice that

$$
\begin{aligned}
&E\left(\left(c+\lambda\sigma_t^{2\alpha}+\varepsilon_t\sigma_t\right)^2\left(c+\lambda\sigma_{t-k}^{2\alpha}+\varepsilon_{t-k}\sigma_{t-k}\right)^2\right)=\\
&=c^4+c^2\lambda^2E\left(\sigma_{t-k}^{4\alpha}\right)+c^2E\left(\varepsilon_{t-k}^2\sigma_{t-k}^2\right)+2c^3\lambda E\left(\sigma_{t-k}^{2\alpha}\right)+2c^3E\left(\varepsilon_{t-k}\sigma_{t-k}\right)\\
&+2c^2\lambda E\left(\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+c^2\lambda^2E\left(\sigma_t^{4\alpha}\right)+\lambda^4E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{4\alpha}\right)+\lambda^2E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}^2\sigma_{t-k}^2\right)\\
&+2c\lambda^3E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{2\alpha}\right)+2c\lambda^2E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}\sigma_{t-k}\right)+2\lambda^3E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+c^2E\left(\varepsilon_t^2\sigma_t^2\right)\\
&+\lambda^2E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{4\alpha}\right)+E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}^2\sigma_{t-k}^2\right)+2c\lambda E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{2\alpha}\right)+2cE\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}\sigma_{t-k}\right)\\
&+2\lambda E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+2c^3\lambda E\left(\sigma_t^{2\alpha}\right)+2c\lambda^3E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{4\alpha}\right)+2c\lambda E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}^2\sigma_{t-k}^2\right)\\
&+4c^2\lambda^2E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha}\right)+4c^2\lambda E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}\sigma_{t-k}\right)+4c\lambda^2E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)\\
&+2c^3E\left(\varepsilon_t\sigma_t\right)+2c\lambda^2E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{4\alpha}\right)+2cE\left(\varepsilon_t\sigma_t\varepsilon_{t-k}^2\sigma_{t-k}^2\right)+4c^2\lambda E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{2\alpha}\right)\\
&+4c^2E\left(\varepsilon_t\sigma_t\varepsilon_{t-k}\sigma_{t-k}\right)+4c\lambda E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+2c^2\lambda E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\\
&+2\lambda^3E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{4\alpha}\right)+2\lambda E\left(\sigma_t^{2\alpha+1}\varepsilon_t\varepsilon_{t-k}^2\sigma_{t-k}^2\right)+4c\lambda^2E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{2\alpha}\right)\\
&+4c\lambda E\left(\sigma_t^{2\alpha+1}\varepsilon_t\varepsilon_{t-k}\sigma_{t-k}\right)+4\lambda^2E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
Cov\left(y_t^2,y_{t-k}^2\right)&=\lambda^4\left[E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{4\alpha}\right)-E^2\left(\sigma_t^{4\alpha}\right)\right]+4c\lambda^3\left[E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{4\alpha}\right)-E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha}\right)\right]\\
&+2\lambda^3\left[E\left(\sigma_t^{4\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{4\alpha}\right)-2E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]\\
&+2c\lambda^2\left[E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}\sigma_{t-k}\right)+E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{4\alpha}\right)-2E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_t\sigma_t\right)\right]\\
&+2c\lambda\left[E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{2\alpha}\right)+E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-2E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\sigma_t^{2\alpha}\right)\right]\\
&+2c\left[E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}\sigma_{t-k}\right)+E\left(\varepsilon_t\sigma_t\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-2E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\varepsilon_t\sigma_t\right)\right]\\
&+2\lambda\left[E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+E\left(\sigma_t^{2\alpha+1}\varepsilon_t\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-2E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]\\
&+4c^2\lambda^2\left[E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha}\right)-E^2\left(\sigma_t^{2\alpha}\right)\right]+4\lambda^2\left[E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)-E^2\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]\\
&+4c^2\lambda\left[E\left(\sigma_t^{2\alpha}\varepsilon_{t-k}\sigma_{t-k}\right)+E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{2\alpha}\right)-2E\left(\sigma_t^{2\alpha}\right)E\left(\varepsilon_t\sigma_t\right)\right]\\
&+4c\lambda^2\left[E\left(\sigma_t^{2\alpha}\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)+E\left(\sigma_t^{2\alpha+1}\varepsilon_t\sigma_{t-k}^{2\alpha}\right)-2E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]\\
&+4c\lambda\left[E\left(\sigma_t^{2\alpha+1}\varepsilon_t\varepsilon_{t-k}\sigma_{t-k}\right)+E\left(\varepsilon_t\sigma_t\sigma_{t-k}^{2\alpha+1}\varepsilon_{t-k}\right)-2E\left(\varepsilon_t\sigma_t\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]\\
&+\lambda^2\left[E\left(\sigma_t^{4\alpha}\varepsilon_{t-k}^2\sigma_{t-k}^2\right)+E\left(\varepsilon_t^2\sigma_t^2\sigma_{t-k}^{4\alpha}\right)-2E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_t^2\sigma_t^2\right)\right]\\
&+4c^2\left[E\left(\varepsilon_t\sigma_t\varepsilon_{t-k}\sigma_{t-k}\right)-E^2\left(\varepsilon_t\sigma_t\right)\right]+E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-k}^2\sigma_{t-k}^2\right)-E^2\left(\varepsilon_t^2\sigma_t^2\right)
\end{aligned}
$$

Hence,

$$Cov\left(y_t^2, y_{t-k}^2\right) = \lambda^4 \left[\exp\left(\frac{4\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right] \exp\left(\frac{4\alpha\omega}{1-\psi} + \frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$

$$+2\mathcal{A}\lambda^3 \exp\left(\frac{3\alpha\omega}{1-\psi} + \frac{20\alpha^2\sigma_\eta^2}{8(1-\psi^2)}\right) + \mathcal{B}\lambda^2 \exp\left(\frac{2\alpha\omega}{1-\psi} + \frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$

$$+\mathcal{C}\lambda^2\rho\sigma_\eta \exp\left(\frac{(4\alpha+1)\,\omega}{2(1-\psi)} + \frac{(8\alpha^2+1)\,\sigma_\eta^2}{8(1-\psi^2)}\right) + \mathcal{D}\lambda\rho\sigma_\eta \exp\left(\frac{(2\alpha+3)\,\omega}{2(1-\psi)} + \frac{\left((2\alpha+1)^2+4\right)\sigma_\eta^2}{8(1-\psi^2)}\right)$$

$$+\mathcal{F}c\lambda\rho\sigma_\eta \exp\left(\frac{(2\alpha+1)\,\omega}{2(1-\psi)} + \frac{(4\alpha^2+1)\,\sigma_\eta^2}{8(1-\psi^2)}\right) + 2\mathcal{G}c\rho\sigma_\eta \exp\left(\frac{3\omega}{2(1-\psi)} + \frac{5\sigma_\eta^2}{8(1-\psi^2)}\right)$$

$$+2\mathcal{H}c\lambda \exp\left(\frac{(\alpha+1)\,\omega}{1-\psi} + \frac{(\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right)$$

$$+c^2\left(\rho\sigma_\eta\right)^2 \left[\left(\psi^k+1\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right) - 1\right] \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$

$$+\left(1+(\rho\sigma_\eta)^2\right)\left\{\begin{array}{c} \left(1+(\rho\sigma_\eta)^2\left(\psi^k+1\right)^2\right)\exp\left(\frac{\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) \\ -\left(1+(\rho\sigma_\eta)^2\right)\end{array}\right\}\exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right)$$

where

$$\mathcal{A} = \rho\sigma_\eta \left[\begin{array}{c} \left[2\alpha\psi^k + (2\alpha+1)\right]\exp\left(\frac{\alpha(2\alpha+1)\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) \\ -(2\alpha+1)\end{array}\right]\exp\left(\frac{\omega}{2(1-\psi)} + \frac{(4\alpha+1)\,\sigma_\eta^2}{8(1-\psi^2)}\right),$$

$$+2c\left[\exp\left(\frac{2\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right],$$

$$\mathcal{B} = \left[\begin{array}{c} \left[(\rho\sigma_\eta)^2\left(\left(2\alpha\psi^k+1\right)^2+1\right)+2\right]\exp\left(\frac{2\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) \\ -2\left(1+(\rho\sigma_\eta)^2\right)\end{array}\right]\exp\left(\frac{\omega}{1-\psi} + \frac{(2\alpha^2+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right)$$

$$+4c^2\left[\exp\left(\frac{\alpha^2\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right],$$

and

$$\mathcal{C} = 2c\left[\left(2\alpha\psi^k + 1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right]\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$

$$+\rho\sigma_\eta\left(2\alpha+1\right)^2\left[\left(\psi^k + 1\right)\exp\left(\frac{\left(2\alpha+1\right)^2\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right]\exp\left(\frac{\omega}{2\left(1-\psi\right)} + \frac{\left(8\alpha+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+4c\left\{\left(\alpha\psi^k + 2\alpha + 1\right)\exp\left(\frac{\alpha\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - \left(2\alpha+1\right)\right\}\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

$$\mathcal{D} = \left\{\begin{array}{c} \left(2\alpha+1\right)\left[1 + \left(\rho\sigma_\eta\right)^2\frac{\left(\left(2\alpha+1\right)\psi^k+2\right)^2}{4}\right] \\ +\left[1+\left(\rho\sigma_\eta\right)^2\right]\left(2\psi^k + 2\alpha + 1\right)\end{array}\right\}\exp\left(\frac{\left(2\alpha+1\right)\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 2\left(1+\left(\rho\sigma_\eta\right)^2\right)\left(2\alpha+1\right),$$

$$\mathcal{F} = \rho\sigma_\eta\left\{\begin{array}{c}\left[\begin{array}{c}\left(\left(2\alpha+1\right)^2+1\right)\psi^k \\ +2\left(2\alpha+1\right)\end{array}\right]\exp\left(\frac{(2\alpha+1)\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right) \\ -2\left(2\alpha+1\right)\end{array}\right\}\exp\left(\frac{\omega}{2\left(1-\psi\right)} + \frac{\left(4\alpha+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

$$+4c\left[\left(\alpha\psi^k+1\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right],$$

$$\mathcal{G} = \left[\left(1+\left(\rho\sigma_\eta\right)^2\right)\frac{\left(2\psi^k+1\right)}{2} + \frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\frac{\left(\psi^k+2\right)^2}{4}\right)\right]\exp\left(\frac{\psi^k\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - \left(1+\left(\rho\sigma_\eta\right)^2\right),$$

$$\mathcal{H} = \left(\left(\rho\sigma_\eta\right)^2\left(1+\left(\alpha\psi^k+1\right)^2\right)+2\right)\exp\left(\frac{\alpha\psi^k\sigma_\eta^2}{(1-\psi^2)}\right) - 2\left(1+\left(\rho\sigma_\eta\right)^2\right).$$

Further

$$Var\left(y_t^2\right) = \lambda^4\left[E\left(\sigma_t^{8\alpha}\right) - E^2\left(\sigma_t^{4\alpha}\right)\right] + \left[E\left(\varepsilon_t^4\sigma_t^4\right) - E^2\left(\varepsilon_t^2\sigma_t^2\right)\right]$$
$$+4c\lambda^3\left[E\left(\sigma_t^{6\alpha}\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha}\right)\right] + 4\lambda^3\left[E\left(\sigma_t^{6\alpha+1}\varepsilon_t\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]$$
$$+4c\left[E\left(\varepsilon_t^3\sigma_t^3\right) - E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\varepsilon_t\sigma_t\right)\right] + 4\lambda\left[E\left(\sigma_t^{2\alpha+3}\varepsilon_t^3\right) - E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]$$
$$+4c\lambda\left[3E\left(\sigma_t^{2\alpha+2}\varepsilon_t^2\right) - 2E\left(\varepsilon_t\sigma_t\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\sigma_t^{2\alpha}\right)\right]$$
$$+2\lambda^2\left[3E\left(\sigma_t^{4\alpha+2}\varepsilon_t^2\right) - 2E^2\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_t^2\sigma_t^2\right)\right] + 4c^2\lambda^2\left[E\left(\sigma_t^{4\alpha}\right) - E^2\left(\sigma_t^{2\alpha}\right)\right]$$
$$+4c^2\left[E\left(\varepsilon_t^2\sigma_t^2\right) - E^2\left(\varepsilon_t\sigma_t\right)\right] + 8c^2\lambda\left[E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right) - E\left(\sigma_t^{2\alpha}\right)E\left(\varepsilon_t\sigma_t\right)\right]$$
$$+4c\lambda^2\left[3E\left(\sigma_t^{4\alpha+1}\varepsilon_t\right) - E\left(\sigma_t^{4\alpha}\right)E\left(\varepsilon_t\sigma_t\right) - 2E\left(\sigma_t^{2\alpha}\right)E\left(\sigma_t^{2\alpha+1}\varepsilon_t\right)\right]$$

and

$$Var\left(y_t^2\right) = \lambda^4\left[\exp\left(\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{4\alpha\omega}{1-\psi}+\frac{4\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$+4c^2\left[\left(1+(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\frac{(\rho\sigma_\eta)^2}{4}\right]\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+4c\lambda^3\left[\exp\left(\frac{2\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{3\alpha\omega}{1-\psi}+\frac{5\alpha^2\sigma_\eta^2}{2(1-\psi^2)}\right)$$
$$+2\lambda^3\rho\sigma_\eta\mathcal{A}\exp\left(\frac{(6\alpha+1)\,\omega}{2(1-\psi)}+\frac{\left((2\alpha+1)^2+16\alpha^2\right)\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$+2c\rho\sigma_\eta\left[3\left(3+\frac{9}{4}(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)\right]\exp\left(\frac{3\omega}{2(1-\psi)}+\frac{5\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$+2\lambda\rho\sigma_\eta\mathcal{B}\exp\left(\frac{(2\alpha+3)\,\omega}{2(1-\psi)}+\frac{\left((2\alpha+1)^2+4\right)\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$+4c\lambda\mathcal{C}\exp\left(\frac{(\alpha+1)\,\omega}{1-\psi}+\frac{(2\alpha^2+1)\,\sigma_\eta^2}{4(1-\psi^2)}\right)+2\lambda^2D\exp\left(\frac{(2\alpha+1)\,\omega}{1-\psi}+\frac{(4\alpha^2+1)\,\sigma_\eta^2}{4(1-\psi^2)}\right)$$
$$+4c^2\lambda^2\left[\exp\left(\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)-1\right]\exp\left(\frac{2\alpha\omega}{1-\psi}+\frac{\alpha^2\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$+\left\{\left[3+24(\rho\sigma_\eta)^2+16(\rho\sigma_\eta)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)^2\right\}\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$+4c^2\lambda\rho\sigma_\eta\left[(2\alpha+1)\exp\left(\frac{\alpha\sigma_\eta^2}{2(1-\psi^2)}\right)-1\right]\exp\left(\frac{(2\alpha+1)\,\omega}{2(1-\psi)}+\frac{(4\alpha^2+1)\,\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$+2c\lambda^2\rho\sigma_\eta\mathcal{H}\exp\left(\frac{(4\alpha+1)\,\omega}{2(1-\psi)}+\frac{(8\alpha^2+1)\,\sigma_\eta^2}{8(1-\psi^2)}\right).$$

where

$$\mathcal{A} = (6\alpha+1)\exp\left(\frac{\alpha(2\alpha+1)\,\sigma_\eta^2}{(1-\psi^2)}\right)-(2\alpha+1),$$
$$\mathcal{B} = (2\alpha+3)\left[3+(\rho\sigma_\eta)^2\frac{(2\alpha+3)^2}{4}\right]\exp\left(\frac{(2\alpha+1)\,\sigma_\eta^2}{2(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)(2\alpha+1),$$

and

$$\mathcal{C} = 3\left(1 + \left(\rho\sigma_\eta\left(\alpha+1\right)\right)^2\right)\exp\left(\frac{\left(4\alpha+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - \frac{\left(\rho\sigma_\eta\right)^2\left(2\alpha+1\right)}{2}\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$
$$- \left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right),$$
$$\mathcal{D} = \left[3\left(1 + \left(\rho\sigma_\eta\right)^2\left(2\alpha+1\right)^2\right)\exp\left(\frac{2\alpha\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - \left(1 + \left(\rho\sigma_\eta\right)^2\right)\right]\exp\left(\frac{\left(4\alpha^2+1\right)\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$
$$- \left(\rho\sigma_\eta\right)^2\frac{\left(2\alpha+1\right)^2}{2}\exp\left(\frac{\alpha\sigma_\eta^2}{\left(1-\psi^2\right)}\right),$$
$$\mathcal{H} = \left[3\left(4\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - 1\right]\exp\left(\frac{\alpha^2\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - 2\left(2\alpha+1\right)\exp\left(\frac{\alpha\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

# References


[1] Andersen, T.G. (1996), Return Volatility and Trading Volume: An Information Flow Interpretation of Stochastic Volatility, *Journal of Finance* 51, 169-204.

[2] Andersen, T.G. and L. Benzoni (2009), Stochastic Volatility in Meyers R. (eds) Complex Systems in Finance and Econometrics, Springer N. York

[3] Andersen, T.G. and B.E. Sorensen (1996), $GMM$ Estimation of a Stochastic Volatility Model: a Monte Carlo Study, *Journal of Business and Economic Statistics* 14, 328–352.

[4] Anyfantaki, S. and A. Demos (2011), Estimation and Properties of a Time-varying $GQARCH(1,1)-M$ Model, *Journal of Probability and Statistics*, Article ID 718647, 39 pages. doi:10.1155/2011/718647.

[5] Anyfantaki, S. and A. Demos (2016), Estimation and Properties of a Time-varying $EGARCH(1,1)$ in Mean Model, *Econometric Reviews* 35, 293-310.

[6] Arvanitis, S. and A. Demos (2004a), Time Dependence and Moments of a family of Time-Varying parameter $GARCH$ in Mean Models, *Journal of Time Series Analysis* 25, 1-25.

[7] Arvanitis, S. and A. Demos (2004b), Conditional Heteroskedasticity in Mean Models, in *Quantitative Methods in Finance in Honor of Prof. A. Kintis*, Editor A. Refenes, Typothito.

[8] Arvanitis, S. and A. Demos (2024), Gaussian Stochastic Volatility, Misspecified Volatility Filters and Indirect Inference Estimation, WP 17-2024, Dept. of Economics, AUEB.

[9] Asai, M. and M. McAleer (2011), Alternative Asymmetric Stochastic Volatility Models, *Econometric Reviews* 30, 548-564.

[10] Bali, T.G. and L. Peng (2006), Is There a Risk-Return Trade-Off? Evidence from High-Frequency Data, *Journal of Applied Econometrics* 21, 1169-1198.

[11] Barndorff-Nielsen, O. .E. (1997), Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling, *The Scandinavian Journal of Statistics* 24, 1-13.

[12] Bera, A.K. and M.L. Higgins (1993), ARCH models: properties, estimation and testing, *Journal of Economic Surveys* 7, 305-366.

[13] Bermudez, P., J.M. Marín and H. Veiga (2020), Data cloning estimation for asymmetric stochastic volatility models. *Econometric Reviews*, DOI: 10.1080/07474938.2020.1770997

[14] Black, F. (1976), Studies of stock, price volatility changes, *Proceedings of the 1976 Meetings of the Business and Economics Statistics Section*, American Statistical Association, 177-181.

[15] Blasques, F., C. Francq and S. Laurent (2023), Quasi score-driven models, *Journal of Econometrics* 234, 251-275.

[16] Bollerslev, T. (1986), Generalized Autoregressive Conditional Heteroskedasticity, *Journal of Econometrics* 31, 307-327.

[17] Bollerslev, T., R. Y. Chou, and K. F. Kroner (1992), ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics* 52, 5-59.

[18] Campbell, J.Y., L. Hentschel (1992), No news is good News, an asymmetric model of changing volatility stock returns, *Journal of Financial Economics* 31, 281-318

[19] Carnero, M.A., D. Pena and E. Ruiz (2004), Persistence and kurtosis in GARCH and Stochastic Volatility Models, *Journal of Financial Econometrics* 2, 319-342.

[20] Chib, S. F. Nardari and N. Shephard (2002), Markov chain Monte Carlo Methods for Stochastic Volatility Models, *Journal of Econometrics* 108, 281-316.

[21] Danielsson, J. (1998), Multivariate Stochastic volatility Models: Estimation and a Comparison with $VGARCH$ Models, *Journal of Empirical Finance* 5, 125-173.

[22] Demos, A. (2000), The Autocorrelation Function of Conditionally Heteroskedastic in Mean Models, Dept. of International and European Economic Studies, Athens University of Economics and Business, Discussion Paper No. 00-07.

[23] Demos, A. (2002), Moments and Dynamic Structure of a Time-Varying-Parameter Stochastic Volatility in Mean Model, *The Econometrics Journal* 5.2, 345-357.

[24] Duffie, D. and K.J. Singleton (1993), Simulated Moments Estimation of Markov Models of Asset Prices, *Econometrica* 61, 929–952.

[25] Engle, R. F. (1982), Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation, *Econometrica* 50, 987-1007.

[26] Engle, R.F. (1990), Stock Volatility and the Crash of '87: Discussion, *The Review of Financial Studies* 3, 103-106.

[27] Engle, R.F. (2002), New Frontiers for *ARCH* Models, *Journal of Applied Econometrics* 17, 425-446.

[28] Engle, R.F. and T. Bollerslev (1986), Modeling the Persistence of Conditional Variances, *Econometric Reviews* 5, 1-50.

[29] Engle, R. F., D. M. Lilien and R. P. Robins (1987), Estimating Time-Varying Risk Premia in the Term Structure: The ARCH-M Model, *Econometrica* 55, 391-407.

[30] Fousekis P., C. Katrakilidis and E. Trachanas (2016), Vertical Price Transmission in the US Beef Sector: Evidence from the Nonlinear ARDL model, *Economic Modelling* 52, 499-506.

[31] Francq, C. and J.-M. Zakoian (2010), *GARCH Models*, J. Wiley & Sons.

[32] French K.R., G.W. Schwert and R.F. Stambaugh (1987), Expected Stock Returns and Volatility, *Journal of Financial Economics* 19, 3-29.

[33] Gallant, R. A. and G. Tauchen (1996), Which Moments to Match, *Econometric Theory* 12, 657-81.

[34] Ghysels, E., A. Harvey, and E. Renault (1996), Stochastic Volatility, in G. Maddala and C. Rao (Eds.), *Handbook of Statistics* 14, 119–191, Elsevier Science.

[35] Glosten, L. R., R. Jagannathan and D. E. Runkle (1993), On the Relation between the Expected Value and the Volatility of the Nominal Excess Return on Stocks, *Journal of Finance* 48, 1779-1801.

[36] Gonzales-Rivera G,. (1996), Time Varying Risk. The Case of The American Computer Industry, *Journal of Empirical Finance* 2, 333-342.

[37] Gourieroux, C., A. Monfort, and E. Renault (1993), Indirect Inference, *Journal of Applied Econometrics* 8, S85-S118.

[38] Gradshteyn, I. and I. Ryzhik (1994. Table of Integrals, Series and Products. Academic Press.

[39] Harvey, A., (2007), Long Memory in Stochastic Volatility, in J. Knight and S. Satchell (Eds.), *Forecasting Volatility in the Financial Markets*, Third Edition, 351-363, Elsevier.

[40] Harvey, A. and Palumbo (2023), Score-driven models for realized volatility, *Journal of Econometrics* 237, art. 105448.

[41] Harvey, A., E. Ruiz and N. Shephard (1994), Multivariate Stochastic Variance Models, *Review of Economic Studies* 61, 247-264.

[42] Harvey, A. and N. Shephard (1996), Estimation of an Asymmetric Stochastic Volatility Model for asset Returns, *Journal of Business and Economic Statistics* 14, 429-434.

[43] He C. and T. Terasvirta (1999), Fourth Moment Structure of the $GARCH(p, q)$ Process, *Econometric Theory* 15, 824-846.

[44] He, C., T. Terasvirta and H. Malmsten (2002), Moment Structure of a Family of First-Order Exponential $GARCH$ Models, *Econometric Theory* 18, 868-885.

[45] Hentschel, L. (1995), All in the Family. Nesting symmetric and asymmetric $GARCH$ models, *Journal of Financial Economics* 39, 71-104.

[46] Horvath, L. , P. Kokoszka and R. Zitikis (2006), Sample and Implied Volatility in GARCH Models, Journal of Financial Econometrics 4, 617–635 .

[47] Jacquier, E., N.G. Polson and P.E. Rossi (1994), Bayesian Analysis of Stochastic Volatility Models, *Journal of Business and Economic Statistics* 12, 371-417.

[48] Jacquier, E., N.G. Polson and P.E. Rossi (2004), Bayesian Analysis of Stochastic Volatility Models with Fat-tails and Correlated Errors, Journal of Econometrics 122, 185-212.

[49] Karanasos, M. (1999), The Second Moment and Autocovariance Function of the Squared Errors of the $GARCH$ Model, *Journal of Econometrics* 90, 63-76.

[50] Kim, S., N. Shephard and S. Chib (1998), Stochastic Volatility: Likelihood Inference and comparison with $ARCH$ Models, *Review of Economic Studies* 65, 361-393.

[51] Koopman, S.J. and E. H. Uspensky (2002), The Stochastic Volatility in Mean Model: Empirical Evidence from International Stock Markets, *Journal of Applied Econometrics* 17, 667-689.

[52] Marin, J.M., E. Romero and H. Veiga (2024), Fitting complex stochastic volatility models using Laplace approximation, WP2024-05, Statistics and Econometrics series, Universidad Carlos III de Madrid.

[53] Mao, X., V. Czellar, E. Ruiz, and H. Veiga (2020), Asymmetric stochastic volatility models: Properties and particle filter-based simulated maximum likelihood estimation, *Econometrics and Statistics* 13, 84–105.

[54] Melino, A. and S.M. Turnbull (1990), Pricing Foreign Currency Options with Stochastic Volatility, *Journal of Econometrics* 45, 239-265.

[55] Merton, R.C. (1980), On Estimating the Expected Return on the Market. An Exploratory Investigation, *Journal of Financial Economics* 8, 323-361.

[56] Meyer, R. and J. Yu (2000), BUGS for a Bayesian Analysis of Stochastic Volatility Models, *Econometrics Journal* 3, 198-215..

[57] Milhoj, A. (1985), The Moment Structure of ARCH Processes, *Scandinavian Journal of Statistics* 12, 281-292.

[58] Nelson, D.B. (1991), Conditional Heteroskedasticity in Asset Returns: A New Approach, *Econometrica* 59, 347-370.

[59] Perez, A. and E. Ruiz (2003), Properties of the Sample Autocorrelations of Nonlinear Transformations in Long Memory Stochastic Volatility Models, *Journal of Financial Econometrics* 1, 420-444.

[60] Perez, E. Ruiz and H. Veiga (2009), A note on the properties of power-transformed returns in long-memory stochastic volatility models with leverage effect, *Computational Statistics and Data Analysis* 53, 3593-3600.

[61] Poon, S. and S.J. Taylor (1992), Stock returns and volatility: an empirical study of the UK stock market, *Journal of Banking and Finance* 16, 37-59.

[62] Romero, E. and E. Ropero-Moriones (2023), Testing Data Cloning as the Basis of an Estimator for the Stochastic Volatility in Mean Model, *Discrete Dynamics in Nature and Society* 2023, Article ID 7657430, https://doi.org/10.1155/2023/7657430.

[63] Rodriguez, M.J. and E. Ruiz (2012), Revisiting Several popular *GARCH* Models with Leverage Effect: Differences and Similarities, *Journal of Financial Econometrics* 10, 637-668.

[64] Ruiz, E. and E. Veiga (2008), Modelling long memory volatilities with leverage effect: A-LMSV versus FIEGARCH, *Computational Statistics and Data Analysis* 52, 2846-2862.

[65] Sentana, E. (1995), Quadratic ARCH Models, *Review of Economic Studies* 62, 639-661.

[66] Smith, A.A. (1993), Estimating Nonlinear Time-Series Models using Si1mulated Vector Autoregressions, *Journal of Applied Econometrics* 8, S63-S84.

[67] Taylor, S.J. (1986), *Modelling Financial Time Series*, J. Wiley, Chichester.

[68] Taylor, S.J. (1994), Modeling Stochastic Volatility: A Review and Comparative Study, *Mathematical Finance* 4, 183-204.

[69] Tsiotas, G. (2007), On the use of the Box–Cox Transformation on Conditional Variance Models, *Finance Research Letters* 4, 28-32.

[70] Tsiotas, G. (2012), On Generalised Asymmetric Stochastic Volatility Models, *Computational Statistics and Data Analysis* 56, 151-172.

[71] Wu, G. (2001), The Determinant of Asymmetric Volatility, *The Review of Financial Studies* 14, 837-859.

[72] Yu, J. (2005), On Leverage in a Stochastic Volatility Model, *Journal of Econometrics* 127, 165-178.