Energy Balance Climate Models, Damage Reservoirs and the Time Profile of Climate Change Policy

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Abstract

A simplified energy balance climate model is considered with the global mean temperature as the state variable, and an endogenous ice line. The movements of the ice line towards the Poles are associated with damage reservoirs where initial damages are high and then eventually vanish as the ice caps vanish and the damage reservoir is exhausted. We couple this climate model with a simple economic growth model and we show that the endogenous ice line induces a nonlinearity. This nonlinearity when combined with two sources of damages - the conventional damages due to temperature increase and the reservoir damages - generates multiple steady states and Skiba points. It is shown that the policy ramp implied by this model calls for high mitigation now. Simulation results suggest that the policy ramp could be U-shaped instead of the monotonically increasing with low starting mitigation gradualist policy ramp.

Keywords: Energy Balance Climate Models, Damage Reservoir, Ice Line, Permafrost, Heat Diffusion, Policy Ramp, Skiba Points

JEL Classification: Q54, Q58

1 Introduction

Energy balance climate models (EBCMs) have been extensively used to study Earth's climate (e.g. [30], [23], [24], [25]), and [37]). The basic components of these models are incoming solar radiation, outgoing infrared radiation, transportation of heat across the globe and the presence of an endogenous ice line where latitudes north (south) of the ice line are solid ice and latitudes south (north) of the ice line are ice free.

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In this paper we study the economics of climate change by coupling a simplified EBCM with an economic growth model. Coupling the two models provides a link between economic activity and climate change, since the increase in the accumulation of atmospheric carbon dioxide due to human generated carbon dioxide emissions reduces outgoing infrared radiation and thus increases the Earth's surface temperature.

In the economics literature the economics of climate change have been mainly studied in the framework of integrated assessment models (IAMs) with carbon cycle (e.g. [22], [19], [20], [21]), but without heat transport or endogenous ice lines. We believe that the approach in this paper, using EBCMs to model the climate of a coupled economic-climatic model, can provide new insights regarding the profile of mitigation policy and the potentially partial distribution of damages. This is because the explicit presence of a spatial dimension and an ice line whose latitude is determined endogenously may help, as we will suggest below, to identify specific damage profiles that cannot be identified by the traditional IAMs, and provide new insights about policy ramps.

An interesting issue for policy design purposes, which can be addressed by EBCMs, is damage reservoirs. Damage reservoirs in the context of climate change can be regarded as sources of climate damages which will eventually cease to exist when the source of the damages is depleted. Ice lines and permafrost can be regarded as such damage reservoirs, which are latitude dependent objects.

Regarding ice lines, there has been a lot of concern about the effects of ice melting, i.e. the ice lines being pushed closer to the North and South Poles by global warming,[1] and how the incorporation of these effects into economic models might affect decisions to engage in large scale mitigation efforts now. To be more precise, when the ice lines move closer to the poles we might expect that marginal damages from this moving will be large at first and then diminish as the ice line approaches the Poles. When there is no ice left on the Poles this damage reservoir will have been exhausted.[2] Hence marginal damages are plausibly higher when the polar ice caps are larger i.e. there's a larger source of ice to melt. Let us explain this argument in more detail. Suppose human effects are causing the ice lines to move closer to the Poles. Suppose damages from this effect are proportional to the amount of ice melting. Let us consider now damages from moving the ice line by dx towards the North Pole. The ice area lost in the Northern Hemisphere when the Northern ice line is at x_s is approximately proportional to $2(1 - x_s)dx$ for small dx. Thus as human activities move the ice line towards the North Pole the ice area lost diminishes and marginal damages diminish also. The presence of an endogenous ice line in the EBCM allows us to model this type of damages explicitly given the relevant information.[3]

[1] Of course these simple models do not capture elements of potentially abrupt changes in ice melting and its impact on coastlines that are stressed by, for example, Oppenheimer ([26], [27]), but nevertheless they provide useful insight into the expected effects of climate change.

[2] The damages which we refer to here are those caused by sea level rise due to the release of water from melting glacial ice sheets. Further sea level rise can also be caused by thermal expansion of warming oceans, as a direct result of a rising global temperature. Which of these effects dominate depends upon the time scale studied. For example, the Intergovernmental Panel on Climate Change's Fourth Assessment Report ([12]) concluded that thermal expansion can explain about 25 percent of observed sea-level rise for 1961-2003 and 50 percent for 1993-2003, but with considerable uncertainty. There may of course also be other damages caused by the increasing loss of the ice caps and their role in regulating the climate.

[3] Scientific evidence seems to support the argument that ice sheets might be seriously

Permafrost is also related to damage reservoirs. Permafrost or permafrost soil is soil at or below the freezing point of water (0 °C or 32 °F) for two or more years. Permafrost regions occupy approximately 22.79 million square kilometers (about 24 percent of the exposed land surface) of the Northern Hemisphere ([39]). Permafrost occurs as far north as 84°N in northern Greenland, and as far south as 26°N in the Himalayas, but most permafrost in the Northern Hemisphere occurs between latitudes of 60°N and 68°N. (North of 67°N, permafrost declines sharply, as the exposed land surface gives way to the Arctic Ocean.) Recent work investigating the permafrost carbon pool size estimates that 1400-1700 Gt of carbon is stored in permafrost soils worldwide. This large carbon pool represents more carbon than currently exists in all living things and twice as much carbon as exists in the atmosphere ([32]).

Thawing of permafrost as high latitudes become warmer can also be modelled in this context. Thawing of permafrost is expected to bring widespread changes in ecosystems, increase erosion, harm subsistence livelihoods, and damage buildings, roads, and other infrastructure. Loss of permafrost will also cause release of greenhouse gasses, methane in wetter areas and CO_2 in dryer areas. Furthermore, permafrost damages are related to damage reservoirs since when permafrost is gone they will vanish provided appropriate adaptation has been implemented.[4]

The permafrost feedback suggests that permafrost carbon emissions could affect long-term projections of future temperature change. An increase in Arctic temperatures could release a large fraction of the carbon stored in permafrost soils. Studies indicate that up to 22% of permafrost could be thawed already by 2100. Once unlocked under strong warming, thawing and decomposition of permafrost can release amounts of carbon until 2300 comparable to the historical anthropogenic emissions up to 2000 (approximately 440 GtC) ([34]).

EBCMs, by explicitly introducing the spatial dimension into the climate module of the problem, can help in the understanding of this type of latitude dependent damages and incorporate them into the decision-making problem related to climate change. In particular, if we allow for reservoir damages, we actually introduce two types of damage functions with different temporal profiles. These are the traditional damage function used by the IAMs, in which damages increase monotonically with temperature, and a damage function associated with damage reservoirs. This damage function will indicate higher damages earlier when the reservoir is 'full,' in the sense that there is a lot of ice north or south of the ice lines and a lot of carbon stored into permafrost soils.

affected by relatively low increases in temperature. Oppenheimer [26] reports a number of results suggesting that both the Greenland Ice Sheet (GIS) and the West Antarctic Ice Sheet (WAIS) could be highly vulnerable to temperature rise within the range studied by the current IAMs. Oppenheimer and Alley [27] report that a 2-4°C global mean warming could be justified for WAIS. Carlson et al. [7] conclude that geologic evidence for a rapid retreat of the Laurentide ice sheet, which is the most recent (early Holocene epoch) and best documented disappearance of a large ice sheet in the Northern Hemisphere, may describe a prehistoric precedent for mass balance changes of the Greenland Ice Sheet over the coming century. In a recent report from the European Energy Agency [8], it was stated that one of the potential large-scale changes likely to affect Europe is the deglaciation of the WAIS and the GIS and that there is already evidence of accelerated melting of the GIS. Further, a sustained global warming in the range of 1-5°C above 1990 temperatures, could generate tipping points leading to at least partial deglaciation of the GIS and WAIS, thus implying a significant rise in sea levels.

[4] For more details see for example [39], [40], [29].

Once the ice caps are gone and the thawed permafrost has released most of its carbon, then reservoir damages will be exhausted.

In the present paper we couple an EBCM with an endogenous ice line, with a simplified growth model with two types of damages from climate change, traditional and damage reservoir type. Our results suggest that endogenous ice lines and damage reservoirs introduce convexities which induce multiple steady states and Skiba points. The policy implication of these results is that when damage reservoirs are ignored we have a unique steady state and the policy ramp is monotonically increasing. That is, carbon taxes start at low levels and increase with time, which is the 'gradualist approach' to climate policy [19], [20], [21]. On the other hand the existence of damage reservoirs and multiple steady states induced by endogenous ice lines results in policy ramps which suggest high mitigation now, the opposite of what is advocated by the gradualist approach. Furthermore by incorporating damage reservoirs into a DICE type model, our simulations suggest a U-shaped policy ramp with high mitigation now.[5]

The rest of the paper is structured as follows. Since EBCMs are new in economics we proceed in steps that we believe make this methodology accessible to economists. In section 2 we present a basic energy balance climate model[6] which incorporates human impacts on climate which result from carbon dioxide emissions that eventually block outgoing radiation. In developing the model we follow North ([23], [24]) and use his notation. Section 3 couples the spatial EBCM with an economic growth model characterized by both traditional and reservoir damages. We show that nonlinearities induced by endogenous ice lines and reservoir damages result in multiple steady states and Skiba points. Furthermore the optimal policy ramps are characterized by high current mitigation. In section 4 we simulate the well known DICE model allowing for damage reservoirs and derive a U-shaped policy ramp. The last section concludes.

2 A Simplified One-dimensional Energy Balance Climate Model

In this section we present a simplified integrated model of economy and climate, with the climate part motivated by one-dimensional energy balance models described in the introduction. The term "one-dimensional" means that there is an explicit one-dimensional spatial dimension in the model, measured in terms of latitudes. The important feature of these models is that they allow for heat diffusion or transportation across latitudes which increases the relevance of these models in describing climate. Let $T(x,t)$ denote the surface temperature at location (or latitude) x and time t measured in °C. Climate dynamics in the

[5]Multiple equilibria and high current mitigation are also suggested by models incorprating uncertain climate thresholds into DICE ([14], [16]). See also Naevdal [18] for an optimal control version featuring uncertain thresholds. More recently Cai et al. [6] have formulated a dynamic stochastic version of DICE which they call DSICE. They also extend their model to include stochastic tipping point possibilities. They show how this additional real world complexity substantially affects the optimal policy results in comparison to DICE.

[6]For more on EBCMs, see for example Pierrehumbert [28].

context of the ECBM (e.g. [23, 24], [25]) are defined as:

$$B\frac{\partial T(x,t)}{\partial t} = QS(x)\alpha(x,x_s) - [A + BT(x,t) - gM(t)] + D\frac{\partial}{\partial x}\left[(1-x^2)B\frac{\partial T(x,t)}{\partial x}\right] \tag{1}$$

$$T_s = T(x_s(t),t) \tag{2}$$

where x denotes the *sine* of the latitude "x", where units of x are chosen so that $x = 0$ denotes the Equator, $x = 1$ denotes the North Pole[7] and to simplify we just refer to x as "latitude". A and B are constants which are used to relate outgoing infrared radiation flux $I(x,t)$ measured in W/m^2 at latitude x at time t with the corresponding surface temperature $T(x,t)$ through the empirical formula,[8]

$$I(x,t) = A + BT(x,t),\ A = 201.4W/m^2,\ B = 1.45W/m^2. \tag{3}$$

Q is the solar constant[9] divided by 4; D is a thermal diffusion coefficient that has been computed as $D = 0.649Wm^{-2\text{o}}\text{C}^{-1}$ ([25]); $S(x,t)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from 0 to 1 is unity; $\alpha(x,x_s(t))$ is the absorption coefficient which is one minus the albedo of the earth-atmosphere system, with $x_s(t)$ being the latitude of the ice line at time t. In (4) below the ice line absorption drops discontinuously because the albedo jumps discontinuously. North [24], page 2034, equation (3) specifies this co-albedo function as:[10]

$$\alpha(x,x_s) = \begin{cases} b_0 = 0.38 & x > x_s \\ \alpha_0 + \alpha_2 P_2(x) & x < x_s \end{cases},\quad \begin{matrix} \alpha_0 = 0.697 \\ \alpha_2 = -0.0779. \end{matrix} \tag{4}$$

In this set-up the ice line is determined dynamically by the condition ([5], [23], [24]):

$$\begin{matrix} T > -10°\text{C} & \text{no ice line present} \\ T < -10°\text{C} & \text{ice present} \end{matrix} \tag{5}$$

and the ice line function $x_s(t)$ solves the equation $-10 = T(x_s(t),t)$.

Although the introduction of heat diffusion adds extra complexity, since it requires the use of partial differential equations, a more simplified approach is to use the so-called two-mode approximation ([23, 24] [25]) that employs the relatively simpler framework of ordinary differential equations. The two-mode approximation is defined as $T(x,t) = T_0(t) + T_2(t)P_2(x)$ where $T_0(t)$, the first

[7] Symmetry for the part $x \in [-1,0]$ is assumed. This assumption is common in EBCMs.

[8] It is important to note that the original Budyko [5] formulation cited by North parameterizes A, B as functions of fraction cloud cover and other parameters of the climate system. North [24] points out that due to non-homogeneous cloudiness A and B should be functions of x. There is apparently a lot of uncertainty involving the impact of cloud dynamics (e.g. [33] versus [17]). Hence robust control in which A, B are treated as uncertain may be called for but this is left for further research.

[9] The solar constant includes all types of solar radiation, not just the visible light. It is measured by satellite to be roughly 1.366 kilowatts per square meter (kW/m^2).

[10] A smoothed version of a co-albedo function is equation (38) of North et al. ([25], (p. 98)).

mode, and $T_2(t)$, the second mode, solve the ordinary differential equations:[11]

$$\frac{BdT_0}{dt} = -(A + BT_0(t)) + \int_0^1 QS_2(x)\alpha(x, x_s(t))dx + g\left(M\left(t\right)\right) \tag{6}$$

$$\frac{BdT_2}{dt} = -(1 + 6D)BT_2(t) + 5\int_0^1 \left[QS_2(x)\alpha(x, x_s(t)) + g\left(M\left(t\right)\right)\right] P_2(x)dx \tag{7}$$

$$T_0(0) = T_{00},\ T_2(0) = T_{02}, P_2(x) = \frac{(3x^2 - 1)}{2}. \tag{8}$$

In (6)-(8), $P_2(x) = (3x^2 - 1)/2$ is the second Legendre polynomial that provides the spatial dimension to the solution.

From the two-mode approximation of the temperature, we obtain the global mean temperature $m_T = T_0(t)$, which is the integral of $\hat{T}(x, t)$ over x from zero to one,[12] and the variance of the temperature,

$$V_T = \int_0^1 \left[\hat{T}(x, t) - T_0(t)\right]^2 dx = \int_0^1 (T_2(t)P_2(x))^2 dx = \frac{(T_2(t))^2}{5} \tag{9}$$

Local temperature means at latitudes $(x, x + dx)$ and the mean temperature over a set of latitudes, $Z = [a, b]$, are defined by

$$\left[T_0(t) + T_2(t)P_2(x)\right] dx, m\left[a, b\right] = \int_a^b \left[T_0(t) + T_2(t)P_2(x)\right] dx \tag{10}$$

while the variance of temperature over the set of latitudes $Z = [a, b]$ is

$$V\left[a, b\right] = \int_a^b \left[T_0(t) + T_2(t)P_2(x) - m\left[a, b; t\right]\right]^2 dx. \tag{11}$$

When the area $Z = [a, b]$ is introduced, it is plausible to assume that utility in each area $[a, b]$ depends upon both the mean temperature and the variance of temperature in that area. For example we may expect increases in mean temperature and variance to have negative impacts on output in any area Z, if it is located in tropical latitudes. In contrast mean temperature increases in some areas Z (e.g. Siberia) may increase rather than decrease utility.[13] Existing dynamic IAMs cannot deal with these kinds of spatial elements, such as impacts of changes in temperature variance, generated by climate dynamics over an area Z.

In the climate model $M\left(t\right)$ is the stock of the atmospheric carbon dioxide. This stock affects the evolution of the temperature through the function g, and evolves through time under the forcing of human inputs in the form of emissions of green house gasses (GHGs) $h\left(x, t\right)$ emitted at latitude x and time t.

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[11] For a detailed derivation of temperature dynamics with hyman imputs and the two-model solution in the context of a one-dimensional EBCM see Brock et al. [2].

[12] This is because $\int_0^1 P_2(x)dx = 0$.

[13] In a stochastic generalization of our model, we could introduce a stochastic process to represent "weather," i.e. very high frequency fluctuations relative to the time scales we are modeling here. Here the "local variance" of high frequency phenomena like "weather" may change with changes in lower frequency phenomena such as mean area Z temperature and area Z temperature variance. We leave this task to future research.

For the human input we assume that emissions $h\left(x,t\right)$ relate to $S\left(t\right)$ by the simple equation

$$\dot{M}\left(t\right)=\int_0^1 h(x,t)dx-mM\left(t\right)=h\left(t\right)-mM\left(t\right) \tag{12}$$

where m is the carbon decay rate. To simplify the exposition we reduce the number of state variables in the problem by assuming that $M\left(t\right)$ has relaxed to a steady state and it relates to $h\left(t\right)$ through the simple linear relation $M\left(t\right)=(1/m)h\left(t\right)$. Thus we approximate $g\left(M\left(t\right)\right)$ by a simple linear relation $\gamma h\left(t\right)$.[14] In this model the latitude of the ice line can move in time in response to changes in human input since the ice line solution depends on $h(t)$. Moving of the ice line towards the poles generates the damages related to damage reservoirs.

The climate model (6)-(8) that incorporates human input, which affects the evolution of temperature can be further simplified by following simplifications proposed by Wang and Stone [35] which suggest that an approximation for the solution equation $T(x,t)=T_0(t)+T_2(t)P_2(x)$ can be achieved by replacing $T_2(t)$ by an appropriate constant. Then $dT(x,t)/dt=dT_0(t)/dt$, where $T_0(t)$, is global mean surface (sea level) temperature. Writing $T\left(t\right)=T_0(t)$ the evolution of the global mean temperature can be approximated by:

$$\frac{dT(t)}{dt}=-\frac{A}{B}-T(t)+\frac{1}{B}\int_0^1\left[QS_2(x)\alpha(x,x_s(t))\right]dx+g\left(S\left(t\right)\right). \tag{13}$$

Thus the Wang-Stone [35] approximation reduces the model to one whose evolution is described by (13). Wang and Stone [35] (equation 3) calibrate the model to get a simple equation for the ice line

$$x_s(t)=(a_{ice}+b_{ice}T(t))^{1/2}, a_{ice}=0.6035, b_{ice}=0.02078. \tag{14}$$

3 The Economic-Climate Model: Damage Reservoirs and Multiple Steady States

We introduce the two types of damages due to climate change mentioned earlier.. Let us define these damages by two functions $D_1(T(t))$ and $D_2(x_s(t))$, where 1 denotes the traditional damages due to temperature rise, and 2 denotes damages due to reservoir damages from movement of the ice line towards the north and permafrost melting. A simplified integrated EBCM can be developed along the following lines.

We consider a simplified economy with aggregate capital stock K. An amount K_2 from this capital stock is diverted to alternative "clean technologies". Output in this economy is produced by capital and emissions h according to a standard production function $F(K-K_2,h+\phi K_2)$, where ϕ is an efficiency parameter for clean technologies.[15] The cost of using a unit of h is $C_h(h)$,with $C_h(0)=0$, $C_h^{'}>0, C_h^{''}>0$. The use of emissions can be reduced by employing

[14] More complicated and probably more realistic approximations will not affect our qualitative results regarding the multiplicity of steady states and the emergence of Skiba points.

[15] See Xepapadeas [38] for different ways in which emissions and environment can be modeled as production factors.

clean technologies at an effective rate ϕK_2. Denoting consumption by C, net capital formation in our simplified economy is described by

$$\frac{dK}{dt} = F(K - K_2, h + \phi K_2) - C - C_h(h) - \delta K \tag{15}$$

where δ is the depreciation rate on the capital stock. Assuming a linear utility function or $U(C) = C$, we consider the problem of a social planner that seeks to maximize discounted life time consumption less damages from climate change subject to (13), (14), and (15).

In this set-up the problem of the social planner can be described, in terms of the following Most Rapid Approach Problem (MRAP) problem,[16]

$$V(T(0)) = \max \int_0^\infty \mathrm{e}^{-\rho t} \left[F(K - K_2, h + \phi K_2) - C_h(h) - (\delta + \rho) K \right. \tag{16}$$

$$\left. - D_1(T(t)) - D_2(x_s(t)) \right] dt$$

subject to (14) and

$$\frac{dT(t)}{dt} = -\frac{A}{B} - T(t) + \frac{\gamma}{B} h(t) + \frac{1}{B} \Psi(T(t)), \tag{17}$$

$$\Psi(T(t)) = \int_0^1 \left[Q S_2(x) \alpha(x, x_s(t)) \right] dx \ , T(0) = T_0, \tag{18}$$

where $V(T(0))$ is the current value state valuation function, ρ is the subjective rate of discount on future utility,and the nonlinear function $\Psi(T(t))$ is an increasing function of T ([23]). Problem (16)-(18), after the successive approximations have been made, has practically been reduced, regarding the climate part, to a zero-dimensional model as found in North et al. [25]. We believe that this exercise is of value because it outlines a pathway to extensions to one-dimensional models and is even suggestive via the Legendre basis method of how one might potentially extend the work to two-dimensional models on the sphere.[17] Problem (16)-(18) is in principle tractable to phase diagram methods with the costate variable on the vertical axis and the state variable on the horizontal axis.

At this point, it should be noted that technical change and population growth could also have been introduced in the form of Harrod neutral (labor augmenting) technical change, a formulation which is required for consistency with balanced growth in the neoclassical context. Balanced growth formulations allow us to conduct phase diagram analysis as in the text below. In this case the production function might be written as $F(K - K_2, h + \phi K_2, AL)$, where F is a constant returns to scale production function and $dA/dt = gA$, $dL/dt = nL$, where g is the rate of exogenous labor augmenting technical change and n is

[16] The assumption of linear utility allows the capital accumulation problem too be written as a MRAP problem. Problem (16) is an approximation of the MRAP problem for very large B and $-B \leq \frac{dK}{dt} \leq B$. In problem (16) capital, K, can thus be eliminated as a state variable. It should also be noted that in this section, damages are modeled using an additive functional form as explained in Weitzman [36]. In section 4 we will revert to the more common multiplicative form. The main qualitative results hold for both these forms.

[17] Research in progress [3] focuses on the development of a two-dimensional spherical coupled climate/economic dynamics model by using a basis of spherical harmonics as in Wu and North [37]. This approach, as well as the Legendre basis approach we are using in this paper for one-dimensional models, fits in nicely with the general approach to approximation methods in Judd ([13], Chapter 6).

the population rate of growth. Output, capital, consumption, emissions and the capital accumulation equation (15) can thus be defined in per effective worker (AL) terms. However the temperature dynamics (18) and (20) now have a non-autonomous term due to exponentially growing emissions. Dealing with this problem while staying within a framework of autonomous dynamics, requires introduction of emission reducing technological progress at an appropriate rate in order to be able to transform the temperature dynamics into a stationary form so that phase diagram techniques of analysis of autonomous systems can still be applied. However, this is beyond the scope of the current paper. In the current paper we wish to show how spatial EBCMs can be integrated with capital accumulation models in economics while preserving analytical tractability. The time stationary analysis developed here indicates that a full analysis of more realistic non-stationary systems is potentially tractable now that we have pointed the way in this paper.

Returning to our time stationary framework, we feel that insights are gained more rapidly by analyzing the following qualitatively similar problem that is strongly motivated by the problem (16)-(18):

$$\begin{aligned} V(T(0)) = \max \int_0^\infty & \mathrm{e}^{-\rho t} \left[F(K - K_2, h + \phi K_2) - C_h(h) - (\delta + \rho) K \right. \qquad (19) \\ & \left. - D_1(T) - D_2(T) \right] dt \\ \text{s.t. } \frac{dT}{dt} & = a_T - b_T T + c_T h \;, (a_T, b_T, c_T) > (0,0,0) \qquad (20) \end{aligned}$$

where $D_1^{'}(T) = a_1 T$, implying increasing marginal damages due to temperature increase, while $D_2^{'}(T)$ is a function increasing at low T reaching a maximum and then decreasing gradually to zero. The shape of $D_2(T)$ is intended to capture initially increasing marginal damages associated with damage reservoirs which reach a maximum as temperature increases, and eventually vanish once the polar ice caps are gone.

The exposition of a number of issues related to damages functions is useful at this point. Assuming a quadratic or a higher degree power function for damages $D_1(T)$ due to temperature increase is consistent with damages related to falling crop yields or reduction to ecosystem services, and this has been the shape adopted in many IAMs. To consider a plausible shape for $D_2(T)$ we have argued in the introduction that as the ice line moves towards the north, there is initially a large quantity of ice to melt which can generate high melt per unit time. As the ice cap is reduced, the melt is reduced and eventually tends to zero when the ice cap disappears. Similar behavior is expected by permafrost. Once permafrost is gone further damages associated with permafrost thawing should vanish. A potential damage function invoking these properties is the gamma function (see Appendix A) which we will be using throughout the paper to capture this type of effect. Another function having similar properties is the S-shaped function used in Brock and Starrett [4] to describe internal loading of phosphorous in a lake system. This functional form yielded give very similar qualitative results to the ones obtained with the gamma function. Further discussion regarding the shape of $D_2(T)$ can be found in Appendix A.1. Furthermore, we argue that the combination of these two damage functions, $D_1(T)$ and $D_2(T)$, each one associated with climate change impacts having different time profiles and being disciplined by scientific evidence, provides a more comprehensive description of the problem.

To further analyze the economic part of the problem, define

$$\pi(h) = \max_{K \geq 0, K_2 \geq 0} \{F(K - K_2, h + \phi K_2) - (\delta + \rho)K\}. \tag{21}$$

Since we assume that $F(\cdot, \cdot)$ is concave increasing, $\pi(h)$ is an increasing concave function of h.[18] We may now write down the current value Hamiltonian and the first order necessary conditions for an optimum,

$$\mathcal{H}(h, T, \lambda_T) = \pi(h) - C_h(h) - D_1(T) - D_2(T) + \lambda_T(a_T - b_T T + c_T h) \tag{22}$$

$$\pi'(h) = C'_h - \lambda_T c_T \Rightarrow h = h^*(\lambda_T)\ ,\ h^{*'}(\lambda_T) > 0, \tag{23}$$

where it is understood in (23) that the inequality conditions of boundary solutions are included, and

$$\frac{dT}{dt} = a_T - b_T T + c_T h^*(\lambda_T)\ ,\ T(0) = T_0 \tag{24}$$

$$\frac{d\lambda_T}{dt} = (\rho + b_T)\lambda_T + a_1 T + D'_2(T). \tag{25}$$

We know that since $\lambda_T(t) = \frac{\partial V(T(t))}{\partial T(t)} := V'(T(t)) < 0$, the costate variable can be interpreted as the shadow cost of temperature. We also know that if a decentralized representative firm pays an emission tax, then the path of the optimal emission tax is $-\lambda_T(t)$. We can study properties of steady states of the problem (16)-(18) by analyzing the phase portrait implied by (24)-(25). The isocline $dT/dt = 0$ is easy to draw for (24). Along this isocline we have $\frac{d\lambda_T}{dT} = \frac{b_T}{c_T h^{*'}} > 0$, by using (23), thus along this isocline λ_T is increasing in T. There is a value λ_{Tc} such that if $\lambda_T(t) < \lambda_{Tc}$ then $h^* = 0$ and $a_T/b_T = T$. If there are no ice line damages, the $d\lambda_T/dt$ isocline is just a linear decreasing function of T that is zero at $T = 0$, or $\lambda_T = -\frac{a_1}{(\rho + b_T)}T$, which implies that $\lambda_T < 0$ for all $T > 0$. Now add the damages emerging from the damage reservoir to this function. The isocline is defined as

$$\lambda_T|_{\frac{d\lambda_T}{dt}=0} = -\frac{a_1 T + D'_2(T)}{(\rho + b_T)}. \ \frac{d\lambda_T}{dT} = -\frac{a_1 + D''_2(T)}{(\rho + b_T)}. \tag{26}$$

With a gamma function representation of $D_2(T)$, $D''_2(T)$ is positive and decreasing, it becomes negative, reaches a minimum and vanishes after becoming positive again. This induces a nonlinearity to the $d\lambda_T/dt = 0$ isocline. In general it is expected that this isocline will have an inverted N-shape, which means that with an increasing $dT/dt = 0$ isocline if a steady state $(\bar{T}, \bar{\lambda}_T)$ exists, there will be either one or three steady states. To study the stability properties of these steady states we form the Jacobian matrix of (24)-(25),

$$J(\bar{T}, \bar{\lambda}_T) = \begin{bmatrix} -b_T & c_T h^{*'}(\bar{\lambda}_T) \\ a_1 + D''_2(\bar{T}) & b_T + \rho \end{bmatrix}. \tag{27}$$

If at a steady state $a_1 + D''_2(\bar{T}) > 0$ so that the $d\lambda_T/dt = 0$ isocline is decreasing then $\det J(\bar{T}, \bar{\lambda}_T) < 0$ and the steady state is a local saddle point.

[18] Note that $\pi'(0) < \infty$ if $\phi > 0$ for the alternative "clean" technology.

If $a_1 + D_2''(\bar{T}) < 0$ so that the $d\lambda_T/dt = 0$ isocline is increasing, the steady state is an unstable spiral.[19] Thus when a unique steady state exists it will be a saddle point. The case of three candidate optimal steady states $\bar{T}_1 < \bar{T}_2 < \bar{T}_3$ is of particular interest. In this case given the shapes of the two isoclines the smallest one and the largest one are saddles and the middle one is an unstable spiral. Thus we have a problem much like the lake problem analyzed by Brock and Starrett [4], and following a similar argument, it can be shown (under modest regularity conditions so that the Hamiltonian is concave-convex in T) that there are two value functions, call them, $V_{mitigate}(T)$ and $V_{adapt}(T)$, and a "Skiba" point $T_s \in (\bar{T}_1, \bar{T}_3)$ such that $V_{mitigate}(T_s) = V_{adapt}(T_s)$. For $T_0 < T_s$, it is optimal to follow the costate/state equations associated with $V_{mitigate}(T)$ and converge to $\bar{T}_1$, while for $T_0 > T_s$ it is optimal to follow the costate/state equations associated with $V_{adapt}(T)$ and converge to $\bar{T}_3$. In Figure 1 we present this situation for an appropriate choice of functional forms and parameters.[20] Besides the solution path the figure also plots the isoclines both with and without ice line damages. Without ice line damages we have the case when the $\dot{\lambda}_T$-isocline is a linear decreasing function of T, implying that we get a unique global saddle point at the crossing of the $\dot{\lambda}_T = 0$, $\dot{T} = 0$ isoclines denoted by $\bar{T}_n$. For the case with ice line damages on the other hand, we get the inverted N-shaped $\dot{\lambda}_T$, isocline giving us a "Skiba" point T_s lying just between the unstable spiral $\bar{T}_2$ and the local saddle point $\bar{T}_3$.

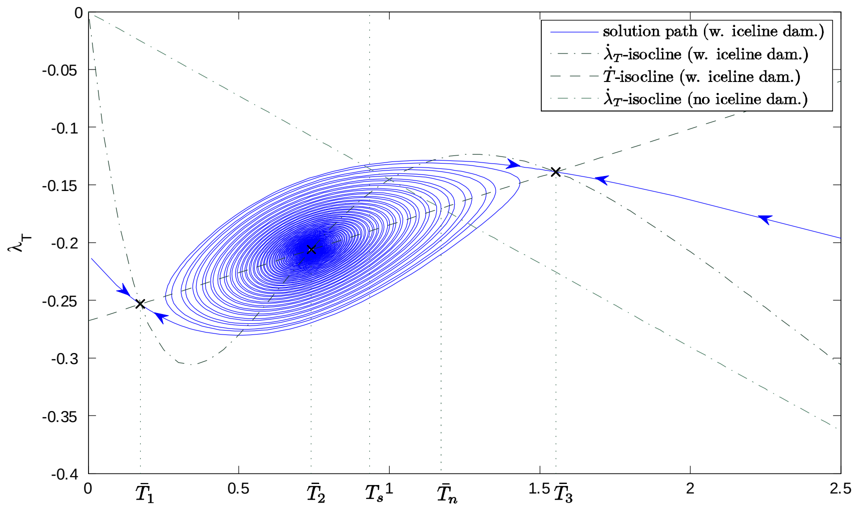


Figure 1: Multiple steady states and Skiba point

Hence, for low initial $T_0 < \bar{T}_1$, it will be optimal to levy a low initial carbon tax even though there is a polar ice cap threat and then gradually increasing

[19] The eigenvalues of J are: $\frac{1}{2}(\rho \pm \sqrt{\Delta})$, where $\Delta = \rho^2 + 4\left[(a_1 + D_2''(\bar{T}))c_T h^{*'} + b_T(b_T + \rho)\right]$. When $a_1 + D_2''(\bar{T}) > 0$ then $\Delta < 0$ and we have two complex eigenvalues with positive real parts which implies an unstable spiral.

[20] The assumed functions, parameters and calculations used in figure 1 are provided in Appendix A.

the carbon tax along a gradualist policy ramp. However, if $T_0 \in (\bar{T}_1, T_s)$, it is optimal to tax carbon higher at T_0 and let the tax gradually fall. But if the initial temperature is large enough, the ice caps are essentially already gone and damage reservoirs have been exhausted. Then the optimal thing to do is to tax carbon initially quite modestly but along an increasing schedule through time to deal with the rising marginal damages due to temperature rise. Figure 1 thus shows how the qualitative picture changes completely when a different shape for the ice line damage function is considered. In particular, the area $T \in (\bar{T}_1, T_s)$ is of interest since, if ice line damages go unaccounted for, the optimal strategy will be to levy a low carbon tax which eventually will raise temperature to $\bar{T}_n$, while in a model with ice line damages included the exact opposite will be true, implying a decrease in temperature to $\bar{T}_1$.

It is important to note that this stationary model is not rich enough to capture the eventual rather sharp increase along the "gradualist" policy ramp of Nordhaus ([19], [20]) because in Nordhaus's case the Business as Usual (BAU) emissions path would be growing because of economic growth. Thus the damages from temperature rise alone, growing quadratically as the quantity of emissions grows, would lead to the gradualist path of carbon taxes "taking off" in the future. However, this simple stationary model does expose the new behavior of a higher initial carbon tax for $T_0 \in (\bar{T}_1, T_s)$. Our runs of the DICE model in section 5 exhibit a sharply higher carbon tax at the beginning due to the extra ice line damages added to Nordhaus's damages.[21]

4 Energy balance - integrated assessment models with damage reservoirs

In this section we incorporate the framework of the simplified energy balance models developed above into a framework similar to well established IAMs such as the DICE/RICE models proposed by Nordhaus. We use notation close to that of Nordhaus for the DICE/RICE part of the model. Consider the continuous time spatial analog of Nordhaus's equations ([19] Appendix 1 or [20], A.1-A.20) where we have made some changes to be consistent with our notation and have suppressed (x, t) arguments to ease typing, unless (x, t) is needed for clarity,

$$W = \int_0^\infty \mathrm{e}^{-\rho t} \int_0^1 \phi(x) U(C) dx dt, \tag{28}$$

where $U(C)$ is utility and C is aggregate consumption at (x, t), and $\phi(x)$ is a Negishi weight function.[22] Furthermore,

$$Y_n = C + \frac{dK}{dt} + \delta K \tag{29}$$

$$Y_n = \Omega(1 - \Lambda) Y, \ Y = F(K) \tag{30}$$

[21] Note that Nordhaus does include damages from ice melt, but the climate model above with moving ice line adds another component of ice melt that has a declining marginal damage function.

[22] The maximization of objective (28) with the "Negishi" $\phi(x)$weighting function is a way of computing a Pareto Optimum competitive equilibrium allocation across latitudes as in Nordhaus's [20] discrete time non-spatial formalization. For a presentation of the use of the Negishi weights in IAMs, see Stanton [31].

where, $Y_n(x,t)$ is output of goods and services at latitude x and time t, net of abatement and damages; $\Omega(T(x,t))$ is the damage function (climate damages as fraction of output) as a function of temperature at (x,t); $\Lambda(x,t)$ is the abatement cost function (abatement costs as fraction of output)[23] at (x,t); and $F(K(x,t))$ is a concave production function of capital. δ is the usual depreciation rate of capital. As explained in the previous section, technology and labor have been removed from the production function in order to avoid problems of non-stationarity in the temperature equation.

Aggregate emissions at time t are defined as:

$$E(t) = \int_0^1 \sigma(1-\mu(x,t))Y(x,t)dx \tag{31}$$

where σ is ratio of uncontrolled industrial emissions to output (metric tons carbon per output at a base year prices), and $\mu(x,t)$ is the emissions-control rate (fraction of uncontrolled emissions) at (x,t). Climate dynamics in the context of the ECBM are given by (1) and (2). Notice that we have replaced Nordhaus's climate equations [20], equations A.14-A.20) with the spatial climate dynamics, (1) and (2).

Maximization of objective (28) subject to the constraints above is a very complicated and difficult optimal control problem of the PDE (1) on an infinite dimensional space $x \in [0,1]$. We reduce this problem to a much simpler approximate problem of the optimal control of a finite number of "modes" using the two-mode approach described earlier.

For the two-mode approximation equations $T(x,t) = T_0(t) + T_2(t)P_2(x)$, (1) and (2) reduce to the pair of ODEs

$$\frac{dT_0}{dt} = \frac{1}{B}\left[-(A+BT_0) + \int_0^1 QS_2(x)\alpha(x,x_s(t))dx + \gamma E\,(t)\right], T_0(0) = T_{00} \tag{32}$$

$$\frac{dT_2}{dt} = \frac{1}{B}\left[-(1+6D)BT_2 + 5\int_0^1 QS_2(x)\alpha(x,x_s(t))P_2(x)dx\right],\ T_2(0) = T_{02} \tag{33}$$

$$T_0(t) + T_2(t)P_2(x_s(t)) = T_s,\ T_s = -10°\text{C}. \tag{34}$$

Before continuing notice that North's two-mode approximation has reduced a problem with a continuum of state variables indexed by $x \in [0,1]$ to a problem where the climate part has only two state variables. We can make yet a further simplification by assuming, as in section 3, that the utility function is linear, i.e. $U(C) = C$. This will allow us to write (28) as the MRAP problem:

$$W = \int_0^\infty \mathrm{e}^{-\rho t}\int_0^1 \phi C dx dt = \int_0^\infty \mathrm{e}^{-\rho t}\int_0^1 \phi\left[\Omega(1-\Lambda)F - (\rho+\delta)K\right]dxdt. \tag{35}$$

Note that for the two mode approximation, the damage function should be defined as:

$$\Omega(T(x,t)) = \Omega(T_0(t) + T_2(t)P_2(x)). \tag{36}$$

[23] With our spatial approach abatement costs could be made site specific which would enable a more comprehensive analysis of issues concerning, e.g., geoengineering. However this goes beyond the scope of the current paper and is left for future research.

To ease notation we introduce the inner product notation $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. We may now write the current value Hamiltonian for the optimal control problem (35) and show how we have drastically simplified the problem by using a two-mode approximation,[24]

$$\mathcal{H} = \int_0^1 \phi \left[\Omega(1-\Lambda)F - (\rho+\delta)K + \frac{\lambda_0}{B}\sigma(1-\mu)F\right] dx \tag{37}$$
$$\frac{\lambda_0}{B}\left[\langle QS\alpha, 1\rangle - A - BT_0\right] + \frac{\lambda_2}{B}\left[5\langle QS\alpha, P_2\rangle - (1+6D)BT_2\right].$$

For the simplified problem (35), the capital stock and the emissions control rate $K^*(x,t), \mu^*(x,t)$ are chosen to maximize $\mathcal{H}$ for each (x,t), which is a relatively simple problem. However there is one complication to be addressed. The absorption function $\alpha(x, x_s(t))$ depends upon the ice line $x_s(t)$ where the ice line is given by a solution of (34), i.e.

$$x_s(t) = P_+^{-1}\left(\frac{T_s - T_0(t)}{T_2(t)}\right) \tag{38}$$

where the subscript "+" denotes the largest inverse function of the quadratic function $P_2(x) := (1/2)(3x^2-1)$. Notice that the inverse function is unique and is the largest one on the set of latitudes $[0,1]$. Equation (38) induces a nonlinear dependence of equations (32) and (33) through the absorption function, but no new state variables are introduced by this dependence. An additional dependence induced by equations (32) and (33) as well as equation (38) is on the damage function which we parameterize as:

$$\Omega = \Omega(T_0(t), T_2^2(t)P_2^2(x); x_s(t), x) \tag{39}$$

The first term in (39) represents damages to output at latitude x as a function of average planetary temperature as in Nordhaus ([19], [20]) and the second term is an attempt to capture extra damages due to climate "variance". Note that the component $P_2^2(x)$ is larger at $x=0$ and $x=1$ than it is at the "temperate" latitude $x=(1/3)^{1/2}$ where $P_2^2(x)=0$. This is an admittedly crude attempt to capture the component of damages due to "wetter places getting wetter" and "dryer places getting dryer" as well as damages to arctic latitudes compared to temperate latitudes. But some of this dependence can be captured also in the "x" term in the parameterization (39). Finally the impact on damages at latitude x due to shifts in the ice line is captured by inclusion of the ice line in (39). This is a fairly flexible parameterization of spatial effects (i.e. latitude specific effects) that are not captured in the traditional non-spatial formulations of integrated assessment models.

4.1 Optimal mitigation and location specific policy ramp

Let us first illustrate optimal mitigation using our two-mode simplification of our original "infinite mode" problem with linear utility by considering a version

[24] The important thing to note about this Hamiltonian compared to the Hamiltonian of the original problem (28) is this. The original problem would generate a Hamiltonian with a continuum of costate variables, one for each $x \in [0,1]$. The two-mode approximation approach developed could be quite easily extended to an n-mode approximation approach. Since however North argues that a two-mode approximation is quite good, we continue with a two-mode approximation here.

of the problem where the impact of policy $\{\mu(x,t)\}$ on the location of the ice line $x_s(t)$ is ignored. That is there is no ice line dependence of any functions of the problem including the absorption function. In this simplified case the albedo function depends only upon x and thus the terms $\langle QS\alpha, 1\rangle, \langle QS\alpha, P_2\rangle$ do not depend upon $T_0(t), T_2(t)$ in (32) and (33). Hence the two costate ODEs would become

$$\frac{d\lambda_0}{dt} = (\rho+1)\lambda_0 - \frac{\partial \mathcal{H}}{\partial T_0} = (\rho+1)\lambda_0 - \int_0^1 \phi \frac{\partial \Omega}{\partial T_0}(1-\Lambda)F dx \tag{40}$$

$$\frac{d\lambda_2}{dt} = (\rho+1+6D)\lambda_2 - \frac{\partial \mathcal{H}}{\partial T_2} = (\rho+1+6D)\lambda_2 - \int_0^1 \phi \frac{\partial \Omega}{\partial T_2}(1-\Lambda)F dx$$

Wang and Stone [35] argue that one can even get a fairly good approximation of T_2 by exploiting how fast mode 2 converges relative to mode zero in equation (33) as compared to (32). Hence we can further simplify the problem by assuming that T_2 has already converged to:

$$T_2 = \frac{5\langle QS\alpha, P_2\rangle}{(1+6D)B} \tag{41}$$

for each $T(t)$.[25] The Hamiltonian (37) for the case when the absorption function and T_2 are constant can thus be written as[26]

$$\mathcal{H} = \int_0^1 \left[\phi(\Omega(1-\psi\mu)F - (\rho+\delta)K) + \frac{\lambda_0}{B}\sigma(1-\mu)F\right] dx \tag{42}$$

$$+ \frac{\lambda_0}{B}\left[Q\alpha - A - BT_0\right]. \tag{43}$$

In this case we obtain the following switching decision rule for $\mu^*(x,t)$[27]

$$\mu^*(x,t) \left\{ \begin{array}{c} =0 \\ \in [0,1] \\ =1 \end{array} \right\} \text{ for } -\lambda_0(t) \left\{ \begin{array}{c} < \\ = \\ > \end{array} \right\} \frac{\phi(x)\psi B}{\sigma(x)}\Omega \tag{44}$$

$$\Omega = \Omega(T_0(t), (T_2 P_2(x))^2, x) \tag{45}$$

$$\lambda_0(t) = \int_{s=t}^{\infty} e^{-(\rho+1)(s-t)} \left[\int_0^1 \Omega(1-\psi\mu^*)F\frac{\partial \Omega}{\partial T_0} dx\right] ds. \tag{46}$$

Suppose some type of institution wanted to implement this social optimum. One way to do it would be to impose a tax $\tau(\lambda) = \frac{-\lambda_0(t)}{B}$ on emissions when individual agents solve the static problems

$$\max_{\{\mu\in[0,1], K\geq 0\}} \left\{\Omega(1-\psi\mu)F - (\rho+\delta)K - \tau(\lambda)\sigma(1-\mu)F\right\}. \tag{47}$$

We see right away that the first order necessary conditions for the problem (47) are the same with those resulting from the Hamiltonian function (42). Since

[25] Note that in the case where the absorption function does not depend upon $x_s(t)$ the RHS of (41) is constant.

[26] Note that with a constant absorption function, $\langle QS\alpha, 1\rangle = \langle Q(1+S_2P_2(x))\alpha, 1\rangle = \langle Q\alpha + QS_2\alpha P_2(x), 1\rangle = \langle Q\alpha, 1\rangle = Q\alpha$, since $\langle QS_2\alpha P_2(x), 1\rangle = 0$.

[27] Here, we have also assumed that abatement costs ($\Lambda = \psi\mu,\ \psi > 0$) are linear, implying that the solution is of the bang-bang type. In section 4.2 we will consider a nonlinear version of abatement costs.

$F(K)$ is a concave increasing function, then setting $\tau(\lambda) = \frac{-\lambda_0(t)}{B}$ implements the social optimum. Note that the socially optimal emissions tax is uniform across all locations as one would expect from Nordhaus ([19], [20]).

An important question arises at this point: What substantive difference does the spatial climate model coupled to the economic model add that is not already captured by non-spatial climate models? There are several important differences regarding policy implications.

The emission reduction policy ramp $\mu^*(x,t)$ is location specific and dictates $\mu^*(x,t) = 1$ for all (x,t) where the relative Negishi weight $\phi(x)$ on welfare at that location is small (recall that $\int_0^1 \phi(x)dx = 1$ by normalization). Assume that the damage function $\Omega = \Omega(T_0(t), (T_2P_2(x))^2, x) = \Omega(T_0(t), (T_2P_2(x))^2)$ is decreasing in both arguments.[28] This crudely captures the idea that damages increase at each latitude as average planetary temperature, $T_0(t)$, increases and as a measure of local climate "variance" $(T_2P_2(x))^2$ increases. Let R denote a set of "at risk latitudes" with low values of $\Omega(T_0(t), (T_2P_2(x))^2)$, i.e. with high values of the arguments. The set R is a crude attempt to capture latitudes that would be relatively most damaged by climate change. A plausible type of objective would be to solve the social problem above but with $\phi(x) > 0, x \in R$, $\phi(x) \simeq 0, x \notin R$. We see right away that this social problem would require all xs not in R to reduce all emissions immediately. In general we have,

$$\mu^*(x,t) = 1, \; for \; -\lambda_0(t) > \frac{\phi(x)\psi B}{\sigma(x)}\Omega \tag{48}$$

and vice versa. This makes good economic sense. The marginal social burden on the planet as a whole of a unit of emissions at date t, no matter from which x it emanates is, $-\lambda_0(t)$. Locations x where the Negishi weight on the location is small, where emissions per unit of output are relatively large (relatively large $\sigma(x)$), and that are already relatively heavily damaged ($\Omega(T_0(t), (T_2P_2(x))^2, x)$ is high) are ordered to stop emitting. Thus our modeling allows plausible specifications of the economic justice argument stemming from geography to shape policy rules.

In the following section, we use this framework to extend our results in the presence of an discontinuous absorption function that changes at the ice line. This is a more realistic model which introduces ice line damages which we develop in the context of a DICE/RICE-type integrated assessment model.

4.2 Optimal mitigation in an IAM-type model with damage reservoirs

We now introduce as the absorption function the version proposed in North ([23]) where

$$\alpha(x, x_s) = 1 - \alpha(x) = \begin{cases} \alpha_1 = 0.38 & x > x_s \\ \alpha_0 = 0.68 & x < x_s \end{cases}, \tag{49}$$

where $\alpha(x)$ is the albedo. With this absorption function, the dynamics $T_0(t)$ in (32) and the T_2 approximation in (41) become respectively

[28] $(T_2P_2(x))^2$ denotes the variance of the average temperature at location x.

$$\frac{dT_0}{dt} = \frac{1}{B}\left[-(A+BT_0) + Q(\alpha_0-\alpha_1)\int_{x=0}^{x=x_s(t)}(1+S_2P_2(x))dx + \gamma E + Q\alpha_1\right] \tag{50}$$

$$T_2 = \frac{1}{(1+6D)B}\left[5Q(\alpha_0-\alpha_1)\int_{x=0}^{x=x_s(t)}(1+S_2P_2(x))P_2(x)dx + Q\alpha_1 S_2\right], \tag{51}$$

where the equation for the ice line is, using (38),

$$x_s(t) = \left[\frac{2}{3}\frac{T_s - T_0(t)}{T_2} + \frac{1}{3}\right]^{\frac{1}{2}}. \tag{52}$$

The objective (28) and the constraints (49)-(52) determine optimal mitigation over time and latitude. The discontinuous absorption function can create a strong nonlinearity where a small change in T_0 can cause a large change in damages at some latitudes. However this nonlinearity makes it difficult to proceed with analytical solutions. To obtain a qualitative idea of the impact of the nonlinearity due to the absorption function and the ice line, we use the climate parametrization used by North [23] ($\alpha_0 = 0.68, \alpha_1 = 0.38, A = 201.4, B = 1.45, S_2 = -0.483, T_s = -10, Q = 334.4$). The heat transport coefficient D is found to be approximately 0.2214 by calibrating the ice line function to the current ice line estimate ($x_s = 0.95$).[29]

The system (50)-(52) is highly nonlinear and can be simplified by deriving a polynomial approximation of x_s as a function of $T_0(t)$. We proceed in the following way. If we substitute $x_s(t)$ from (52) into (51), then T_2 is a fixed point of (51). We solve numerically the fixed point problem (51) for values of $T_0 \in \left[-\bar{T}_0, \bar{T}_0\right]$, obtaining the solution $\hat{T}_2(T_0)$. Substituting this back into equation (52) gives us the $\hat{x}_s(\hat{T}_2(T_0), T_0)$ which is then used to fit a quadratic curve on $(T_0, \hat{x}_s)$ by using least squares. Thus $\hat{x}_s$ is approximated by a convex curve $\hat{x}_s = \zeta_0 + \zeta_1 T_0 + \zeta_2 T_0^2 = \zeta(T_0)$, $(\zeta_0, \zeta_1, \zeta_2) > 0$.[30] Making use of this approximation, the system (50)-(52) can thus be written as:

$$\frac{dT_0}{dt} = \frac{1}{B}\left[-(A+BT_0) + Q(\alpha_0-\alpha_1)\theta(T_0) + E + Q\alpha_1\right] \tag{53}$$

$$\text{where } \theta(T_0) := \left[\hat{x}_s + \frac{S_2}{2}(\hat{x}_s^3 - \hat{x}_s)\right] \text{ with } \hat{x}_s := \zeta_0 + \zeta_1 T_0 + \zeta_2 T_0^2$$

Assuming linear utility once again, the Hamiltonian can be written as:

$$\mathcal{H} = \int_0^1 \left[\phi[K^\beta \Omega(T_0)(1-\Lambda) - (\rho+\delta)K] + \frac{\lambda_0}{B}\sigma(1-\mu)K^\beta\right]dx \tag{54}$$
$$+ \frac{\lambda_0}{B}\left[-A - BT_0 + Q(\alpha_0-\alpha_1)\theta(T_0) + Q\alpha_1\right].$$

[29] The calibration procedure is explained in detail by North ([24] p.2035-2037).

[30] The estimated quadratic function was

$$\hat{x}_s = 0.7126 + 0.0098T_0 + 0.0003T_0^2, \; R^2 = 0.99.$$

We now assume that abatement costs are increasing in abatement activities, $\Lambda = \psi\mu^2$. The optimal μ and K will thus be defined as:

$$\mu^*(x,t) = -\frac{\lambda_0 \sigma}{2B\phi\psi\Omega(T_0)}, \forall x \in [0,1] \tag{55}$$

$$K^*(x,t) = \left(\frac{\rho+\delta}{\beta}\right)^{\frac{1}{\beta-1}} \left[\Omega(T_0)(1-\psi\mu^{*2}) - \frac{\lambda_0}{\phi B}\sigma(1-\mu^*)\right]^{\frac{-1}{\beta-1}}. \tag{56}$$

and the canonical system becomes:

$$\frac{dT_0}{dt} = \left[-A - BT_0 + Q(\alpha_0 - \alpha_1)\theta(T_0) + \int_0^1 \sigma(1-\mu^*)K^{*\beta}dx\right] \tag{57}$$

$$\frac{d\lambda_0}{dt} = (\rho + 1 - \frac{Q}{B}(\alpha_0 - \alpha_1)\theta'(T_0))\lambda_0 - \int_0^1 \left[K^{*\beta}\Omega'(T_0)(1-\psi\mu^{*2})\right] dx \tag{58}$$

which can be solved numerically given a specific shape of $\phi(x)$.

To proceed further we need a more detailed specification for the damage function, which as explained above should contain a temperature component denoted by $D_1(T_0)$ and an ice line component, denoted by $D_2(T_0)$. We specify the damage function in the following way. Lost output from temperature induced damages is: $Y - \frac{Y}{1+D_1(T_0)} = \frac{YD_1(T_0)}{1+D_1(T_0)} := Yd_1(T_0)$. Lost output from ice line movement towards the poles written as a function of T_0 is: $Y - \frac{Y}{1+D_2(T_0)} = \frac{YD_2(T_0)}{1+D_2(T_0)} := Yd_2(T_0)$. The sum of lost output from both sources is: Lost$Y = Yd_1(T_0) + Yd_2(T_0)$. Thus net output available for consumption and mitigation is: $Y -$ Lost$Y = (1 - d_1(T_0) - d_2(T_0))Y$.

If we define $\Omega_i(T_0) = \frac{1}{1+D_i(T_0)}, i = 1, 2$, then the term $(1 - d_1(T_0) - d_2(T_0))$ can be written as the damage function Ω of the system (55)-(58) in the form

$$\Omega(T_0) = \Omega_1(T_0) + \Omega_2(T_0) - 1. \tag{59}$$

As the global warming problem concerns damages resulting from temperature increases rather than decreases, we restrict the state space to include only temperatures $T_0 > 15°$C, i.e. in the vicinity of the present average global temperature level.[31] In the spatial model used in this section, this temperature level is found by setting $E = 0$ and solving (53), which gives us $T_0 \approx 15.27$. Hence, 15°C can be viewed as a rough ballpark estimate of the preindustrial global temperature average. Damages are assumed to start at 15°C and we thus write our normalized damage function as $\Omega(T_0 - 15)$. Furthermore, we will use the same functional forms for the damage functions as used in section 3.

The EBCM that we presented in this section, resulting from the concepts developed in the earlier part of the paper, has many similarities to the traditional IAMs but also two potentially important differences. The first is the discontinuous absorption function and the second is an alternative shape for ice line damages as opposed to other temperature related damages. Together they introduce complex nonlinearities into the temperature dynamics. The question

[31] During the development of many energy balance models in the 1960s and 1970s the main concern was usually not that of global warming, but rather that of drastic global cooling that could result due to a slight decrease in the solar constant. This hypothesis was later coined "Snowball earth" by Kirschvink [15].

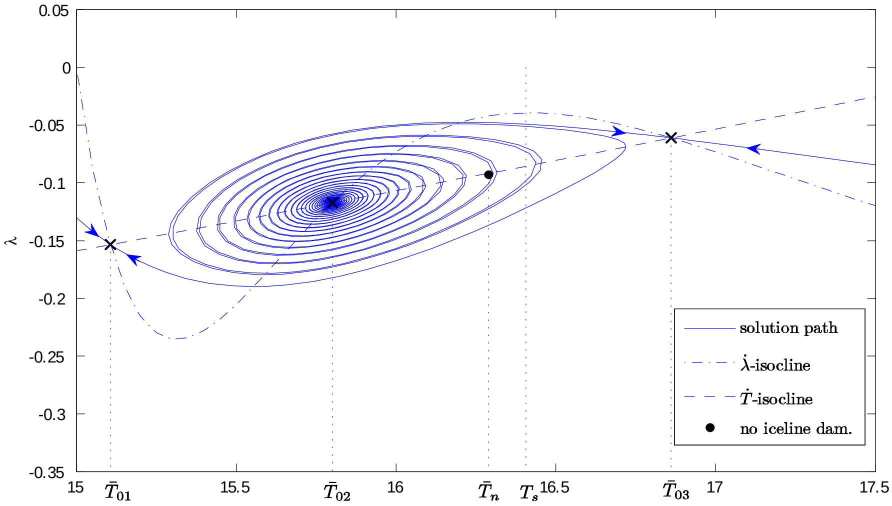


Figure 2: IAM: Multipe steady states and Skiba point

of whether these differences imply significant deviations from the model's predictions, cannot be answered analytically due to the high complexity of the models. So we resort to numerical simulations.

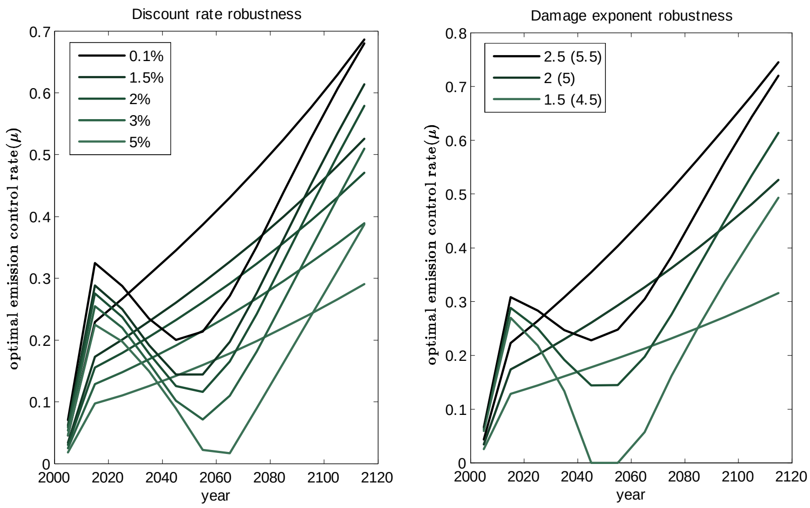
Figure 2 shows the results for the spatial climate model presented in this section. As in section 2 this model also gives us 3 candidate optimal steady states, $\bar{T}_{01} < \bar{T}_{02} < \bar{T}_{03}$, where the largest and the smallest ones are saddles while the middle one is an unstable spiral.[32] Between the unstable spiral $\bar{T}_2$ and the saddle $\bar{T}_3$ we have a Skiba point $\bar{T}_s$ similar to that of section2.[33] Hence, for low initial temperatures $T_{00} < \bar{T}_1$ a low but gradually increasing carbon tax is optimal, while for $T_{00} < T_s$ we get the case where it is optimal to levy a high carbon tax at T_{00} and then gradually decrease it. Furthermore, figure 2 also depicts the case when ice line damages are omitted, $\bar{T}_n$. In contrast to section 2, both of the isoclines are now affected and in order to keep the figure from becoming too messy, we have chosen to plot only the single equilibrium at the crossing of these isoclines, which is denoted by the black dot at $\bar{T}_n$ in figure 3. The qualitative behavior is however the same as in section 2, i.e. the "damage reservoir - no ice line damage equilibrium" is a saddle having a positive slope for the $\dot{T}$-isocline and a negative slope for the $\dot{\lambda}$-isocline.

[32] The corresponding eigenvalues are approximated numerically as $e_{01} = [-0.3974, 0.4174]$, $e_{02} = [0.0100 \pm 0.2045i]$ and $e_{03} = [-0.1946, 0.2146]$.

[33] Greiner et al. [11] find multiple equilibria in a zero-dimensional EBCM, where albedo is modeled by a continuous S-shaped function of temperature. The derived multiple-equilibria and Skiba planes, however, only apply for fixed levels of abatement, i.e. there is just a single control variable (consumption). If, however, the social planner can control both consumption and abatement then there exists only a single stable saddle. Our approach, apart from explicitly addressing the more appropriate one-dimensional model also differs in the sense that we obtain multiple equilibria and Skiba points when controlling both consumption and abatement.

5 The DICE Model with Damage Reservoirs

Both the relatively simple model of section 2 and the more complex model of section 4 strongly suggest that the explicit modeling of ice line damages shows the need for strong mitigation now. In order to further demonstrate that this result is robust to the choice of model, we now turn to the DICE model. The purpose of this exercise is to show how the introduction of ice line damages into the damage function, along the lines suggested by the EBCMs, will affect the optimal emission policy implied by DICE. The DICE model, the most well known of the IAMs, assumes that all damages to the economy evolve according to the quadratic equation (A.5) in Nordhaus [19]. This equation has been calibrated to a 2.5 degree warming based on an aggregate of impact studies from a variety of different sources.[34] In order to separate out the ice line component from the total amount of damages, we follow the procedure shown in section 4.2. We thus simply replace (A.5) with equation (59) from this section. Hence, we have two separate damage components, $D_1(T)$ and $D_2(T)$, which can be calibrated independently according to different impact assessments. Nordhaus [19] finds the aggregate impact of a 2.5 degree warming to be roughly 2% of GDP. Since, it is not possible to determine exactly how much of this 2% fall in GDP is due to ice line specific damages, we simply make a crude assumption that approximately half of these damages are attributable to the ice line component $D_2(T)$.[35] Next, we make the following assumptions regarding the shapes of the temperature and ice line specific components, i.e. we set $D_1(T) = a_1 T^5$ and $D_2(T) = a_2 e^{-2T} T^2$. In a manner consistent with Nordhaus ([19]) we then proceed by calibrating the parameters a_1 and a_2 so that $D_1(2.5) = 0.01$ and $D_2(2.5) = 0.01$. In this way our new damage function produces an amount of damage at a 2.5 degree warming which is equivalent to that in the original model but will differs from it for all other temperature levels.



[34]See Nordhaus ([19] accompanying notes p.23-25).

[35]On page 24 of the accompanying notes of the DICE 2007 ([19]) model there is an impact assessment by region and impact type. These are then weighted based on GDP estimates for 2105. As these weights are not provided, it is not possible to determine a specific region or impact type.

Figure 3: U-shaped policy ramp

This new damage function has the property that the temperature component, having a larger exponent than the original quadratic function, makes the impact of GDP on the temperature much larger when temperature levels start to rise above 3 degrees. On the other hand when temperature levels are lower, the damages from the ice line are the ones that dominate.[36] Figure 3 plots the optimal emission control rate resulting from the DICE-2007 model with and without ice line damages. As can be seen from this graph, the separation of different damage structures gives us a U-shaped policy where it is optimal to mitigate more initially as opposed to the normal gradualist policy ramp. The figure also displays a simple robustness check, showing how the results are affected by changing the values for the discount rate and damage exponent. As can be seen from the left graph, raising the discount rate seems to strengthen the case for an act now policy as opposed to the more gradualist path at the same level of discounting. Although these results remain specific to our assumptions regarding the shape of the damage function for the ice line as well as the temperature component, they still exemplify the sensitivity of the model to structural changes in the damage function and the impact of incorporating insights from energy balance models.

6 Summary, Conclusions, and Suggestions for Future Research

In this paper we introduce the economics profession to spatial Energy Balance Climate Models (EBCMs) and show how to couple them to economic models while deriving analytical results of interest to economists and policy makers. While we believe this contribution is of importance in its own right, we also show how introduction of the spatial dimension incorporated into the EBCMs leads to new ways of looking at climate policy.

In particular, by accounting for an endogenous ice line and paying attention to the associated damage reservoirs and albedo effects we show that due to nonlinearities even simple economic-EBCMs generated multiple steady states and policy ramps which do not in general follow the "gradualist" predictions. These results carry over to more complex models where the economic module has an IAM structure. The interesting issue from the emergence of multiple steady states, is that when the endogenous ice line and discontinuous albedo are ignored, as in traditional IAMs, the policy prescription of these models could be the opposite of the policy dictated by the economic-EBCMs. Furthermore the spatial aspect of the EBCMs allows arguments associated with the spatial structure of climate change damages to shape policy rules. When we applied the damage function implied by the EBCMs and calibrated appropriately simulations in the DICE model gave results interpretable as a U-shaped policy ramp indicating an important deviation from the gradualist policy ramp derived from the standard DICE model. Thus a rapid mitigation policy can be justified on the new insights obtained by coupling the economy with the EBCMs.

Areas for further research could range from making the economics more so-

[36] See Ackerman et al. [1] for a discussion regarding different values for the exponent of the damage function used in DICE.

phisticated by abandoning the simplifying assumption of linear utility; allowing for technical change and knowledge spillovers across latitudes; or introducing strategic interactions among regions,[37] to extending the EBCMs. Future work that needs to be done regarding EBCMs is extension to two-dimensional spherical EBCMs because Earth is a sphere, not a line. [3] are attempting to make a dent in this problem. They frame the problem as a recursive dynamic programming problem where the state vector includes a number of "spherical modes" that are analogs of the modes in this paper as well as economic state variables. Another possible extension could be the consideration of new policy instruments. Emissions reduction acts on the outgoing radiation in the sense that by reducing emissions the outgoing radiation increases through the second term of the right hand side of (1). Another kind of policy could act on the first term of the right hand side of (1) in the sense of reducing the incoming radiation. This type of policy might be associated with geoengineering options. Finally a policy which acts on the damage function in the sense of reducing damages for any given level of temperature and radiation balance might be associated with adaptations options. Unified economic-EBCMs might be a useful vehicle for analyzing the structure and the trade offs among these different policy options.

[37] These extensions will undoubtedly increase the complexity and the computational needs for solving the economic-EBCMs.

A Appendix: derivations and assumptions

This section drafts some of the more specific assumptions on which figure 1 is based. The production function in (21) is assumed to take the following form:

$$F(K-K_2, h+\phi K_2) = (K-K_2)^{\beta_1}(h+\phi K_2))^{\beta_2} \tag{60}$$

with $\beta_1 > 0, \beta_2 > 0$. The solution to problem (21) is derived from the first order conditions:

$$\frac{\partial F}{\partial K} = \beta_1 (K-K_2)^{\beta_1 - 1}(h+\phi K_2))^{\beta_2} - (\delta+\rho) = 0 \tag{61}$$

$$\frac{\partial F}{\partial K_2} = -\beta_1 (K-K_2)^{\beta_1 - 1}(h+\phi K_2))^{\beta_2} + \beta_2 \phi (K-K_2)^{\beta_1}(h+\phi K_2))^{\beta_2 - 1} = 0 \tag{62}$$

Solving the system (61) and (62) for K and K_2 gives the solution to problem (21).

$$K_2^*(h) = \frac{1}{\phi}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{1}{\beta_1 - 1 + \beta_2}} - \frac{h}{\phi}$$

$$K^*(h) = \frac{\beta_1}{\phi\beta_2}h + \left(1 + \frac{\beta_1}{\beta_2}\right)K_2^*(h)$$

Plugging these values back into (21) allows us to write $\pi(h)$ as a linear function of h:

$\pi(h) = \tilde{A} + \tilde{B}h$

with

$$\tilde{A} := \left(\frac{\beta_1}{\phi\beta_2}\right)^{\beta_1}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{\beta_1+\beta_2}{\beta_1-1+\beta_2}} - (\delta+\rho)\frac{(1+\phi)}{\phi}\left(\frac{(\delta+\rho)}{\beta_1}\left(\frac{\beta_1}{\phi\beta_2}\right)^{1-\beta_1}\right)^{\frac{1}{\beta_1-1+\beta_2}}$$

$$\tilde{B} := -(\delta+\rho)\left(\frac{\beta_1}{\phi\beta_2} - \frac{(1+\phi)}{\phi}\right)$$

which is increasing in h given that $\beta_1/\beta_2 < (1+\phi)$. Assuming also that $D_1(T) = a_1 T^2$, $D_2(T) = a_2 \exp(-2T)T^2$ and $C_h(h) = c_h h^2$, where $a_1, a_2, c_h > 0$. [38] Substituting this into (22) and using the first order condition we can thus derive the canonical system:

$$\frac{dT}{dt} = a_T - b_T T + c_T \frac{\tilde{B} + \lambda_T c_T}{2c_h}\ ,\ T(0) = T_0 \tag{63}$$

$$\frac{d\lambda_T}{dt} = (\rho + b_T)\lambda_T + a_1 T - 2a_2 e^{-2T}(T-1)T \tag{64}$$

From (63) and (64) it is easy to confirm the shape of the isoclines depicted in figure 1. For the numerical calculations of the solution paths and the Skiba point we used numerical methods described in [10], [9]. The parameter values used for the numerical calculations are given in the table below:

Parameter	Value	Description
ρ	0.02	discount rate
β_1	0.3	elasticity of capital with respect to output
β_2	0.6	elasticity of energy with respect to output
δ	0.1	depreciation rate of capital
ϕ	0.9	efficiency parameter of clean energy
a_1	0.09	damage parameter of $D_1(T)$
a_2	0.7	damage parameter of $D_2(T)$
a_T	0.8	parameter of temperature equation
b_T	0.6	parameter of temperature equation
c_T	0.85	parameter of temperature equation
c_h	0.05	parameter of cost function

Table 1: The parameter values of figure 1

We also tested the following S-shaped functional form for ice line damages:

$$D_2(T) = \theta \frac{T^\gamma}{\varphi + T^\gamma}, \text{ with } \{\gamma, \theta, \varphi\} > 0$$

For appropriate values of the parameters we got the same qualitative results as displayed in the phase plots of both section 2 and 4.

A.1 Damage reservoirs

Although we have already provided some intuitive arguments regarding the shape of $D_2(T)$ and strengthened these arguments with empirical findings from the literature, further rigor might be called for. Consider the following line of argumentation:

Define $x_s(T) = \min\{1, (a_i + b_i T)^{0.5}\}$ to modify the Wang/Stone equation (14) for the ice line so it can't go above 1 where the ice caps are completely gone. Tune the a_i, b_i parameters of Wang/Stone so that $x_s(T)$ reaches one at a very large value of T, call it T^*. Note that $D_2(T^*)$ is very small for $T > T^*$. This is good enough to motivate the right hand part of the specification of $D_2(T)$ for large T. Now motivate $D_2(T)$ by specifying a function $g(\cdot)$ such that $D_2(T)$ is an approximation to $g(x_s(T))$. Note that since $g(x_s(T))$ must be zero for $T > T^*$, $D_2(T)$ can't be exactly represented by $g(x_s(T))$ but is close enough for $T > T^*$ to serve as a "good enough" approximation in return for its tractability as shown by our phase diagrams in section 2 and 4.

One could also argue for an alternative formulation of the $D_2(T)$ that does not include adaptation abilities to damages which are implicitly assumed in our current formulation. Such a damage function, would look similar to the gamma function in our paper except that damages would not go to zero but instead level off at some T^* implying that a certain fraction of output is lost forever for $T > T^*$. Such an S-shaped function has been used frequently in the literature

[38] The shape of $D_1(T)$ has become fairly standard in the literature. Still, in a recent review by [1], they uncovered no rationale, whether empirical or theoretical, for adopting a quadratic form for the damage function. The shape of $D_2(T)$ is motivated in the text and in appendix A.1.

describing non-convexities adherent in ecosystems, see e.g. [4]. We tested such an S-shaped functional form which proved to give very similar qualitative results to the ones we present in this paper. From a technical point of view the choice between this form and the gamma function is thus only a matter of preference.

Of course, no one really knows exactly what the damages to world welfare as a whole are as the ice lines retreat from their present position to the poles, but it seems plausible to expect the damages to initially increase, perhaps at an increasing rate as people struggle to deal with the large adjustment costs of dealing with melting of a large ice mass, but as the ice mass gets smaller, the adjustment costs should get smaller until the costs start dropping due to smaller and smaller ice masses melting. As the remaining ice mass shrinks to zero with increasing T one could thus argue that $D_2(T)$ goes to zero.

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