Energy balance climate models and the economics of climate change

By William A. Brock, Gustav Engström and Anastasios Xepapadeas*

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Abstract

Spatial energy balance models are used by climate scientists to help understand climate dynamics and to assist in construction of more complex general circulation models. In this paper we present the first, to our knowledge, coupled spatial energy balance and economic growth model. This leads to new insights regarding: (i) The contentious issue of whether a gradualist approach to mitigation, is preferable to an initially more aggressive approach, (ii) The effect of polar ice melting on optimal policy, (iii) Robustness of optimal climate policy to spatial damage uncertainty, and (iv) Economic justice considerations raised by variation in latitude specific damages.

Keywords: Spatial energy balance models, climate change, ice line, economic growth

This paper presents the first, to our knowledge, coupled spatial energy balance climate model (EBCM) integrated with an economic growth model. It introduces solution methods for spatial climate models that may be new to economics and it integrates these methods with the standard methods of solving economic models. Before we proceed further we believe that it is useful to point out why this is worth doing by providing an important example at the outset.

It appears that much of the current scientific discussion about climate change concentrates around the calculation of the true costs of global climate change and the implications of these calculations for policy design, an issue which relates directly to the decision to undertake or not policy action and its time profile. It seems that among economists there is no longer a debate on whether action should be taken or not. Carey (2011) quotes Robert Mendelsohn as stating that:

> "The debate is how much and when to start. If you believe that there are large damages, you would want more drastic immediate action. The Nordhaus camp, however, says we would start modestly and get tougher over time".

Thus the debate among economists in dealing with climate change on the mitigation side has basically settled on whether to increase mitigation efforts (e.g. carbon taxes)

*Brock: Department of Economics, University of Wisconsin, 1180 Observatory Drive, Madison, WI, USA, and Beijer Fellow. e-mail:wbrock@ssc.wisc.edu. Engström: Beijer Institute of Ecological Economics, Royal Swedish Academy of Sciences, Sweden and Department of Economics, Stockholm University, Stockholm, Sweden, e-mail: gustav.engstrom@beijer.kva.se. Xepapadeas: Athens University of Economics and Business, Department of International and European Economic Studies, Greece and Beijer Fellow. Brock's research was supported by the Vilas Trust.

gradually (e.g. Nordhaus (2007, 2010, 2011)) versus those who believe we should mitigate rapidly. Stern (2006) justifies the call for immediate action on the normative grounds of using a low discount rate to discount the future costs of climate change. Weitzman (2009*a*) and in his rebuttal of Nordhaus (Weitzman, 2009*b*) argues that the possibility of low probability climate catastrophes strengthens the case for quick action now to mitigate potential catastrophic climate change. His argument is based on bad fat tails in the distribution of future damages from climate change. Sterner and Persson (2008) justify strong and urgent action by accounting for non-market damages from climate change, while Weitzman (2010) based on two risk aversion axioms discusses policy implications stemming from the distinction between additive and multiplicative dis-utility damages.[1]

This paper attempts to provide new insights regarding the debate of "how much and when to start" using as starting point the temporal and the spatial structure of damages from climate change which is implied by the science of climate change, without resorting to arguments regarding the choice of discount rate, the structure of uncertainty, or the rising relative prices for environmental amenities. Although all these factors are important in deciding "how much and when to start", we believe that by framing the problem in a way that climate science implies the structure, the spatial, and the time profile of damages provides a sound and potentially empirically justified approach to policy making. Thus the coupling of dynamic economic growth models with dynamic spatial EBCMs that we undertake in this paper enables us, as we will make clear in the rest of the paper, to obtain new insights about the inter temporal shape and the spatial shape of the distribution function of damages and to translate these insights into policy rules regarding the time and spatial paths of mitigation efforts.[2]

A popular class of EBCMs which we focus upon, are the models of North (North (1975*a*), North (1975*b*)), North, Cahalan and Coakley (1981), and Wu and North (2007).[3] A common feature among these models, is the presence of an endogenous ice line where latitudes north (south) of the ice line are solid ice and latitudes south (north) of the ice line are ice free. There has been a lot of concern about the effects of ice melting, i.e. the ice lines being pushed closer to the North and South Poles by global warming,[4] and how the incorporation of these effects into economic models might affect decisions to engage

[1]Judd and Lontzek (2011) have formulated a dynamic stochastic version of DICE which they call, DSICE. They also extend their model to include stochastic tipping point possibilities. They show how this additional real world complexity substantially affects the optimal policy results in comparison to DICE. See also Nævdal (2006) for an optimal control version featuring an uncertain threshold with application to the instability and possible disintegration of the West Antarctic ice sheet.

[2]We hasten to add that the basic argument of economists, e.g., Nordhaus (2007), that taxing carbon in a harmonized manner is the efficient policy still stands in our spatial setting.

[3]Although the EBCMs that we use are simple climate models, many useful insights into climate dynamics can arise from these simple models (Pierrehumbert, 2008).

[4]Of course these simple models do not capture elements of potentially abrupt changes in ice melting and its impact on coastlines that are stressed by, for example, Michael Oppenheimer and his co-authors (Oppenheimer and Alley (2000); Oppenheimer (2005), Little, Gnanadesikan and Oppenheimer (2009), but nevertheless they provide useful insight into the expected effects of climate change.

in large scale mitigation efforts now.

To be more precise, when the ice lines move closer to the poles marginal damages from moving will be large at first and then diminish as the ice line approaches the Poles.[5] This makes sense. When there is a lot of ice to melt damages would be larger than when there is almost no ice left to melt. Hence marginal damages are plausibly higher when the polar ice caps are larger i.e. there's a larger source of ice to melt. Let us explain this argument in more detail. Suppose human effects are causing the ice lines to move closer to the Poles. Suppose damages from this effect are proportional to the amount of ice melting. Let x denote the sine of the latitude as in North (1975*a*,*b*) and assume that the ice line is at latitude x_s from the North Pole (at the the North Pole $x = 1$). Let us consider now damages from moving the ice line by dx towards the North Pole. The ice area lost in the Northern Hemisphere when the Northern ice line is at x_s is approximately proportional to $2\pi(1 - x_s)dx$ for small dx. Thus as human activities move the ice line towards the North Pole the ice area lost diminishes and marginal damages diminish also.

Apart from the arguments concerning changes in the mass of ice, the actual damage, in terms of sea level rise, will also depend upon the characteristics of the coastline and the economic activities located there. Li et al. (2009) apply GIS methods to assess and visualize the global impacts of potential inundation from sea level rise. They find an approximately logarithmic relationship between inundation and sea level rise i.e. that the first meter of sea level rise will affect a larger area of land than the following and so on. Coastal regions also have the greatest concentration of economic activities (Nicholls and Tol, 2006; Nicholls and Cazenave, 2010).

Using the arguments sketched above, we investigate how the damages imposed by a moving ice line affect decisions on optimal mitigation policy. We test this using two different types of damage functions having functional properties consistent with evidence and apply these using both additive and multiplicative damage structures in order to assure the robustness of the qualitative results we obtain in this paper.[6] The analysis done so far supports arguments for a rapid ramping up of mitigation efforts (e.g. Weitzman, Stern) and is thus suggestive of the value added from developing unified economic and energy balance climate models.[7]

[5]The damages which we refer to here are damages caused by sea level rise due to the release of water from melting glacial ice sheets. The presence of an endogenous ice line allows us to model this explicitly. Further sea level rise can also be caused by thermal expansion of warming oceans, as a direct result of a rising global temperature. Which of these effects that dominate will depend upon the time scale studied. For example, the Intergovernmental Panel on Climate Change's Fourth Assessment Report (IPCC, 2007) concluded that thermal expansion can explain about 25 percent of observed sea-level rise for $1961 - 2003$ and 50 percent for $1993 - 2003$, but with considerable uncertainty. There may of course also be other damages caused by the increasing loss of the ice caps and their role in regulating the climate.

[6]See Weitzman (2010) for a good discussion on additive and multiplicative damages.

[7]Scientific evidence seems to support the argument that ice sheets might be seriously affected with relatively low increases in temperature. Oppenheimer (2005) reports a number of results suggesting that both the Greenland Ice Sheet (GIS) and the West Antarctic Ice Sheet (WAIS) could be highly vulnerable to temperature rise within the range studied by the current integrated assessment models (IAMs). Oppenheimer and Alley (2004) report that a $2 - 4°C$ global mean warming could be justified

Another issue that economic-EBCMs could provide new insights relates to the argument that the gradualist policy ramp may not be robust to other plausible specifications is the economic justice argument of Rawls, i.e. that global policy should be to maximize the welfare of the worst off region. For example, Nordhaus (2007) and Dell, Jones and Olken (2008) point out that poorer (and more tropical) regions are projected to suffer more damages from climate change than wealthier (and more temperate) regions. A Rawlsian objective would maximize the welfare of the least well off region. In our spatial model this objective could be formalized by maximization of the least well off latitude.[8]

Remaining with the spatial aspects of the EBCMs this is a good point to further discuss what kinds of questions we may hope to address with a spatial climate model in coupled climate economic modeling that can not be addressed with models like that of Nordhaus (2007, 2010). For example, Nordhaus's RICE 2010 divides the world into US, EU, Japan, Russia, Eurasia, China, India, Middle East, Africa, Latin America, Other high income, Other developing Asia. The climate dynamics of RICE 2010 are

> "mass of carbon in reservoir for atmosphere, upper oceans, and lower oceans,... global mean surface temperature, of upper oceans, temperature of lower oceans." Nordhaus (2010)

Dynamics of these quantities are distributed lag equations of past quantities and the global mean surface temperature dynamics is also a function of current radiative forcing, but there is no spatial geography. It is probably useful to think of Nordhaus's quantities on the climate side of the model as some sort of aggregates over spatial dimensions. In his book, Nordhaus (2007) states that the damage function continues to be a major source of modeling uncertainty in the DICE model. A recent study of climate damages due to

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for WAIS. Carlson et al. (2008) conclude that geologic evidence for a rapid retreat of the Laurentide ice sheet, which is the most recent (early Holocene epoch) and best constrained disappearance of a large ice sheet in the Northern Hemisphere, may describe a prehistoric precedent for mass balance changes of the Greenland ice sheet over the coming century. In a recent report, from the European Energy Agency (EEA, 2010) it was stated that one of the potential large-scale changes likely to affect Europe is the deglaciation of the WAIS and the GIS and that there is already evidence of accelerated melting of the GIS. Further, a sustained $1-2°C$, respectively $3-5°C$, global warming above 1990 temperatures could be tipping points leading to at least partial deglaciation of the GIS and WAIS thus implying a significant rise in sea levels. Hansen (2005) discusses ice sheet disintegration as a wet process, spurred by positive feedbacks, which once underway can be explosively rapid. Concerning the terminology "explosively" he refers to Kienast et al. (2003) which find that in melt-water pulse 1A, about 14,000 years ago, sea level rose about 20 meters in approximately 400 years, which is an average of 1 meter of sea level rise every 20 years. Many studies of global sea-level rise also discuss how the contribution from melting ice to sea level rise will decrease over time as the amount of ice diminishes (See Rahmstorf and Vermeer (2009) for a good discussion).

[8]To do a more accurate job of finding the optimal policy ramp of, say, carbon taxes, under a Rawlsian objective, we would need an spherical type model like that of Brock and Judd (2010). However the models considered here provide useful insights at a much lower degree of complexity. Rawlsian objectives may strike the reader as rather "starry eyed" from the point of view of wealthier parts of the world. However, elements such as national security concerns may drive enlightened self interest on the part of wealthier regions to act more like Rawlsians. At the very minimum one should design policy to be robust against uncertainty in the specification of the proper policy objective function as well as uncertainty in climate and economic dynamics.

temperature and precipitation changes is Dell, Jones and Olken (2008) which found that levels and growth rates of the economies of poorer parts of the world were damaged more than levels and growth rates of the wealthier parts of the world. The wetter regions of the world are expected to become wetter and the dryer regions of the world are expected to become dryer (GFDL, 2008). We proxy this kind of effect of climate change in this paper by a damage function for an area where damages increase as the mean temperature of the area increases and the temperature variance of the area increases. More will be said about this below.

To summarize we believe that the main contribution of our paper is to couple spatial climate models having endogenous ice lines, with economic models, and use these spatial climate science models to discipline the structure and the shape of potential damage functions, in order to provide new insights regarding the optimal time profile for current and future mitigation. To put it another way this paper couples the economic models we use all the time in economics with a class of spatial climate models used by climate scientists. We believe this endeavor apart from being valuable in its own right, provides new insights regarding the temporal and spatial paths of policies designed to address climate change.

Since energy models are new in economics we proceed in steps that we believe make this methodology accessible to economists. In section I we present a basic energy balance climate model[9] which incorporates human impacts on climate. In developing the model we follow North (1975*a*,*b*) and use his notation. We use the model to expose solution methods and especially the two mode approach which transforms systems of partial differential equations (PDEs) in infinite dimensional spaces resulting from spatial modeling, into systems of ordinary differential equations (ODEs) in finite dimensional spaces. The two mode approach will be extensively used to solve the integrated economic-EBCM. In section II we couple a simplified version of the energy balance model, with a simple economic model and show that ice line damages explicitly introduced through the EBCMs, suggest even at this very simple level, the possibility of multiple steady states, history dependence in the optimal paths and a rapid now, instead of gradual mitigation policy. Section III uses the insights of the previous sections to couple a spatial EBCM with an economic model that has the structure of the well known integrated assessment model RICE (Nordhaus and Boyer, 2000; Nordhaus, 2010). We use this approach to discipline the temporal and spatial shape of the damage function. In this more traditional, on the part of the economics modeling, we obtain results similar to the more simplified model of section II, regarding multiple steady states and history dependence of the optimal paths, and insights about the spatial and temporal structure of optimal mitigation policies. Motivated by this modeling exercise we turn, in section IV to the analysis of DICE, the most

[9]For more on EBCMs see for example Pierrehumbert (2008) (chapters 3 and 9, especially sections 9.2.5 and 9.2.6 and surrounding material). North, Cahalan and Coakley (1981) is a very informative review of EBCMs while Wu and North (2007) is a very recent paper on EBCMs.

popular of the integrated assessment models. We provide numerical results by running the DICE model with explicit ice line damages which have a time profile consistent with the profile implied by the EBCMs we have developed earlier in the paper. Our numerical results suggest a U-shaped optimal policy ramp with rapid immediate mitigation, to defend against the ice-cap loss, which then slow down as damages from the ice caps are reduced and then once again increases when damages from the overall increase in temperature starts to catch up. The final section concludes.

I. A Basic Energy Balance Climate Model

In this section we develop a one-dimensional Energy Balance Climate Model with human inputs. The term "one-dimensional" means that there is an explicit one dimensional spatial dimension in the model so that our unified model of the climate and the economy evolves both in time and space.[10] We follow North (1975*a*,*b*) and North, Cahalan and Coakley (1981) in this development.

Let x to denote the *sine* of the latitude. We shall abuse language and just refer to x as "latitude". Following North (1975*a*,*b*) let $I(x,t)$ denote outgoing infrared radiation flux measured in W/m^2 at latitude x at time t, $T(x,t)$ denote surface (sea level) temperature measured in °C at latitude x at time t. The outgoing radiation and surface temperature can be related through the empirical formula.[11]

$$I(x,t) = A + BT(x,t), \;\; A = 201.4W/m^2, \;\; B = 1.45W/m^2 \tag{1}$$

Following North (North (1975*a*), equation (29)) the basic energy balance equation with a human input can be written as:

$$\frac{\partial I(x,t)}{\partial t} = QS(x,t)\alpha(x,x_s(t)) - [I(x,t) - h(x,t)] + D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right] \tag{2}$$

where units of x are chosen so that $x=0$ denotes the Equator, $x=1$ denotes the North Pole, and $x=-1$ denotes the South Pole; Q is the solar constant[12] divided by 4; $S(x,t)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from 0 to 1 is unity; $\alpha(x,x_s(t))$ is the absorption coefficient which is one minus the albedo of the earth-atmosphere system, with $x_s(t)$ being the latitude of the ice line at time t; and D is a thermal diffusion coefficient that has been computed as

[10] In contrast, the "zero-dimensional "model does not explicitly account for the spatial dimension.

[11] It is important to note that the original Budyko (1969) formulation cited by North parameterizes A, B as functions of fraction cloud cover and other parameters of the climate system. North (1975*b*) points out that due to non-homogeneous cloudiness A and B should be functions of x. There is apparently a lot of uncertainty involving the impact of cloud dynamics (e.g. Trenberth et al. (2010) versus Lindzen and Choi (2009)). Hence robust control in which A, B are treated as uncertain may be called for but this is left for further research.

[12] The solar constant includes all types of solar radiation, not just the visible light. It is measured by satellite to be roughly 1.366 kilowatts per square meter (kW/m^2).

$D = 0.649 Wm^{-2\circ}\mathrm{C}^{-1}$ (North, Cahalan and Coakley (1981))

Equation (2) states that the rate of change of outgoing radiation is determined by the difference between the incoming absorbed radiant heat $QS(x,t)\alpha(x, x_s(t))$ and the outgoing radiation $[I(x,t) - h(x,t)]$. Note that the outgoing radiation is reduced by the human input $h(x,t)$. Thus the human input at time t and latitude x, can be interpreted as the generation of greenhouses gases (GHGs) that reduce outgoing radiation. Since GHGs can be regarded as a function f of produced output at latitude x, we may write $h(x,t) = f(Y(x,t))$ where $Y(x,t)$ is produced output at (x,t). As pointed out by North (1975*b*), in equilibrium at a given latitude the incoming absorbed radiant heat is not matched by the net outgoing radiation and the difference is made by the meridional divergence of heat flux which is modelled by the term $D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right]$. This term explicitly introduces the spatial dimension into the climate model. The energy balance equation (2) incorporates, for the first time to our knowledge, economic variables - output production - in an energy balance model. The importance of this is that by modeling ice line damages and discontinuous albedo, issues which are not taken into account in standard integrated assessment models (IAMs), we identify the existence of nonlinearities and multiple steady state for the unified economy-climate model which could be important in policy design and the identification of new policy ramps.[13]

Returning to the description of (2), above the ice line absorption drops discontinuously because the albedo jumps discontinuously. We will follow North (1975*b*), page 2034, equation (3) and put

$$\alpha(x, x_s) = \begin{cases} b_0 = 0.38 & x > x_s \\ \alpha_0 + \alpha_2 P_2(x) & x < x_s \\ \alpha_0 = 0.697 & \\ \alpha_2 = -0.0779 & \end{cases} \tag{3}$$

where $P_2(x) = (3x^2 - 1)/2$ is the second Legendre polynomial.[14] In this set up the ice line is determined dynamically by the condition: (Budyko (1969), North (1975*a*), North (1975*b*))

$$\begin{array}{ll} T > -10°\mathrm{C} & \text{no ice line present} \\ T < -10°\mathrm{C} & \text{ice present} \end{array} \tag{4}$$

The ice line function $x_s(t)$ solves the equation $I_s = I(x_s(t), t)$. Thus the latitude of the ice line can move in time in response to changes in human input since the ice line solution depends on $h(x,t)$. Moving of the ice line towards the poles generates the damages we

[13]Note that at this stage output is regarded as an exogenous forcing parameter in order to introduce the EBCM in a clear way. Output will be endogenized in the unified economy-EBCMs that we develop in the next sections

[14]A smoothed version of (3) is Equation (38) of North, Cahalan and Coakley (1981), (p. 98).

discussed in the introduction. Using 1 and 4 the outgoing radiation at the latitude of the ice line for each date t is

$$I(x_s) = I_s = 186.8 \; W/m^2 \tag{5}$$

A steady state for the outgoing radiation is a function of latitude $\bar{I}(x)$ which satisfies the equation

$$0 = QS(x)\alpha(x, \bar{x}_s) - \left[\bar{I}(x) - \bar{h}(x)\right] + D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial \bar{I}(x)}{\partial x}\right] \tag{6}$$

for a steady state human input $\bar{h}(x)$ while the steady state ice line will satisfy $I_s = \bar{I}(\bar{x}_s)$ with I_s determined by (5).

The way to approach this problem would be to solve (2) for a given human input function $h(x,t)$ and to obtain a solution function $I(x,t)$. Then using (1) the temperature and the ice line at each date and latitude can be determined. When human input changes, this solution can then be used to trace the impact of the human input on outgoing radiation, the surface temperature and the ice line at each latitude. Since temperature and ice line changes are associated with damages this type of modeling allows us to incorporate spatial impacts and different sources of climate damages into the damage functions used in the economics of climate change.

We turn now to a more detailed analysis of the solution process. Equation (2) is a PDE. One might think that we are going to have to deal with the complicated mathematical issues of the solution or the optimal control of PDEs when we need to discuss the social optimization problems over space and time. But, as we shall see, the climate problem reduces to the optimal control of a small number of "modes" where each "mode" follows a simple ODE. We believe this decomposition is another important and new contribution of our paper to the study to coupled economic and climate models. Let us continue with the development of the solution procedure for equation (2) before turning to optimization.

North (1975*b*) approached the solution of (2) by using approximation methods (Judd (1998) Chapter 6). Thus the solution is approximated as:

$$I(x,t) = \sum_{n \text{ even}} I_n(t) P_n(x) \tag{7}$$

where $I_n(t)$ are solutions to appropriately defined ODEs and $P_n(x)$ are even numbered Legendre polynomials. A satisfactory approximation of the solution for (2) can be obtained by the so called two mode solution where $n = \{0, 2\}$. We develop here a two mode solution given the human forcing function $h(x,t)$. We do it for the Northern Hemisphere only since, following North, we treat the Southern Hemisphere symmetrically.[15] The two mode solution is defined as

$$\hat{I}(x,t) = I_0(t) + I_2(t)P_2(x) \tag{8}$$

$$\frac{dI_0}{dt} = -I_0(t) + \int_0^1 [QS_2(x)\alpha(x, x_s(t)) + h(x,t)]\, dx,\ \ I_0(0) = I_{00} \tag{9}$$

$$I_0(t) = e^{-t}\left[I_{00} + \int_0^t e^u \left[QS_0(x_s(u)) + h_0(u)\right] du\right] \tag{10}$$

$$\frac{dI_2}{dt} = -(1+6D)I_2(t) + 5\int_0^1 [QS_2(x)\alpha(x, x_s(t)) + h(x,t)]\, P_2(x)dx, \tag{11}$$

$$I_2(0) = I_{02} \tag{12}$$

$$I_2(t) = e^{-(1+6D)t}\left[I_{02} + \int_0^t e^{(1+6D)u}\left[QS_2(x_s(u)) + h_2(u)\right] du\right] \tag{13}$$

$$S_n(x_s) = \int_0^1 S(x)\alpha(x, x_s)P_n(x)dx\ \ ,\ \ h_n(t) = \int_0^1 h(x,t)P_n(x)dx,\ \ \ n = 0, 2 \tag{14}$$

$$S(x) = 1 + S_2P_2(x)\ ,\ \ S_2 = -0.482,\ \ \ n = 0, 2 \tag{15}$$

$$P_0(x) = 1, P_2(x) = \frac{(3x^2-1)}{2} \tag{16}$$

The derivation of the solution is presented in Appendix A.[16] Given the definitions of the functional forms the two mode solution is tractable and can be calculated given initial conditions I_{00}, I_{02} which are determined by the initial climate state. As shown below, the two mode solution can be used to obtain tractable solutions regarding the ice line and temperature $T(x,t)$.

A. *The two mode approximation of ice line function*

This is a function $x_s(t)$ that solves

$$I_s = I_0(t) + I_2(t)P_2(x_s(t)) \tag{17}$$

To determine the two mode ice line function through (17) the discontinuity in the albedo expressed by (3) and (4) should be taken into account. This can be done by applying to the two mode solution for the ice and the ice free areas, value matching, smooth pasting and appropriate boundary conditions at the pole and the equator North (1975*a*). This function, which may not be unique, will depend on the human input $h(x,t)$.

[15] Of course the two hemispheres are very different in reality, but we abstract from that complexity here.

[16] The two mode solution is an approximating solution. We can develop a series of approximations of increasing accuracy by solving this problem for expansions using (a "two mode" solution) and using (a "three mode" solution) and so on. North's results suggest that the two mode solution is an adequate approximation. Approximations will be denoted with a ˆ sign here on after.

To obtain the two mode approximation steady-state ice line (9) and (11) are used. The steady state values for the $I's$ are given by

$$\bar{I}_0 = QS_0(\bar{x}_s) + \bar{h}_0 \quad , \quad \bar{I}_2 = \frac{5\left[QS_2(\bar{x}_s) + \bar{h}_2\right]}{1+6D} \tag{18}$$

where it is assumed that

$$\text{as } t \to \infty, \int_0^1 h(x,t)dx \to \bar{h}_0 \text{ and } \int_0^1 h(x,t)P_2(x)dx \to \bar{h}_2. \tag{19}$$

The two mode steady state ice line is the solution of $I_s = \bar{I}_0 + \bar{I}_2 P_2(\bar{x}_s)$,and can be obtained by using value matching, smooth pasting and appropriate boundary conditions. It is important to note that there may be more than one solution to the ice line.

B. The two mode approximation of the surface temperature

In the context of the two mode approximation, we may use the two mode expression for $I(x,t)$ to obtain a two mode expression for surface (sea level) temperature $T(x,t)$, i.e $\hat{T}(x,t) = T_0(t) + T_2(t)P_2(x)$ where $T_0(t)$ and $T_2(t)$ solve the ordinary differential equations.

$$\frac{BdT_0}{dt} = -(A + BT_0(t)) + \int_0^1 \left[QS_2(x)\alpha(x,x_s(t)) + h(x,t)\right]dx \tag{20}$$

$$\frac{BdT_2}{dt} = -(1+6D)BT_2(t) + 5\int_0^1 \left[QS_2(x)\alpha(x,x_s(t)) + h(x,t)\right]P_2(x)dx \tag{21}$$

$$T_0(0) = T_{00},\ T_2(0) = T_{02} \tag{22}$$

The ice line function $x_s(t)$ in terms of the temperature solves

$$T_0(t) + T_2(t)P_2(x_s(t)) = T_s,\ T_s = -10°\text{C} \tag{23}$$

and can be determined using the value matching conditions described above. From the two mode approximation of the temperature, we obtain the global mean temperature $m_T = T_0(t)$, which is the integral of $\hat{T}(x,t)$ over x from zero to one[17], and the variance of the temperature,

$$V_T = \int_0^1 \left[\hat{T}(x,t) - T_0(t)\right]^2 dx = \int_0^1 (T_2(t)P_2(x))^2 dx = \frac{(T_2(t))^2}{5} \tag{24}$$

Local temperature means at latitudes $(x, x+dx)$ and the mean of temperature over the set of latitudes $Z = [a,b]$ are defined by

$$[T_0(t) + T_2(t)P_2(x)]\,dx, m\,[a,b] = \int_a^b [T_0(t) + T_2(t)P_2(x)]\,dx \tag{25}$$

[17]This is because $\int_0^1 P_2(x)dx = 0$.

while the variance of temperature over the set of latitudes $Z = [a, b]$ is

$$V[a,b] = \int_a^b \left[T_0(t) + T_2(t)P_2(x) - m[a,b;t]\right]^2 dx \tag{26}$$

When the area $Z = [a, b]$ is introduced, It is plausible to assume that utility in each area $[a, b]$ depends upon both the mean temperature and the variance of temperature in that area. For example we may expect increases in mean temperature and variance to have negative impacts on output in any area Z, if it is located in tropical latitudes. Whereas mean temperature increases in some areas Z (e.g. Siberia) may increase utility rather than decrease utility.[18] Existing dynamic integrated models of climate and economy, (e.g. Nordhaus's well known work (2007), (2010)) can not deal with these kinds of spatial elements, such as impacts of changes in temperature variance, generated by climate dynamics over an area Z.

The two mode approximate solutions (8)-(16) and (20)-(22) are equivalent because they are related by $I = A + BT$. Since the existing models of climate and economy, model climate in terms of temperature we are going to use this equivalence to develop energy balance models of economy and climate using temperature as the state variable directly associated with climate. We introduce such a model in the next section.

II. A Simple Integrated Dynamic Economic - Climate Model

In this section we develop a simplified integrated model of economy and climate, with the climate part motivated by the energy balance models described above. The climate part should incorporate state variables related to the two mode temperature solution and an ice line equation. The two-mode temperature solution is $\hat{T}(x,t) = T_0(t) + T_2(t)P_2(x)$. Wang and Stone (1980) argue that an approximation for this solution equation can be achieved by replacing $T_2(t)$ by an appropriate constant, which we shall denote by $\bar{T}$. Then $d\hat{T}(x,t)/dt = dT_0(t)/dt$. Recall that $T_0(t)$ is global mean surface (sea level) temperature. Then the evolution of the mean temperature is given by (20) or, by setting $T_0(t) = T(t)$

$$\frac{dT(t)}{dt} = -\frac{A}{B} - T(t) + \frac{1}{B}\int_0^1 \left[QS_2(x)\alpha(x, x_s(t)) + h(x,t)\right] dx \tag{27}$$

Thus the Wang-Stone approximation reduces the state variables from two, in the model (20)-(21), to one whose evolution is described by (27). Wang and Stone (1980) (equation 3) calibrate the model by best fitting the two mode solution to data and use this

[18] In a stochastic generalization of our model, we could introduce a stochastic process to represent "weather," i.e. very high frequency fluctuations relative to the time scales we are modeling here. Here the "local variance" of high frequency phenomena like "weather" may change with changes in lower frequency phenomena such as mean area Z temperature and area Z temperature variance. We leave this task to future research.

approximation to get a simple equation for the ice line

$$x_s(t) = (a_{ice} + b_{ice}T(t))^{1/2}, a_{ice} = 0.6035, b_{ice} = 0.02078 \tag{28}$$

Damages from climate change emerge both from temperature increase and movement of the ice line towards the north. Let us define these damages by two functions $D_1(T(t))$ and $D_2(x_s(t))$, where 1 denotes damages due to temperature rise and 2 denotes damages due to ice line movement. A simplified integrated economic climate model can be developed along the following lines.

We associate human input with greenhouse gas emissions on a one-to-one basis and thus denote emissions by $h(x,t)$. These emissions affect the temperature dynamics of our simplified climate model. We further assume, as is plausible, that at each latitude emissions disperse rapidly, relative to the longer time scale of our analysis across latitudes, so that $\int_0^1 h(x,t)dx = h(t)$. We consider a simplified economy with aggregate capital stock K. An amount K_2 from this capital stock is diverted to alternative "clean technologies". Output in this economy is produced by capital and emissions h according to a standard production function $F(K - K_2, h + \phi K_2)$, where ϕ is an efficiency parameter for clean technologies.[19] The cost of using a unit of h is $C_h(h)$,with $C_h(0) = 0$, $C_h^{'} > 0, C_h^{''} > 0$. The use of emissions can be reduced by employing clean technologies at an effective rate ϕK_2. Denoting consumption by C, net capital formation in our simplified economy is described by

$$\frac{dK}{dt} = F(K - K_2, h + \phi K_2) - C - C_h(h) - \delta K \tag{29}$$

where δ is the depreciation rate on the capital stock. Assuming a linear utility function or $U(C) = C$ the problem of a social planner that seeks to maximize discounted life time consumption subject to (27), (28), and (29) can be described, in the context of an integrated economic/climate model, in terms of the following Most Rapid Approach

[19]See Xepapadeas (2005) for different ways in which emissions and environment can be modeled as production factors.

Problem (MRAP) problem.[20]

$$\text{(30a)} \qquad V(T(0)) = \max \int_0^{\infty} \mathrm{e}^{-\rho t} \left[F(K - K_2, h + \phi K_2) - C_h(h) - (\delta + \rho) K \right.$$
$$\left. - D_1(T(t)) - D_2(x_s(t)) \right] dt$$

subject to (28) and

$$\text{(30b)} \qquad \frac{dT(t)}{dt} = -\frac{A}{B} - T(t) + \frac{\gamma}{B} h(t) + \frac{1}{B} \Psi(T(t)),$$

$$\text{(30c)} \qquad \Psi(T(t)) = \int_0^1 \left[Q S_2(x) \alpha(x, x_s(t)) \right] dx \;, T(0) = T_0$$

where $V(T(0))$ is the current value state valuation function, ρ is the subjective rate of discount on future utility,and the nonlinear function $\Psi(T(t))$ is an increasing function of T (North (1975*a*)). Problem (30a)-(30c) after the successive approximations we have made, has practically been reduced, regarding the climate part, to a zero-dimensional model as found in North, Cahalan and Coakley (1981). We still believe that this exercise is of value because it outlines a pathway to extensions to one-dimensional models and is even suggestive via the Legendre basis method of how one might potentially extend the work to two-dimensional models on the sphere.[21] Problem (30a)-(30c) is in principle tractable to one dimensional phase diagram methods with the costate variable on the vertical axis and the state variable on the horizontal axis.

At this point, it should be noted that technical change and population growth could also have been introduced in the form of Harrod neutral (labor augmenting) technical change, a formulation which is required for consistency with balanced growth in the neoclassical context. Balanced growth formulations allow us to conduct phase diagram analysis as in the text below. In this case the production function might be written as $F(K - K_2, h + \phi K_2, AL)$, where F is a constant returns to scale production function and $dA/dt = gA$, $dL/dt = nL$, where g is the rate of exogenous labor augmenting technical change and n is the population rate of growth. Output, capital, consumption, emissions and the capital accumulation equation (29) can thus be defined in per effective worker (AL) terms. However the temperature dynamics (30c) and (31b) now have a non-autonomous term due to exponentially growing emissions. Dealing with this problem and staying within a framework of autonomous dynamics, requires introduction of emission

[20]The assumption of linear utility allows one to write a capital accumulation problem as a MRAP problem. Problem (30a) is an approximation of the MRAP problem for very large B and $-B \leq \frac{dK}{dt} \leq B$. In problem (30a) capital, K, can thus be eliminated as a state variable. It should also be noted that in this section, damages are modeled using an additive functional form as explained in Weitzman (2010). In section III we will revert to the more common multiplicative form. The main qualitative results hold for both these forms.

[21]Brock and Judd (2010) are developing a two-dimensional spherical coupled climate/economic dynamics model by using a basis of spherical harmonics as in Wu and North (2007). This approach, as well as the Legendre basis approach we are using in this paper for one-dimensional models fits in nicely with the general approach to approximation methods in Judd's book (Judd (1998), Chapter 6)

reducing technological progress at an appropriate rate in order to be able to transform the temperature dynamics into a stationary form so that phase diagram techniques of analysis of autonomous systems can still be applied. However, this is beyond the scope of the current paper. In the current paper we wish to show how spatial energy balance models can be integrated with capital accumulation models in economics while preserving analytical tractability. We plan to undertake future research where we introduce realistic technical change and thus solve non-stationary versions of our model using a combination of analytical and computational methods. The time stationary analysis developed here indicates that a full analysis of more realistic non-stationary systems is potentially tractable now that we have pointed the way in this paper.

Returning to our time stationary framework, we feel that insights are gained more rapidly by analyzing the following qualitatively similar problem that is strongly motivated by the problem (30a)-(30c).

(31a) $$V(T(0)) = \max \int_0^\infty \mathrm{e}^{-\rho t} \left[F(K - K_2, h + \phi K_2) - C_h(h) - (\delta + \rho)K \right.$$
$$\left. -D_1(T) - D_2(T)\right] dt$$

(31b) $$\text{s.t.} \ \frac{dT}{dt} = a_T - b_T T + c_T h \ , (a_T, b_T, c_T) > (0,0,0)$$

where $D_1^{'}(T) = a_1 T$, implying increasing marginal damages due to temperature increase, while $D_2^{'}(T)$ is a function increasing at low T reaching a maximum and the decreasing gradually to zero. The shape of $D_2(T)$ is intended to capture initially increasing marginal damages from ice line rise (induced by temperature rise) which reach a maximum, as temperature increases, and eventually vanish once the polar ice caps are gone.[22] Define

(32) $$\pi(h) = \max_{K \geq 0, K_2 \geq 0} \{F(K - K_2, h + \phi K_2) - (\delta + \rho)K\}$$

Since we assume that $F(\cdot, \cdot)$ is concave increasing, $\pi(h)$ is an increasing concave function of h.[23] We may now write down the current value Hamiltonian and the first order necessary

[22]Assuming a quadratic or a higher degree power function for damages $D_1(T)$ due to temperature increase is consistent with damages related to falling crop yields or reduction to ecosystem services, and this has been the shape adopted in many IAMs. To consider a plausible shape for $D_2(T)$ we have argued in the introduction that as the ice line moves towards the north, there is initially a large quantity of ice to melt which can generate high melt per unit time. As the ice cap is reduced the melt is reduced and eventually tends to zero when the ice cap disappears. A potential damage function invoking these properties is the gamma function (see Appendix B) which we will be using throughout the paper to capture this type of effect. Another function having similar properties is the S-shaped function used in Brock and Starrett (2003) to describe internal loading of phosphorous in a lake system. This functional form proved to give very similar qualitative results to the ones obtained with the gamma function. Further discussion regarding the shape of $D_2(T)$ can be found in Appendix B.A. Furthermore, we argue that the combination of these two damage functions, $D_1(T)$ and $D_2(T)$, each one associated with climate change impacts having different time profiles and being disciplined by scientific evidence provides a more comprehensive description of the problem.

[23]Note that $\pi'(0) < \infty$ if $\phi > 0$ for the alternative "clean" technology.

conditions for an optimum,

$$\mathcal{H}(h, T, \lambda_T) = \pi(h) - C_h(h) - D_1(T) - D_2(T) + \lambda_T(a_T - b_T T + c_T h) \tag{33}$$

$$\pi'(h) = C_h' - \lambda_T c_T \Rightarrow h = h^*(\lambda_T) \ , \ h^{*'}(\lambda_T) > 0 \tag{34}$$

where it is understood in (34) that the inequality conditions of boundary solutions are included, and

$$\frac{dT}{dt} = a_T - b_T T + c_T h^*(\lambda_T) \ , \ T(0) = T_0 \tag{35}$$

$$\frac{d\lambda_T}{dt} = (\rho + b_T)\lambda_T + a_1 T + D_2'(T) \tag{36}$$

We know that since $\lambda_T(t) = \frac{\partial V(T(t))}{\partial T(t)} := V'(T(t)) < 0$ the costate variable can be interpreted as the shadow cost of temperature. We also know that if a decentralized representative firm pays an emission tax then the path of the optimal emission tax is $-\lambda_T(t)$. We can study properties of steady states of the problem (30a)-(30c) by analyzing the phase portrait implied by (35)-(36). The isocline $dT/dt = 0$ is easy to draw for (35). Along this isocline we have $\frac{d\lambda_T}{dT} = \frac{b_T}{c_T h^{*'}} > 0$, by using (34), thus along this isocline λ_T is increasing in T. There is a value λ_{Tc} such that if $\lambda_T(t) < \lambda_{Tc}$ then $h^* = 0$ and $a_T/b_T = T$. If there are no ice line damages, the $d\lambda_T/dt$ isocline is just a linear decreasing function of T that is zero at $T = 0$, or $\lambda_T = -\frac{a_1}{(\rho+b_T)}T$, which implies that $\lambda_T < 0$ for all $T > 0$. Now add the ice line damage to this function. The isocline is defined as

$$\lambda_T|_{\frac{d\lambda_T}{dt}=0} = -\frac{a_1 T + D_2'(T)}{(\rho + b_T)}. \ \frac{d\lambda_T}{dT} = -\frac{a_1 + D_2''(T)}{(\rho + b_T)} \tag{37}$$

With a gamma function representation of $D_2(T)$, $D_2''(T)$ is positive and decreasing, it becomes negative, reaches a minimum and vanishes after becoming positive again. This induces a nonlinearity to the $d\lambda_T/dt = 0$ isocline. In general it is expected that this isocline will have an inverted N-shape, which means that with an increasing $dT/dt = 0$ isocline if a steady state $(\bar{T}, \bar{\lambda}_T)$ exists, there will be either one or three steady states. To study the stability properties of these steady states we form the Jacobian matrix of (35)-(36)

$$J(\bar{T}, \bar{\lambda}_T) = \begin{bmatrix} -b_T & c_T h^{*'}(\bar{\lambda}_T) \\ a_1 + D_2''(\bar{T}) & b_T + \rho \end{bmatrix} \tag{38}$$

If at a steady state $a_1 + D_2''(\bar{T}) > 0$ so that the $d\lambda_T/dt = 0$ isocline is decreasing then $\det J(\bar{T}, \bar{\lambda}_T) < 0$ and the steady state is a local saddle point. If $a_1 + D_2''(\bar{T}) < 0$ so that the $d\lambda_T/dt = 0$ isocline is increasing the steady state is an unstable spiral.[24] Thus when a unique steady state exits it will be a saddle point. The case of three candidate optimal steady states $\bar{T}_1 < \bar{T}_2 < \bar{T}_3$ is of particular interest. In this case given the shapes of the two isocline's the smallest one and the largest one are saddles and the middle one is an unstable spiral. Thus we have a problem much like the lake problem analyzed by Brock and Starrett (2003). Following an argument like that in Brock and Starrett (2003) it can be shown (under modest regularity conditions so that the Hamiltonian is concave-convex in T) that there are two value functions, call them, $V_{mitigate}(T)$ and $V_{adapt}(T)$, and a "Skiba" point $T_s \in (\bar{T}_1, \bar{T}_3)$such that $V_{mitigate}(T_s) = V_{adapt}(T_s)$ and for $T_0 < T_s$, it is optimal to follow the costate/state equations associated with $V_{mitigate}(T)$ and converge to $\bar{T}_1$, while for $T_0 > T_s$ it is optimal to follow the costate/state equations associated with $V_{adapt}(T)$ and converge to $\bar{T}_3$. In Figure 1 we present this situation for an appropriate choice of functional forms and parameters.[25] Besides the solution path the figure also plots the isocline's both with and without ice line damages. Without ice line damages we have the case when the $\dot{\lambda}_T$-isocline is a linear decreasing function of T implying that we get a unique global saddle point at the crossing of the $\dot{\lambda}_T = 0$, $\dot{T} = 0$ isocline's denoted by $\bar{T}_n$. For the case with ice line damages on the other hand, we get the inverted N-shaped $\dot{\lambda}_T$, isocline giving us a "Skiba" point T_s lying just between the unstable spiral $\bar{T}_2$ and the local saddle point $\bar{T}_3$. Hence, for low initial $T_0 < \bar{T}_1$, it will be optimal to levy a low initial carbon tax even though there is a polar ice cap threat (but it is not discontinuous as in Oppenheimer and his coauthors' work) and then gradually increasing the carbon tax along a gradualist policy ramp. However, if $T_0 \in (\bar{T}_1, T_s)$ it is optimal to tax carbon higher at T_0 and let the tax gradually fall. But if initial temperature is large enough the ice caps are essentially already trashed and the optimal thing to do is to tax carbon initially quite modestly but along an increasing schedule through time to deal with the rising marginal damages due to temperature rise. Figure 1 thus shows how the qualitative picture changes completely when a different shape for the ice line damage function is considered. In particular, the area $T \in (\bar{T}_1, T_s)$ is of interest since, if ice line damages go unaccounted for, the optimal strategy will be levy a low carbon tax which eventually will raise temperature to $\bar{T}_n$, while in a model with ice line damages included the exact opposite will be true implying a decrease in temperature to $\bar{T}_1$.

It is important to note that this stationary model is not rich enough to capture the eventual rather sharp increase along the "gradualist" policy ramp of Nordhaus (2007, 2010) because in Nordhaus's case the Business as Usual (BAU) emissions path would be growing because of economic growth. Thus the damages from temperature rise alone,

[24]The eigenvalues of J are: $\frac{1}{2}(\rho \pm \sqrt{\Delta})$, where $\Delta = \rho^2 + 4\left[(a_1 + D_2''(\bar{T}))c_T h^{*'} + b_T(b_T + \rho)\right]$. When $a_1 + D_2''(\bar{T}) > 0$ then $\Delta < 0$ and we have two complex eigenvalues with positive real parts which implies an unstable spiral.

[25]The assumed functions, parameters and calculations used in figure 1 are provided in Appendix B.

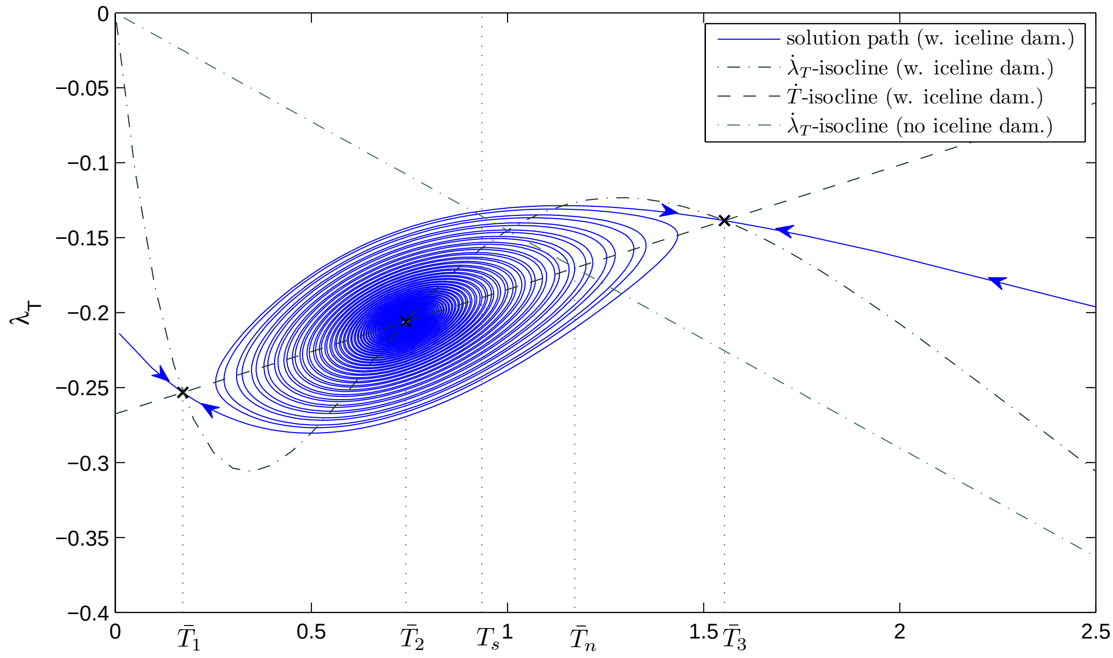

FIGURE 1. PHASE DIAGRAM FOR THE SYSTEM (35)-(36).

growing quadratically as the quantity of emissions grows, would lead to the gradualist path of carbon taxes "taking"off in the future. However, this simple stationary model does expose the "new" behavior of a higher initial carbon tax for $T_0 \in (\bar{T}_1, T_s)$. Our runs of the DICE model in section IV below exhibit a sharply higher carbon tax at the beginning due to the "extra" ice line damages added to Nordhaus's damages.[26]

III. Spatial Energy Balance Integrated Assessment Models

In this section we incorporate the framework of the energy balance models developed above into a framework similar to well established IAMs such as the DICE/RICE models proposed by Nordhaus. We use notation close to that of Nordhaus (2010) for the DICE/RICE part of the model. Consider the continuous time spatial analog of Nordhaus's equations (2007 Appendix 1 or 2010, A.1-A.20) where we have made some changes to be consistent with our notation and have suppressed (x,t) arguments to ease typing, unless (x,t) is needed for clarity

$$W = \int_0^\infty \mathrm{e}^{-\rho t} \int_0^1 \phi(x) U(C) dx dt \tag{39}$$

[26]Note that Nordhaus does include damages from ice melt, but the climate model above with moving ice line adds another component of ice melt that has a declining marginal damage function.

where $U(C)$ is utility and C is aggregate consumption at (x,t), and $\phi(x)$ is a Negishi weight function.[27] Furthermore,

$$Y_n = C + \frac{dK}{dt} + \delta K \tag{40}$$

$$Y_n = \Omega(1-\Lambda)Y,\ Y = F(K) \tag{41}$$

where, $Y_n(x,t)$: output of goods and services at latitude x and time t, net of abatement and damages, $\Omega(T(x,t))$: damage function (climate damages as fraction of output) as a function of temperature at (x,t), $\Lambda(x,t)$ abatement cost function (abatement costs as fraction of output)[28] at (x,t) and $F(K(x,t))$ is a concave production function of capital. δ is the usual depreciation rate of capital.As explained in the previous section, technology and labor have been removed from the production function in order to avoid problems of non-stationarity in the temperature equation.

Aggregate emissions at time t are defined as:

$$E(t) = \int_0^1 \sigma(1-\mu(x,t))Y(x,t)dx \tag{42}$$

where σ : ratio of uncontrolled industrial emissions to output (metric tons carbon per output at a base year prices), $\mu(x,t)$: emissions-control rate (fraction of uncontrolled emissions) at (x,t). Climate dynamics in the context of the ECBM developed in the previous sections are defined as:

$$B\frac{\partial T(x,t)}{\partial t} = QS(x)\alpha(x,x_s) - [A + BT(x,t) - E(t)] + D\frac{\partial}{\partial x}\left[(1-x^2)B\frac{\partial T(x,t)}{\partial x}\right] \tag{43}$$

$$T_s = T(x_s(t),t) \tag{44}$$

Notice that we have replaced Nordhaus's climate equations (2010, equations A.14-A.20) with the spatial climate dynamics, (43), (44). Maximization of objective (39) subject to the constraints above is a very complicated and difficult optimal control problem of the PDE (43) on an infinite dimensional space $x \in [0,1]$. We reduce this problem to a much simpler approximate problem of the optimal control of a finite number of "modes" using the two mode approach described earlier.

For the two mode approximation equations $T(x,t) = T_0(t) + T_2(t)P_2(x)$, (43) and (44). reduce to the pair of ODEs.

[27] The maximization of objective (39) with the "Negishi" $\phi(x)$weighting function is a way of computing a Pareto Optimum competitive equilibrium allocation across latitudes as in Nordhaus's discrete time non-spatial formalization in Nordhaus (2010). For a presentation of the use of the Negishi weights in IAMs see Stanton (2010).

[28] With our spatial approach abatement costs could be made site specific which would enable a more comprehensive analysis of issues concerning e.g. geoengineering. This goes beyond the scope of the current paper and is left for future research.

$$(45) \qquad \frac{dT_0}{dt} = \frac{1}{B}\left[-(A+BT_0) + \int_0^1 QS_2(x)\alpha(x,x_s(t))dx + E\right], T_0(0) = T_{00}$$

$$(46) \qquad \frac{dT_2}{dt} = \frac{1}{B}\left[-(1+6D)BT_2 + 5\int_0^1 QS_2(x)\alpha(x,x_s(t))P_2(x)dx\right],\ T_2(0) = T_{02}$$

$$(47) \qquad T_0(t) + T_2(t)P_2(x_s(t)) = T_s,\ T_s = -10°\mathrm{C}$$

Before continuing notice that North's two mode approximation has reduced a problem with a continuum of state variables indexed by $x \in [0,1]$ to a problem where the climate part has only two state variables. We can make yet a further simplification by assuming, as in section II, that the utility function is linear, i.e. $U(C) = C$. This will allow us to write (39) as the MRAP problem:

$$(48) \qquad W = \int_0^\infty \mathrm{e}^{-\rho t}\int_0^1 \phi C dx dt = \int_0^\infty \mathrm{e}^{-\rho t}\int_0^1 \phi\left[\Omega(1-\Lambda)F - (\rho+\delta)K\right] dx dt$$

Note that for the two mode approximation, the damage function should be defined as:

$$(49) \qquad \Omega(T(x,t)) = \Omega(T_0(t) + T_2(t)P_2(x))$$

To ease on the notation we introduce the inner product notation $\langle f,g\rangle = \int_0^1 f(x)g(x)dx$. We may now write down the current value Hamiltonian for the optimal control problem (48) and show how we have drastically simplified the problem by using a two mode approximation,[29]

$$(50) \qquad \mathcal{H} = \int_0^1 \phi\left[\Omega(1-\Lambda)F - (\rho+\delta)K + \frac{\lambda_0}{B}\sigma(1-\mu)F\right]dx$$
$$\frac{\lambda_0}{B}\left[\langle QS\alpha, 1\rangle - A - BT_0\right] + \frac{\lambda_2}{B}\left[5\langle QS\alpha, P_2\rangle - (1+6D)BT_2\right]$$

For the simplified problem (48) the capital stock and the emissions control rate $K^*(x,t), \mu^*(x,t)$ are chosen to maximize $\mathcal{H}$ for each (x,t), which is a relatively simple problem. However there is one complication to be addressed. The absorption function $\alpha(x, x_s(t))$ depends upon the ice line $x_s(t)$ where the ice line is given by a solution of (47), i.e.

$$(51) \qquad x_s(t) = P_+^{-1}\left(\frac{T_s - T_0(t)}{T_2(t)}\right)$$

[29]The important thing to note about this Hamiltonian compared to the Hamiltonian of the original problem (39) is this. The original problem would generate a Hamiltonian with a continuum of costate variables one for each $x \in [0,1]$. The two-mode approximation approach developed could be quite easily extended to an n-mode approximation approach. Since however North argues that a two mode approximation is quite good, we continue with a two mode approximation here.

Where the subscript “+” denotes the largest inverse function of the quadratic function $P_2(x) := (1/2)(3x^2 - 1)$. Notice that the inverse function is unique and is the largest one on the set of latitudes $[0, 1]$. Equation (51) induces a nonlinear dependence of equations (45) and (46) through the absorption function , but no new state variables are introduced by this dependence. An additional dependence induced by equations (45) and (46) as well as equation (51) is on the damage function which we parameterize as:

$$\Omega = \Omega(T_0(t), T_2^2(t)P_2^2(x); x_s(t), x) \tag{52}$$

The first term in (52) represents damages to output at latitude x as a function of average planetary temperature as in Nordhaus (2007,2010), the second term is an attempt to capture extra damages due to climate “variance”, Note that the component $P_2^2(x)$ is larger at $x = 0$ and $x = 1$ than it is at the “temperate” latitude $x = (1/3)^{1/2}$ where $P_2^2(x)$ is zero. This is an admittedly crude attempt to capture the component of damages due to “wetter places getting wetter” and “dryer places getting dryer” as well as damages to arctic latitudes compared to temperate latitudes. But some of this dependence can be captured also in the “x” term in the parameterization (52). Finally the impact on damages at latitude x due to shifts in the ice line is captured by inclusion of the ice line in (52). This is a fairly flexible parameterization of spatial effects (i.e. latitude specific effects) that are not captured in the received non-spatial formulations of integrated assessment models.

A. Optimal mitigation and location specific policy ramp in a spatial climate model

Let us first illustrate optimal mitigation using our two mode simplification of our original “infinite mode” problem with linear utility by considering a version of the problem where the impact of policy $\{\mu(x, t)\}$ on the location of the ice line $x_s(t)$ is ignored. I.e. there is no ice line dependence of any functions of the problem including the absorption function. In this simplified case the albedo function depends only upon x and thus the terms $\langle QS\alpha, 1\rangle, \langle QS\alpha, P_2\rangle$ do not depend upon $T_0(t), T_2(t)$ in (45) and (46). Hence the two costate ODEs would become

$$\begin{aligned} \frac{d\lambda_0}{dt} &= (\rho + 1)\lambda_0 - \frac{\partial \mathcal{H}}{\partial T_0} = (\rho + 1)\lambda_0 - \int_0^1 \phi \frac{\partial \Omega}{\partial T_0}(1 - \Lambda)F dx \\ \frac{d\lambda_2}{dt} &= (\rho + 1 + 6D)\lambda_2 - \frac{\partial \mathcal{H}}{\partial T_2} = (\rho + 1 + 6D)\lambda_2 - \int_0^1 \phi \frac{\partial \Omega}{\partial T_2}(1 - \Lambda)F dx \end{aligned} \tag{53}$$

Wang and Stone (1980) argue that one can even get a fairly good approximation of T_2 by exploiting how fast mode 2 converges relative to mode zero in equation (46) as compared to (45). Hence we can further simplify the problem by assuming that T_2 has already converged to:

$$T_2 = \frac{5 \langle QS\alpha, P_2 \rangle}{(1+6D)B} \tag{54}$$

for each $T(t)$.[30] The Hamiltonian (50) for the case when the absorption function and T_2 are constant can thus be written as[31]

$$\mathcal{H} = \int_0^1 \left[\phi(\Omega(1-\psi\mu)F - (\rho+\delta)K) + \frac{\lambda_0}{B}\sigma(1-\mu)F \right] dx \tag{55}$$

$$+ \frac{\lambda_0}{B}\left[Q\alpha - A - BT_0\right] \tag{56}$$

In this case we obtain the following switching decision rule for $\mu^*(x,t)$[32]

$$\mu^*(x,t) \left\{ \begin{array}{c} = 0 \\ \in [0,1] \\ = 1 \end{array} \right\} \text{ for } -\lambda_0(t) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \frac{\phi(x)\psi B}{\sigma(x)}\Omega \tag{57}$$

$$\Omega = \Omega(T_0(t), (T_2P_2(x))^2, x) \tag{58}$$

$$\lambda_0(t) = \int_{s=t}^{\infty} e^{-(\rho+1)(s-t)} \left[\int_0^1 \Omega(1-\psi\mu^*)F \frac{\partial \Omega}{\partial T_0} dx \right] ds \tag{59}$$

Suppose some type of institution wanted to implement this social optimum. One way to do it would be to impose a tax $\tau(\lambda) = \frac{-\lambda_0(t)}{B}$ on emissions when individual agents solve the static problems

$$\max_{\{\mu \in [0,1], K \geq 0\}} \{\Omega(1-\psi\mu)F - (\rho+\delta)K - \tau(\lambda)\sigma(1-\mu)F\} \tag{60}$$

We see right away that the first order necessary conditions for the problem (60) are the same with those resulting from the Hamiltonian function (55). Since $F(K)$ is a concave increasing function, then setting $\tau(\lambda) = \frac{-\lambda_0(t)}{B}$ implements the social optimum. Note that the socially optimal emissions tax is uniform across all locations as one would expect from Nordhaus (2007, 2010).

The reader might ask at this point: What substantive difference does the spatial climate model coupled to the economic model add that is not already captured by non-spatial climate models? There are several important differences regarding policy implications.

The emission reduction policy ramp $\mu^*(x,t)$, is location specific and dictates $\mu^*(x,t) =$

[30] Note that in the case where the absorption function does not depend upon $x_s(t)$ that the RHS of (54) is constant.

[31] Note that with a constant absorption function, $\langle QS\alpha, 1\rangle = \langle Q(1+S_2P_2(x))\alpha, 1\rangle = \langle Q\alpha + QS_2\alpha P_2(x), 1\rangle = \langle Q\alpha, 1\rangle = Q\alpha$, since $\langle QS_2\alpha P_2(x), 1\rangle = 0$.

[32] Here, we have also assumed that abatement costs ($\Lambda = \psi\mu,\ \psi > 0$) are linear implying that the solution is of bang-bang type. In section III.B we will consider a nonlinear version of abatement costs.

1 for all (x,t) where the relative "Negishi" weight $\phi(x)$ on welfare at that location is small (recall that $\int_0^1 \phi(x)dx = 1$ by normalization). For example, if a Rawlsian social objective is imposed, as mentioned in the introduction, where the social welfare of the worst off latitude, call it x^0, is maximized, then $\phi(x) = 0$ for all latitudes different then x^0. Hence all latitudes other than x^0 would be immediately ordered to reduce their emissions to zero. Consider a more plausible scenario. Assume that $\Omega = \Omega(T_0(t), (T_2P_2(x))^2, x) = \Omega(T_0(t), (T_2P_2(x))^2)$ is decreasing in both arguments. This crudely captures the idea that damages increase at each latitude as average planetary temperature, $T_0(t)$ increases and as a measure of local climate "variance" $(T_2P_2(x))^2$ increases. Let R denote a set of "at risk latitudes" with low values of $\Omega(T_0(t), (T_2P_2(x))^2)$, i.e. with high values of the arguments. The set R is a crude attempt to capture latitudes that would be relatively most damaged by climate change. A more plausible type of "Rawlsian" objective would be to solve the social problem above but with $\phi(x) > 0, x \in R$, $\phi(x) = 0, x \notin R$. We see right away that this social problem would require all x's not in R to reduce all emissions immediately. In general we have,

$$\mu^*(x,t) = 1, \; for \; -\lambda_0(t) > \frac{\phi(x)\psi B}{\sigma(x)}\Omega \tag{61}$$

and vice versa. This makes good economic sense. The marginal social burden on the planet as a whole of a unit of emissions at date t, no matter from which x it emanates is, $-\lambda_0(t)$. Locations x where the "Negishi" weight on the location is small, where emissions per unit of output are relatively large (relatively large $\sigma(x)$), and that are already relatively heavily damaged ($\Omega(T_0(t), (T_2P_2(x))^2, x)$ is high) are ordered to stop emitting. Thus our modeling allows plausible specifications of the economic justice argument stemming from geography to shape policy rules.

In the following section, can now use this framework to extend our results in the presence of an discontinuous absorption function that changes at the ice line. This is a more realistic model which introduces ice line damages which we will now develop in the context of a DICE/RICE-type integrated assessment model.

B. *Optimal mitigation in a spatial IAM-type climate model*

We introduce now as the absorption function the version proposed in North (North (1975a)) where

$$\alpha(x, x_s) = 1 - \alpha(x) = \begin{cases} \alpha_1 = 0.38 & x > x_s \\ \alpha_0 = 0.68 & x < x_s \end{cases} \tag{62}$$

where $\alpha(x)$ is the albedo. With this absorption function the dynamics $T_0(t)$ in (45) and the T_2 approximation in (54) become respectively

$$(63) \qquad \frac{dT_0}{dt} = \frac{1}{B}\left[-(A + BT_0) + Q(\alpha_0 - \alpha_1)\int_{x=0}^{x=x_s(t)} (1 + S_2P_2(x))dx + E + Q\alpha_1\right]$$

$$(64) \qquad T_2 = \frac{1}{(1 + 6D)B}\left[5Q(\alpha_0 - \alpha_1)\int_{x=0}^{x=x_s(t)} (1 + S_2P_2(x))P_2(x)dx + Q\alpha_1S_2\right]$$

where the equation for the ice line is, using (51):

$$(65) \qquad x_s(t) = \left[\frac{2}{3}\frac{T_s - T_0(t)}{T_2} + \frac{1}{3}\right]^{\frac{1}{2}}$$

The objective (39) and the constraints (62)-(65) determine optimal mitigation over time and latitude. The discontinuous absorption function can create a strong nonlinearity where a small change in T_0 can cause a large change in damages at some latitudes. This nonlinearity makes it however difficult to proceed with analytical solutions. To obtain a qualitative idea of the impact of the nonlinearity due to the absorption function and the ice line we use the climate parametrization used by North (1975*a*) ($\alpha_0 = 0.68, \alpha_1 = 0.38, A = 201.4, B = 1.45, S_2 = -0.483, T_s = -10, Q = 334.4$). The heat transport coefficient D is found to be approximately 0.2214 by calibrating the ice line function to the current ice line estimate ($x_s = 0.95$).[33]

The system (63)-(65) is highly nonlinear and can be simplified by deriving a polynomial approximation of x_s as a function of $T_0(t)$. We proceed in the following way. If we substitute $x_s(t)$ from (65) into (64), then T_2 results as a fixed point of (64). We solve numerically the fixed point problem (64) for values of $T_0 \in \left[-\bar{T}_0, \bar{T}_0\right]$, obtaining the solution $\hat{T}_2(T_0)$. Substituting this back into equation (65) gives us the $\hat{x}_s(\hat{T}_2(T_0), T_0)$ which is then used to fit using least squares, a quadratic curve on $(T_0, \hat{x}_s)$. Thus $\hat{x}_s$ is approximated by a convex curve $\hat{x}_s = \zeta_0 + \zeta_1 T_0 + \zeta_2 T_0^2 = \zeta(T_0)$, $(\zeta_0, \zeta_1, \zeta_2) > 0$,.[34] Making use of this approximation the system (63)-(65) can thus be written as:

$$(67) \qquad \frac{dT_0}{dt} = \frac{1}{B}\left[-(A + BT_0) + Q(\alpha_0 - \alpha_1)\theta(T_0) + E + Q\alpha_1\right]$$

$$\text{where } \theta(T_0) := \left[\hat{x}_s + \frac{S_2}{2}(\hat{x}_s^3 - \hat{x}_s)\right] \text{ with } \hat{x}_s := \zeta_0 + \zeta_1 T_0 + \zeta_2 T_0^2$$

[33] The calibration procedure is explained in detail by North (1975*b*) p.2035-2037.

[34] The estimated quadratic function was

$$(66) \qquad \hat{x}_s = 0.7126 + 0.0098T_0 + 0.0003T_0^2\ ,\ R^2 = 0.99$$

Assuming once again, linear utility, the Hamiltonian can thus be written as:

$$\begin{aligned}(68)\qquad \mathcal{H} = &\int_0^1 \left[\phi[K^\beta \Omega(T_0)(1-\Lambda) - (\rho+\delta)K] + \frac{\lambda_0}{B}\sigma(1-\mu)K^\beta\right] dx \\ &+ \frac{\lambda_0}{B}\left[-A - BT_0 + Q(\alpha_0-\alpha_1)\theta(T_0) + Q\alpha_1\right]\end{aligned}$$

We now assume that abatement costs are increasing in abatement activities: $\Lambda = \psi\mu^2$. The optimal μ and K will thus be defined as:

$$(69)\qquad \mu^*(x,t) = -\frac{\lambda_0\sigma}{2B\phi\psi\Omega(T_0)}$$

$$(70)\qquad K^*(x,t) = \left(\frac{\rho+\delta}{\beta}\right)^{\frac{1}{\beta-1}} \left[\Omega(T_0)(1-\psi\mu^{*2}) - \frac{\lambda_0}{\phi B}\sigma(1-\mu^*)\right]^{\frac{-1}{\beta-1}}$$

$\forall x \in [0,1]$ and the canonical system becomes:

$$(71)\qquad \frac{dT_0}{dt} = \left[-A - BT_0 + Q(\alpha_0-\alpha_1)\theta(T_0) + \int_0^1 \sigma(1-\mu^*)K^{*\beta}dx\right]$$

$$(72)\qquad \frac{d\lambda_0}{dt} = (\rho + 1 - \frac{Q}{B}(\alpha_0-\alpha_1)\theta'(T_0))\lambda_0 - \int_0^1 \left[K^{*\beta}\Omega'(T_0)(1-\psi\mu^{*2})\right] dx$$

which can be solved numerically given a specific shape of $\phi(x)$.

To proceed further we need a more detailed specification for the damage function, which as explained above should contain a 'temperature component' denoted by $D_1(T_0)$ and an 'ice line component', denoted by $D_2(T_0)$. We specify the damage function in the following way. Lost output from temperature induced damages is: $Y - \frac{Y}{1+D_1(T_0)} = \frac{YD_1(T_0)}{1+D_1(T_0)} := Yd_1(T_0)$. Lost output from ice line moving towards the poles written as a function of T_0 is: $Y - \frac{Y}{1+D_2(T_0)} = \frac{YD_2(T_0)}{1+D_2(T_0)} := Yd_2(T_0)$. The sum of lost output from both sources is: $\text{Lost}Y = Yd_1(T_0) + Yd_2(T_0)$. Thus net output available for consumption and mitigation is: $Y - \text{Lost}Y = (1 - d_1(T_0) - d_2(T_0))Y$.

If we define $\Omega_i(T_0) = \frac{1}{1+D_i(T_0)}, i = 1,2$, then the term $(1 - d_1(T_0) - d_2(T_0))$ can be written as the damage function Ω of the system (69)-(72) in the form

$$(73)\qquad \Omega(T_0) = \Omega_1(T_0) + \Omega_2(T_0) - 1$$

As the global warming problem concerns damages resulting from temperature increases, rather than decreases, we restrict the state space to include only temperatures $T_0 > 15°C$ i.e. in the vicinity of the present average global temperature level.[35] In the spatial model used in this section this temperature level is found by setting $E = 0$ and solving (67),

[35]During the development of many energy balance models in the 1960's and 70's the main concern was usually not that of global warming but rather that of drastic global cooling that could result due to a slight decrease in the solar constant. This hypothesis was later coined 'Snowball earth' by Kirschvink (1992).

which gives us $T_0 \approx 15.27$. Hence, $15°C$ can thus be thought of as a rough ballpark estimate of the preindustrial global temperature average. Damages are assumed to start at $15°C$ and we will thus write our normalized damage function as $\Omega(T_0 - 15)$. Furthermore, we will use the same functional forms for the damage functions as used in section II.[36]

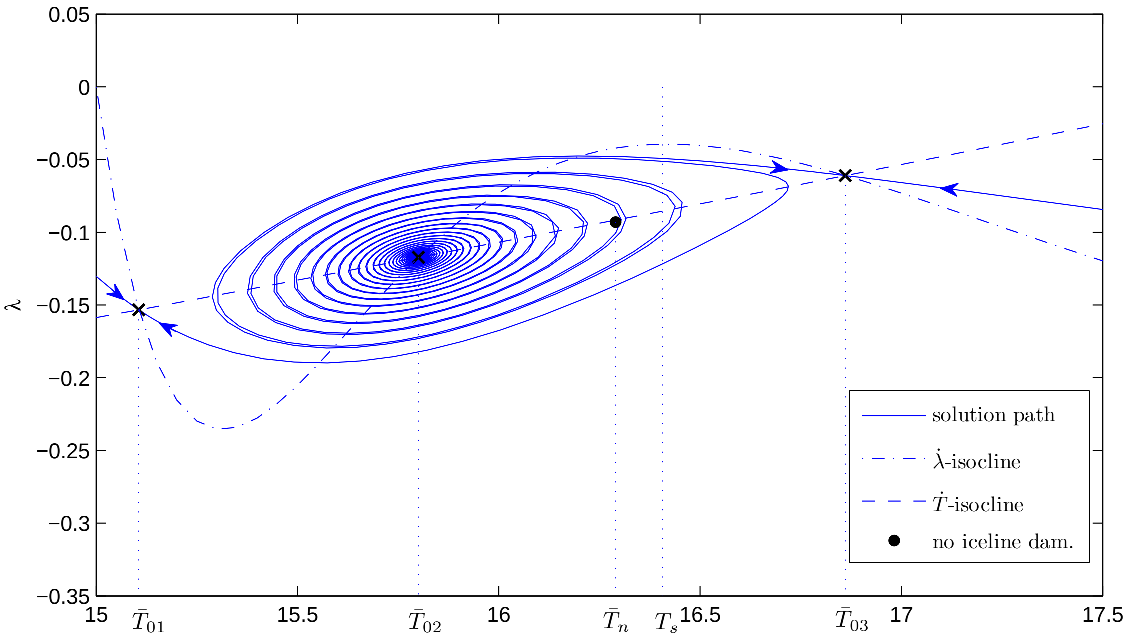

FIGURE 2. PHASE DIAGRAM FOR THE SYSTEM (71)-(72).

The energy balance spatial climate model that we presented in this section as the result of the concepts developed in the earlier part of the paper, has many similarities to the traditional IAMs but also two potentially important differences. The first is that of the discontinuous absorption function, the second is an alternative shape for ice line damages as opposed to other temperature related damages. Together they introduce complex nonlinearities into the temperature dynamics. The question of whether these differences imply significant deviations of the model's predictions, cannot be answered analytically due to the high complexity of the models. So we resort to numerical simulations.

Figure 2 shows the results for the spatial climate model we have presented in this section. As in section II this model also gives us 3 candidate optimal steady states $\bar{T}_{01} < \bar{T}_{02} < \bar{T}_{03}$ where the largest and the smallest ones are saddles while the middle one is an unstable spiral.[37] Between the unstable spiral $\bar{T}_2$ and the saddle $\bar{T}_3$ we have a Skiba

[36]The parameters estimates are taken to be $\rho = 0.02, a_1 = 0.002, a_2 = 0.4, \psi = 0.01, \sigma = 0.2, \beta = 0.5, \delta = 0.1$ and the temperature and ice line components are $D_1(T_0) = a_1 T_0^2$ and $D_2(T_0) = a_2 e^{-2T_0} T_0^2$.

[37]The corresponding eigenvalues are approximated numerically as $e_{01} = [-0.3974, 0.4174]$, $e_{02} = [0.0100 \pm 0.2045i]$ and $e_{03} = [-0.1946, 0.2146]$.

point $\bar{T}_s$ similar to that of section II.[38] Hence, for low initial temperatures $T_{00} < \bar{T}_1$ a low but gradually increasing carbon tax is optimal, while for $T_{00} < T_s$ we get the case where it is optimal to levy a high carbon tax at T_{00} and then gradually decrease it. Further, the figure also depicts the case when ice line damages are omitted $\bar{T}_n$. As opposed to section II both of the isocline's are now affected and in order to to keep the figure from getting too messy we have chosen only to plot the single equilibrium at the crossing of these isocline's, which is denoted by the black dot at $\bar{T}_n$ of figure 2. The qualitative behavior is however the same as in section II, i.e. the "no ice line damage equilibrium " is a saddle having a positive slope for the $\dot{T}$-isocline and a negative slope for the $\dot{\lambda}$-isocline.

IV. DICE model results with ice line damages

Both the relative simple model of section II and the more complex model of section III strongly suggest that the implications of explicitly modeling ice line damages is to call for strong mitigation now. In order to further demonstrate that this result is robust to the choice of model we now turn to the DICE model. The purpose of this exercise is to show how the introduction of ice line damages into the damage function, along the lines suggested by the EBCMs will affect the optimal emission policy implied by DICE the most well know of the IAMs. The DICE model assumes that all damages to the economy evolve according to the quadratic equation (A.5) of Nordhaus (2007). This equation has been calibrated to a 2.5 degree warming based upon an aggregate of impact studies from a variety of different sources.[39] In order to separate out the ice line component from the total amount of damages we follow the procedure shown in section III.B. We thus simply replace (A.5) with equation (73) from this section. Hence, we have two separate damage components $D_1(T)$ and $D_2(T)$ that can be calibrated independently according to different impact assessments. Nordhaus (2007) finds the aggregate impact of a 2.5 degree warming to be roughly 2% of GDP. Since, it is not possible to back out exactly how much of this 2% fall in GDP from a 2.5 degree warming is due to ice line specific damages, we simply make a crude assumption that approximately 50% of these damages are attributable to the ice line component $D_2(T)$.[40] Next, we make the following assumptions regarding the shapes of the temperature and ice line specific components, i.e. we set $D_1(T) = a_1 T^5$ and $D_2(T) = a_2 e^{-2T} T^2$. In a manner consistent with Nordhaus (2007) we then proceed by calibrating the parameters a_1 and a_2 so that $D_1(2.5) = 0.01$ and $D_2(2.5) = 0.01$. In this

[38] Greiner, Grüne and Semmler (2009) find multiple equilibria in a zero-dimensional EBM, where albedo is modeled by a continuous S-shaped function of temperature. The derived multiple-equilibria and Skiba planes, however, only apply for fixed levels of abatement i.e. there is just a single control variable (consumption). If however, the social planner can control both consumption and abatement then there exists only a single stable saddle. Our approach apart from explicitly addressing the more appropriate one-dimensional model also differ in the sense that we obtain multiple equilibria and Skiba points when controlling both consumption and abatement.

[39] See Nordhaus (2007) accompanying notes (p.23-25).

[40] On page 24 of the accompanying notes of the DICE 2007 model there is an impact assessment by region and impact type. These are then weighted based on GDP estimates for 2105. As these weights are not provided it is thus not possible to back out a specific region or impact type.

way our new damage function produces an equivalent amount of damage at a 2.5 degree warming as in the original model but will differ for all other temperature levels. This

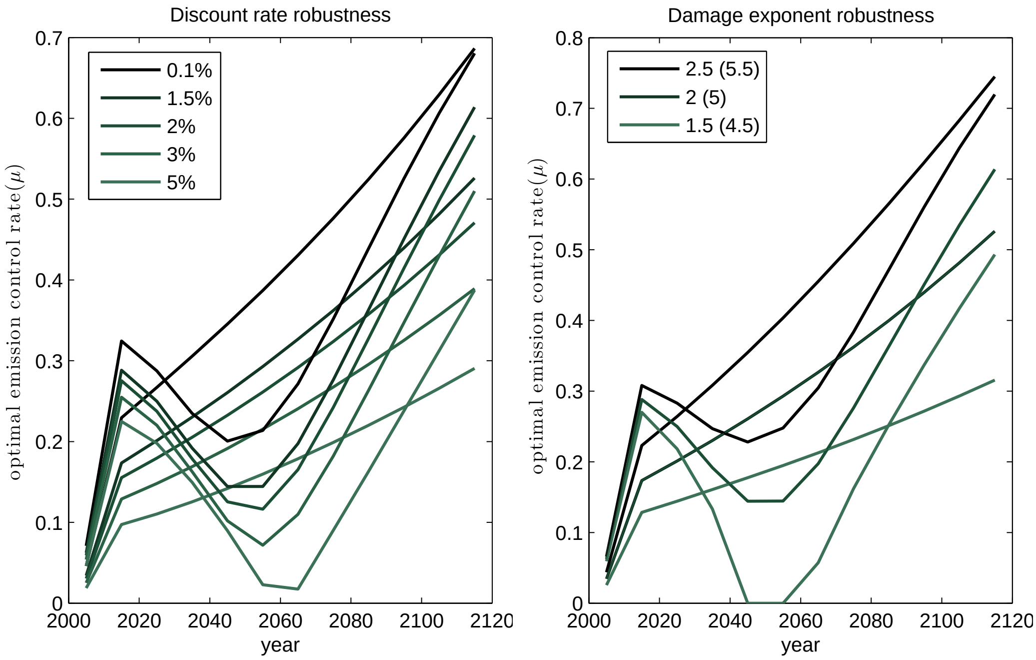

FIGURE 3. OPTIMAL EMISSION CONTROL RATE WITH AND WITHOUT ICE LINE DAMAGES FOR A SET OF DIFFERENT DISCOUNT RATES (LEFT) AND DAMAGE EXPONENTS (RIGHT). THE U-SHAPED PATHS CORRESPOND TO THE CASE WHEN ICE LINE DAMAGES ARE INCLUDED WHILE THE UPWARD SLOPING PATHS ARE THE ORIGINAL DICE MODEL PATHS. THE DAMAGE EXPONENTS OF D_1 VARY FROM 1.5-2.5 FOR THE NORDHAUS CASE AND 4.5-5.5 FOR THE ICE LINE DAMAGES (IN PARENTHESIS).

new damage function thus has the property that the temperature component, having a larger exponent than the original quadratic function, punishes GDP to a much larger extent when temperature levels start to rise above 3 degrees. When temperature levels on the other hand are lower, the damages from the ice line are the ones that dominate.[41] Figure 3 plots the optimal emission control rate the in the DICE-2007 model with and without ice line damages. As can be seen from this graph the separation of different damage structures gives us a U-shaped policy where it is optimal to mitigate harder initially as opposed to the normal gradualist policy ramp. The figure also displays a simple robustness check showing how the results are affected by changing the values for the discount rate and damage exponent. As can be seen from the left graph, raising the discount rate seems to strengthen the case for an act now policy as opposed to the more gradualistic path at the same level of discounting. Although, these results remain specific to our assumptions regarding the shape of the damage function for the ice line as well as the temperature component, it still exemplifies the sensitivity of the model to structural

[41]See Ackerman et al. (2009) for a discussion regarding different values for the exponent of the damage function used in DICE.

changes in the damage function and the impact of incorporating insights from energy balance models.

V. Summary, Conclusions, and Suggestions for Future Research

In this paper we introduce the economics profession to spatial Energy Balance Climate Models (EBCMs) and show how to couple them to economic models while deriving analytical results of interest to economists and policy makers. While we believe this contribution is of importance in its own right, we also show how introduction of spatial considerations leads to new ways of looking at climate policy.

In particular, by accounting for an endogenous ice line and paying attention to the associated ice line damages and albedo effects we show that due to nonlinearities even simple economic-EBCMs generated multiple steady states and policy ramps which do not in general follow the "gradualist" predictions. These results carry over to more complex models where the economic module has an IAM structure. The interesting issue from the emergence of multiple steady states, is that when the endogenous ice line and discontinuous albedo are ignored, as in traditional IAMs, the policy prescription of these models could be the opposite of the policy dictated by the economic-EBCMs. Furthermore the spatial aspect of the EBCMs allows economic justice argument associated with the spatial structure of climate change damages to shape policy rules. When we applied the damage function implied by the EBCMs and calibrated appropriately simulations in the DICE model gave results interpretable as a U-shaped policy ramp indicating an important deviation from the gradualist policy ramp derived from the standard DICE model. Thus a rapid mitigation policy can be justified on the new insights obtained by coupling the economy with the EBCMs.

We consider this paper as a first attempt to bring together EBCMs and economic models and to show how these models can provide new insights which have not been obtained by the traditional IAMs, and furthermore that these new insights could be important for policy design. Being a first attempt also means that there are many areas for future research. These areas range from making the economics more sophisticated by abandoning the simplifying assumption of linear utility; allowing for technical change and knowledge spillovers across latitudes; or introducing strategic interactions among regions,[42] to extending the EBCMs. Future work that needs to be done regarding EBCMs is extension to two-dimensional spherical EBCMs because Earth is a sphere, not a line. Brock and Judd (2010) are attempting to make a dent in this problem. They frame the problem as a recursive dynamic programming problem where the state vector includes a number of "spherical modes" that are analogs of the modes in this paper as well as economic state variables. Another possible extension could be the consideration of new

[42]These extensions will undoubtedly increase the complexity and the computational needs for solving the economic-EBCMs.

policy instruments. Emissions reduction acts on the outgoing radiation in the sense that by reducing emissions the outgoing radiation increases through the second term of the right hand side of (2). Another kind of policy could act on the first term of the right hand side of (2) in the sense of reducing the incoming radiation. This type of policy might be associated with geoengineering options. Finally a policy which acts on the damage function in the sense of reducing damages for any given level of temperature and radiation balance might be associated with adaptations options. Unified economic-EBCMs might be a useful vehicle for analyzing the structure and the trade offs among these different policy options.

A. Appendix: The two mode solution

In this appendix we show how to derive the two mode solution (8)-(16). We start with the basic PDE

$$\frac{\partial I(x,t)}{\partial t} = QS(x,t)\alpha(x,x_s(t)) - [I(x,t) - h(x,t)] + D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial I(x,t)}{\partial x}\right] \tag{74}$$

The two mode solution is defined as:

$$\hat{I}(x,t) = I_0(t) + I_2(t)P_2(x), P_2(x) = \frac{(3x^2-1)}{2} \tag{75}$$

then

$$\frac{\partial I(x,t)}{\partial t} = \frac{dI_0(t)}{dt} + \frac{dI_2(t)}{dt}P_2(x) \tag{76}$$

$$\frac{\partial I(x,t)}{\partial x} = I_2(t)\frac{dP_2(x)}{dx} = I_2(t)3x \tag{77}$$

Substitute the above derivatives into (74) to obtain:

$$\frac{dI_0(t)}{dt} + \frac{dI_2(t)}{dt}P_2(x) = QS(x,t)\alpha(x,x_s(t)) - \tag{78}$$
$$[I_0(t) + I_2(t)P_2(x) - h(x,t)] + D\frac{\partial}{\partial x}\left[(1-x^2)I_2(t)\frac{\partial P_2(x)}{\partial x}\right] \quad \text{, or}$$

$$\frac{dI_0(t)}{dt} + \frac{dI_2(t)}{dt}P_2(x) = QS(x,t)\alpha(x,x_s(t)) - \tag{79}$$
$$I_0(t) - I_2(t)P_2(x) + h(x,t) - 6DI_2(t)P_2(x)$$

Use:

$$\int_0^1 P_n(x)P_m(x)dx = \langle P_n(x), P_m(x)\rangle = \frac{\delta_{nm}}{2n+1} \tag{80}$$
$$\delta_{nm} = 0 \text{ for } n \neq m, \delta_{nm} = 1 \text{ for } n = 1$$

and note that $P_0(x) = 1$, $P_2(x) = \frac{(3x^2-1)}{2}$

Multiply (79) by $P_0(x)$ and integrate from 0 to 1 to obtain

$$\frac{dI_0(t)}{dt} + \frac{dI_2(t)}{dt}\langle P_0(x), P_2(x)\rangle = \int_0^1 QS(x,t)\alpha(x,x_s(t))P_0(x)dx - \tag{81}$$
$$I_0(t) - I_2(t)\langle P_0(x), P_2(x)\rangle + \int_0^1 h(x,t)dx -$$
$$6DI_2(t)\langle P_0(x), P_2(x)\rangle \quad \text{, or}$$

$$\frac{dI_0(t)}{dt} = -I_0(t) + \int_0^1 \left[QS(x,t)\alpha(x,x_s(t)) + h(x,t)\right]dx \tag{82}$$

Multiply (79) by $P_2(x)$ and integrate from 0 to 1 noting that $\int_0^1 P_2(x)dx = 0$, and $\langle P_2(x), P_2(x)\rangle = \frac{1}{5}$ to obtain

$$\frac{dI_0(t)}{dt}\int_0^1 P_2(x)dx + \frac{dI_2(t)}{dt}\langle P_2(x), P_2(x)\rangle = \tag{83}$$

$$\int_0^1 QS(x,t)\alpha(x,x_s(t))P_2(x)dx- \tag{84}$$

$$I_0(t)\int_0^1 P_2(x)dx - I_2(t)\langle P_2(x), P_2(x)\rangle - \tag{85}$$

$$\int_0^1 h(x,t)P_2(x)dx - 6DI_2(t)\langle P_2(x), P_2(x)\rangle \text{ , or}$$

$$\frac{1}{5}\frac{dI_2(t)}{dt} = \left[\int_0^1 QS(x,t)\alpha(x,x_s(t)) + h(x,t)\right] P_2(x)dx-$$

$$\frac{1}{5}I_2(t) - \frac{6}{5}DI_2(t) \text{ , or}$$

$$\frac{dI_2(t)}{dt} = -(1+6D)I_2(t) + 5\int_0^1 \left[QS(x,t)\alpha(x,x_s(t)) + h(x,t)\right] P_2(x)dx \tag{86}$$

The ODEs (82) and (86) are the ODEs (9), (11) of the two mode solution (8)-(16). The solutions of these ODEs shown in (10) and (13) follow from standard methods.

B. Appendix: derivations and assumptions

This section drafts some of the more specific assumptions on which figure 1 is based. The production function in (32) is assumed to take the following form:

$$F(K-K_2,h+\phi K_2)=(K-K_2)^{\beta_1}(h+\phi K_2))^{\beta_2} \tag{87}$$

with $\beta_1>0,\beta_2>0$. The solution to problem (32) is derived from the first order conditions:

$$\frac{\partial F}{\partial K}=\beta_1(K-K_2)^{\beta_1-1}(h+\phi K_2))^{\beta_2}-(\delta+\rho)=0 \tag{88}$$

$$\frac{\partial F}{\partial K_2}=-\beta_1(K-K_2)^{\beta_1-1}(h+\phi K_2))^{\beta_2}+\beta_2\phi(K-K_2)^{\beta_1}(h+\phi K_2))^{\beta_2-1}=0 \tag{89}$$

Solving the system (88) and (89) for K and K_2 gives the solution to problem (32).

$$K_2^*(h)=\frac{1}{\phi}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{1}{\beta_1-1+\beta_2}}-\frac{h}{\phi}$$

$$K^*(h)=\frac{\beta_1}{\phi\beta_2}h+\left(1+\frac{\beta_1}{\beta_2}\right)K_2^*(h)$$

Plugging these values back into (32) allows us to write $\pi(h)$ as a linear function of h:

$\pi(h)=\tilde{A}+\tilde{B}h$

with

$$\tilde{A}:=\left(\frac{\beta_1}{\phi\beta_2}\right)^{\beta_1}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{\beta_1+\beta_2}{\beta_1-1+\beta_2}}-(\delta+\rho)\frac{(1+\phi)}{\phi}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{1}{\beta_1-1+\beta_2}}$$

$$\tilde{B}:=-(\delta+\rho)\left(\frac{\beta_1}{\phi\beta_2}-\frac{(1+\phi)}{\phi}\right)$$

which is increasing in h given that $\beta_1/\beta_2<(1+\phi)$. Assuming also that $D_1(T)=a_1T^2$, $D_2(T)=a_2\exp(-2T)T^2$ and $C_h(h)=c_hh^2$, where $a_1,a_2,c_h>0$. [43] Substituting this into (33) and using the first order condition we can thus derive the canonical system:

$$\frac{dT}{dt}=a_T-b_TT+c_T\frac{\tilde{B}+\lambda_Tc_T}{2c_h}\ ,\ T\left(0\right)=T_0 \tag{90}$$

$$\frac{d\lambda_T}{dt}=\left(\rho+b_T\right)\lambda_T+a_1T-2a_2e^{-2T}(T-1)T \tag{91}$$

[43]The shape of $D_1(T)$ has become fairly standard in the literature. Still, in a recent review by Ackerman et al. (2009), they uncovered no rationale, whether empirical or theoretical, for adopting a quadratic form for the damage function. The shape of $D_2(T)$ is motivated in the text and in appendix B.A.

From (90) and (91) it is easy to confirm the shape of the isoclines depicted in figure 1. For the numerical calculations of the solution paths and the Skiba point we used numerical methods described in Grass et al. (2008); Grass (2010). The parameter values used for the numerical calculations are given in the table below:

Parameter	Value	Description
ρ	0.02	discount rate
β_1	0.3	elasticity of capital with respect to output
β_2	0.6	elasticity of energy with respect to output
δ	0.1	depreciation rate of capital
ϕ	0.9	efficiency parameter of clean energy
a_1	0.09	damage parameter of $D_1(T)$
a_2	0.7	damage parameter of $D_2(T)$
a_T	0.8	parameter of temperature equation
b_T	0.6	parameter of temperature equation
c_T	0.85	parameter of temperature equation
c_h	0.05	parameter of cost function

TABLE 1—THE PARAMETER VALUES OF FIGURE 1.

We also tested the following S-shaped functional form for ice line damages:

$$D_2(T) = \theta \frac{T^\gamma}{\varphi + T^\gamma}, \text{ with } \{\gamma, \theta, \varphi\} > 0$$

For appropriate values of the parameters we got the same qualitative results as displayed in the phase plots of both section II and III.

A. *Ice line damages*

Although we have already provided some intuitive arguments regarding the shape of $D_2(T)$ and strengthened these arguments with empirical findings from the literature, further rigor might be called for. Consider the following line of argumentation:

Define $x_s(T) = \min\{1, (a_i + b_i T)^{0.5}\}$ to modify the Wang/Stone equation (28) for the ice line so it can't go above 1 where the ice caps are completely gone. Tune the a_i, b_i parameters of Wang/Stone so that $x_s(T)$ reaches one at a very large value of T, call it T^*. Note that $D_2(T^*)$ is very small for $T > T^*$. This is good enough to motivate the right hand part of the specification of $D_2(T)$ for large T. Now motivate $D_2(T)$ by specifying a function $g(\cdot)$ such that $D_2(T)$ is an approximation to $g(x_s(T))$. Note that since $g(x_s(T))$ must be zero for $T > T^*$, $D_2(T)$ can't be exactly represented by $g(x_s(T))$ but is close enough for $T > T^*$ to serve as a "good enough" approximation in return for its tractability as shown by our phase diagrams in section II and III.

One could also argue for an alternative formulation of the $D_2(T)$ that does not include adaptation abilities to damages which are implicitly assumed in our current formulation.

Such a damage function, would look similar to the gamma function in our paper except that damages would not go to zero but instead level off at some T^* implying that a certain fraction of output is lost forever for $T > T^*$. Such an S-shaped function has been used frequently in the literature describing non-convexities adherent in ecosystems, see e.g. Brock and Starrett (2003). We tested such an S-shaped functional form which proved to give very similar qualitative results to the ones we present in this paper. From a technical point of view the choice between this form and the gamma function is thus only a matter of preference.

Of course, no one really knows exactly what the damages to world welfare as a whole are as the ice lines retreat from their present position to the poles, but it seems plausible to expect the damages to initially increase, perhaps at an increasing rate as people struggle to deal with the large adjustment costs of dealing with melting of a large ice mass, but as the ice mass gets smaller, the adjustment costs should get smaller until the costs start dropping due to smaller and smaller ice masses melting. As the remaining ice mass shrinks to zero with increasing T one could thus argue that $D_2(T)$ goes to zero.

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