Optimal Agglomerations in Dynamic Economics

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Abstract

We study rational expectations equilibrium problems and social optimum problems in infinite horizon spatial economies in the context of a Ramsey type capital accumulation problem with geographical spillovers. We identify sufficient local and global conditions for the emergence (or not) of optimal agglomeration, using techniques from monotone operator theory and spectral theory in infinite dimensional Hilbert spaces. Our analytical methods can be used to systematically study optimal potential agglomeration and clustering in dynamic economics.

Keywords: Agglomeration, spatial spillovers, spillover induced instability, rational expectations equilibrium, social optimum, monotone operators.

1 Introduction

This paper shows how monotone operator theory can be used to study rational expectations equilibrium problems and social optimum problems in infinite horizon, infinite dimensional spatial economies. Our analysis is applied to an illustrative infinite horizon, infinite dimensional spatial Ramsey type capital accumulation problem where borrowing and lending on world capital markets at a rate of interest equal to the rate of discount on subjective utility are the same and quadratic adjustment costs penalize rapid movements of capital. We locate sufficient conditions on primitives that may cause potential agglomerations to form and to not form for both problems. Furthermore, we show how the spectral theory of compact operators allows decomposition of the infinite dimensional problem into a countable collection of tractable finite dimensional problems. Using this technique we provide explicit local stability criteria for the linearized system.

Related literature includes work by Krugman (1996), Fujita et al. (2001), Lucas (2001), Quah (2002), Desmet and Rossi-Hansberg (2007), Ioannides and Overman (2007), Lucas and Rossi-Hansberg (2007), and others. However, to our knowledge, no one has yet provided a concise framework in which the combination of monotone operator theory, the theory of compact operators, and the decomposition techniques we develop here can be applied to infinite horizon, infinite dimensional spatial economies to study endogenous agglomeration (or non-agglomeration) for rational expectations equilibrium and the social optimum in terms of local and global analysis as we do here.

There is a large literature in mathematical biology (e.g., Murray, 2003) that studies spatial agglomeration problems in infinite dimensional spaces. However, as far as we know, none of this literature deals with optimization problems as we do here. There are many differences between the "backward-looking" dynamics in mathematical biology problems and other natural science problems, and the "forward-looking" dynamics of economic problems. It is not just a simple adaptation of dynamical systems techniques to two-point boundary value problems in analogy with the familiar phase diagrams in textbook analysis of Ramsey type optimal growth problems and Ramsey type rational expectations problems in finite dimensional spaces. For example, our development of techniques from operator theory mentioned above allows us to locate sufficient conditions on primitives for all potential agglomerations to be removed in infinite horizon optimization problems. Intuitively this is a generalization of classical turnpike theory of finite dimensional economic models to infinite dimensional spatial models. Thus, contrary to the spirit of the Turing instability which provides local results for the linearized dynamical systems, we obtain global results valid for the fully nonlinear optimized dynamical system. Global analysis based on monotone operator theory, combined with local analysis based on spectral theory, provides valuable insights regarding the endogenous emergence (or not) of optimal agglomerations at a rational expectations equilibrium and the social optimum of dynamic economic systems. The possibility of a potential agglomeration at a rational expectations equilibrium is related to the incomplete internalization of the spatial externality by optimizing agents, while the "no agglomerations" result at the social optmum stems from the full internalization of the spatial externality by a social planner and the strict concavity of the production function.

The paper is organized as follows: Section 2 introduces the model and Section 3 characterizes equilibria with spatial spillovers. Sections 4 and 5 provide global and local analysis for the emergence (or not) of optimal potential agglomerations while Section 6 presents a detailed analytic and numerical example. Section 7 discusses intuition, shows how our methods can be used to study generalizations to spatial domains of similarly structured, economic problems - in this case the well known investment problem of the firm with adjustment costs - and outlines other ways in which the present paper can be extended. So as not to disrupt the flow of the presentation, all proofs are contained in Section 8 which serves as an Appendix.

2 Geographical Spillovers in Forward-Looking Optimizing Economies

Consider a spatial economy occupying a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. It is worth noting that space may be considered as either geographical (physical) space or as economic space (space of attributes related to economic quantities of interest). Without loss of generality we may assume $d = 1$.

Capital stock is assumed to be a scalar quantity that evolves in time and depends on the particular point z of the domain $\mathcal{O}$ under consideration. Thus capital is described as a function of time t and space z, i.e. $x : I \times \mathcal{O} \to$

$\mathbb{R}$ where $I = (0, T)$ is the time interval over which the temporal evolution of the phenomenon takes place. We assume an infinite horizon model, i.e. $I = \mathbb{R}_+$, and denote the capital stock at point $z \in \mathcal{O}$ at time t by $x\,(t, z)$. The spatial behavior of x is modelled by assuming that the functions $x(t, \cdot)$ belong for all t to an appropriately chosen function space $\mathbb{H}$. Therefore, as is common in the abstract theory of evolution equations, we assume that x is described by a vector-valued function $\breve{x} : I \to \mathbb{H}$, where $I = (0, T)$, and $\mathbb{H}$ is the function space that describes the spatial properties of the function x.[1] Different choices for $\mathbb{H}$ are possible. A convenient choice is to let $\mathbb{H}$ be a Hilbert space,[2] e.g., $\mathbb{H} = L^2(\mathcal{O})$, the space of square integrable functions on $\mathcal{O}$, or an appropriately chosen subspace, e.g. $L^2_{per}(\mathcal{O})$, the space of square integrable functions on $\mathcal{O} = [-L, L]$ satisfying periodic boundary conditions (this would model a circular economy).

Consumption is assumed to be a local procedure and modeled by a vector valued function $c : I \to \mathbb{H}$ interpreted in a similar fashion to the capital stock function x discussed above. By the scalar quantity $c(t, z)$ we denote consumption at time $t \in I$ at the spatial point $z \in \mathcal{O}$. Consumption is associated with a utility function $U : I \times \mathbb{H} \to \mathbb{R}$. The utility of consumption at time $t \in I$ and at point $z \in \mathcal{O}$ is given by $U(c(t, z))$.

Production at each location is determined by local inputs and by nonlocal procedures. At time t, output production at each location z is described by the production function f with inputs being capital $x(t, z)$ and labour $\ell(t, z)$, at this location, and also spatial effects describing the effect that capital stocks on locations $s \in \mathcal{O}$ at time t have on production at location z. Without loss of generality and to concentrate on the impact of geographical spillovers, we assume that labour input is normalized to unity $\ell(t, z) = 1$.

Spillover effects play a very important role in this study. We adopt the notation $\check{X}(t, z)$ for the spillover effects at time t on site z. Production at time t and site z is given by the production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in terms of $f(x(t, z), \check{X}(t, z))$.

[1] This function is defined such that $(\breve{x}(t))(z) := x(t, z)$; to avoid cumbersome notation in the sequel we denote the $\mathbb{H}$-valued function $\breve{x}$ using the same notation x and therefore by $x(t)$ we denote an element of $\mathbb{H}$, which is in fact a function $x(t) : \mathcal{O} \to \mathbb{R}$ which describes the spatial structure of the capital stock at time t.

[2] Other choices are possible, where $\mathbb{H}$ is a Banach space, e.g., $\mathbb{H} = C(\mathcal{O})$ the set of continuous functions on $\mathcal{O}$ or $\mathbb{H} = L^p(\mathcal{O})$, $p \neq 2$, $1 \leq p \leq \infty$, the set of p-integrable functions on $\mathcal{O}$. In the present paper, we restrict our attention to Hilbert spaces, though many of our results may be extended to Banach space.

Clearly the modelling of spillover effects is crucial. We will adopt two alternative ways:

(a) as exogenously given $\check{X}(t,z) = X^e(t,z)$ where $X^e : I \to \mathbb{H}$ is a known function or

(b) as endogenously determined by the state of the system, i.e. $\check{X}(t,z) = (\mathsf{T}x)(t,z)$ where $\mathsf{T} : \mathbb{H} \to \mathbb{H}$ is a mapping (operator) taking the state of the system at time t, $x(t,.) \in \mathbb{H}$ and providing the spillovers $\check{X}(t,\cdot) \in \mathbb{H}$.

If we regard spillovers as the spatial externality case, (b) indicates internalization of the externality. When adopting modelling strategy (b), spillover effects at time t and site z are given by the intermediate quantity:

$$X(t,z) = \int_{\mathcal{O}} w(z,s)x(t,s)ds \tag{1}$$

where $w : \mathcal{O}\times\mathcal{O} \to \mathbb{R}$ is an integrable kernel function modeling the effect that position s has on position z. This introduces nonlocal (spatial) effects, and may be understood as defining a mapping which takes an element $x(t,\cdot) \in \mathbb{H}$ and maps it to a new element $X(t,\cdot) \in \mathbb{H}$ such that (1) holds for every $z \in \mathcal{O}$. This mapping is understood as an operator $\mathsf{T} : \mathbb{H} \to \mathbb{H}$.

Some comments are due on the interpretation of the intermediate variable X. The quantity $X(t,z)$ will have different interpretations in different contexts. If $X(t,z)$ represents a type of knowledge which is produced proportionately to capital usage, it is natural to assume that the kernel $w(\zeta), \zeta = z - s$ is single peaked bell-shaped, with a maximum at $\zeta = 0$, and with $w(\zeta) \to 0$ for sufficiently large ζ. If $X(t,z)$ reflects aggregate benefits of knowledge produced at (t,s) for producers at (t,z) and damages to production at (t,z) from usage of capital at (t,s), then non-monotonic shapes of w, with for example a single peak at $\zeta = 0$ and two local minima located symmetrically around $\zeta = 0$, with negative values indicating damages to production at z from usage of capital at s, are plausible. This production function could be considered as a spatial version of a neoclassical production function with Romer (1986) and Lucas (1988) externalities modelled by geographical spillovers given by a Krugman (1996), Chincarini and Asherie (2008) specification.

Let us now fix a time t and consider a site $z \in \mathcal{O}$. Let $x(t,z)$ be the capital stock at this site and $\check{X}(t,z)$ the spillover effects at site z from all the other sites in $\mathcal{O}$. We treat the site as analytically equivalent to an agent located on the site who has access to valuable technology $f\left(x(t,z), \check{X}(t,z)\right)$ that generates rents. The individual (or the site) has access to the world capital market and can borrow x at an exogenous interest rate $r(t)$ against the present value of future rents from operating $f\left(x(t,z), \check{X}(t,z)\right)$. The agent faces quadratic adjustment costs, $\frac{\alpha}{2}\left[\frac{\partial x(t,z)}{\partial t}\right]^2$, to adjusting the capital stock and experiences geographical spillovers $\check{X}$, while the capital stock depreciates at a fixed rate η. The instantaneous budget constraint facing the individual or the site, z, at t can therefore be written as:

$$c(t,z) + \frac{\partial x(t,z)}{\partial t} = f\left(x(t,z), \check{X}(t,z)\right) - \eta x(t,z) - \frac{\alpha}{2}\left[\frac{\partial x(t,z)}{\partial t}\right]^2. \quad (2)$$

Assuming for simplicity constant r, the lifetime budget constraint expressed in present discounted value form for the agent is obtained if (2) is multiplied by e^{-rt} , integrated over t from $t = 0$ to ∞ (assuming momentarily the existence of exponentially bounded solutions of (2)) and all debts are required to be paid off.[3] Defining[4]

$$\lambda = r + \eta, \quad u(t,z) = \frac{\partial x}{\partial t}(t,z) = x'(t,z)$$

leads to a reformulation of the instantaneous budget constraint in a static

[3]We assume that the agent (or the site) has discounted future income greater than any desired borrowing at any point in time. Thus the capitalized, at the rate r, sum of the site's future income is large enough to pay off the debt incurred by borrowing. To put it differently, we assume that each site z has enough capital so that it is "solvent" in the present value sense at each point in time. If for example initial capital is zero and initial bonds are zero, then the solvency condition is obtained by multiplying both sides of (2) by e^{-rt} and using $x(0,z) = b(0,z) = 0$, as:

$$\int_0^\infty e^{-rt}\left[f\left(x(t,z), \check{X}(t,z)\right) - (r+\eta)\,x(t,z) - \frac{a}{2}\frac{d^2x}{dt^2}(t,z)\right] dt \geq 0$$

where $b(t,z)$ is "bonds" held by z at time t, $b(t,z) < 0$ is debt, and $b(t,z) > 0$ is assets.

[4]By $'$ we denote the derivative with respect to time of the Hilbert space valued function $x : I \to \mathbb{H}$.

form as:

$$0 = \mathcal{C}(z) := \qquad (3)$$
$$\int_0^\infty e^{-rt}[x_0 + f\left(x(t,z), \check{X}(t,z)\right) - \lambda x(t,z) - c(t,z) - \frac{\alpha}{2}u^2(t,z)]dt$$

which holds a.e. in $\mathcal{O}$. The same constraint over the whole domain $\mathcal{O}$ takes the form:

$$0 = \mathcal{C}^\Diamond := \qquad (4)$$
$$\int_{\mathcal{O}}\int_0^\infty e^{-rt}[x_0 + f\left(x(t,z), \check{X}(t,z)\right) - \lambda x(t,z) - c(t,z) - \frac{\alpha}{2}u^2(t,z)]dtdz.$$

To summarize, in this model the use or production of capital on a site affects other sites through the geographical spillovers, while a site can borrow or lend capital using the world capital markets. Depending on the type of the agent we can specify $\check{X}$ accordingly. An individual located at z treats geographical spillovers as parametric and exogenously given, $\check{X}(t,z) = X^e\,(t,z)$, while a social planner fully internalizes geographical spillovers so that $\check{X}(t,z) = X(t,z) = \int_{\mathcal{O}} w(z-s)x(t,s)ds$.

3 Equilibria with Geographical Spillovers

3.1 Rational expectations and social optimum equilibria

The objective is to maximize the utility of consumption either locally or globally. Both cases are considered in this work: the maximization of local consumption when spillovers are exogenous will be called a rational expectations (RE) problem, while the maximization of global utility with endogenous spillovers will be called a social optimum (SO) problem.

Given the (local) utility function U we now define the functionals $J_{RE} : \mathbb{H} \to \mathbb{R}$ and $J_{SO} : \mathbb{H} \to \mathbb{R}$ whose action on the consumption function c is as follows:

$$(J_{RE}(c))(z) := \int_0^\infty e^{-\rho t}U(c(t,z))dt, \qquad (5)$$
$$J_{SO}(c) := \int_{\mathcal{O}} \psi(z)(J_s c)(z)dz = \int_{\mathcal{O}}\int_0^\infty e^{-\rho t}\psi(z)U(c(t,z))dtdz. \qquad (6)$$

The functional J_{RE} provides the discounted - by a subjective utility discount rate $\rho > 0$ - utility of consumption $c(t,z)$ in the infinite horizon at location z. On the other hand, the functional J_{SO} provides the discounted utility of consumption averaged over the whole domain $\mathcal{O}$, with a weight function ψ which will be set to one without loss of generality.

We are now in a position to define the two optimization problems faced by either an arbitrary representative agent at location z (RE problem) or a social planner (SO problem).

Definition 1 (RE and SO problems)**.**

$$\text{RE problem: } \max_{c\in\mathcal{A}} J_{RE} \quad \text{subject to (3) with } \check{X}(t,z) = X^e(t,z) \tag{7}$$

$$\text{SO problem: } \max_{c\in\mathcal{A}} J_{SO} \quad \text{subject to (4) with } \check{X}(t,z) = X(t,z) \tag{8}$$

where J_{RE} and J_{SO} are the functionals defined in (5) and (6) respectively and $\mathcal{A}$ is the acceptable consumption set (typically $c(t,z) \geq 0$ a.e. in $\mathcal{O}$ would suffice).

3.2 Standing assumptions

In developing our model we make the four assumptions below, which will be assumed to hold through the paper unless explicitly stated otherwise. Note that some of these assumptions can be relaxed considerably for some of our results. To simplify the exposition, we assume the stronger conditions that guarantee that all of our results hold uniformly, and we make specific remarks concerning the possibility of relaxing them in the particular cases where this is feasible.

When RE equilibria are concerned, we need to make an assumption on X^e.

Assumption 1. The exogenously given spillover function $X^e \in \mathbb{H}$.

We make the following assumptions on the primitives of the economy.

Assumption 2. Assume that

(a) The influence kernel function $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is continuous and symmetric, i.e. $w(z,s) = w(s,z) = w(z-s)$.

(b) The production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly increasing, strictly concave function of the (real) variables (x, X).

(c) The utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave C^2 function in consumption c and satisfies the Inada conditions[5]

$$\lim_{c\to 0} \partial_c U(c) = +\infty, \quad \lim_{c\to +\infty} \partial_c U(c) = 0.$$

Under Assumption 2(a), equation (1) defines an integral operator $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ whose action on a function x is defined as:

$$(\mathsf{K}x)(t,z) := \int_{\mathcal{O}} w(z-s)x(t,s)ds. \tag{9}$$

Since $\mathcal{O}$ is a bounded domain, the continuity assumption leads us to the result that $w \in L^2(\mathcal{O})$ so that by standard results in the theory of integral operators, K is a compact bounded operator which, furthermore, by the symmetry of the kernel function w, is a self-adjoint operator.

We impose the following smoothness assumptions on the production function.

Assumption 3.

(a) The production function is a C^2 function of the (real) variables (x, X) such that f_{xX}, f_{XX} are uniformly bounded below in $\mathcal{O}$,

$$-\mu := \inf_{(x,X)\in\mathbb{R}^2} f_{XX}, \quad \xi := \inf_{(x,X)\in\mathbb{R}^2} f_{xX}, \quad \mu, \xi \in \mathbb{R}_+.$$

(b) Furthermore, it holds that

$$\lim_{(x,X)\to(0,0)} f_x(x,X) > C, \text{ and } \lim_{(x,X)\to(0,0)} f_X(x,X) > C,$$

for a positive constant C.

The positive constant C in Assumption 3(b) will be chosen typically larger than λ (in the RE case, or a multiple of that depending on the choice of the kernel w in the SO case). This will be needed to guarantee the existence of steady-state solutions (see Theorem 2 and its proof). Assumption 3(b) holds for typical production functions, e.g., for the Cobb-Douglas production function.[6]

[5] $\partial_u \phi$, $\partial_{uv}\phi$ denote first and second order partial derivatives of a function ϕ, with respect to variables u, v.

[6] In fact for the Cobb-Douglas, these limits are infinite.

Finally, we impose a positivity assumption on the spillover operator K. This assumption is needed for the monotonicity results that are important in the study of the global behaviour of the system. This assumption is not imposed in Section 5, where the local behaviour is studied.

Assumption 4.

(a) The operator $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ is strictly positive.[7]

(b) It holds that $\mu/\xi < \mu_1$ where μ_1 is the largest (positive) eigenvalue of operator K.

The economic interpretation of positivity is that spatial spillovers have overall positive effects. This observation stems from the interpretation of the inner product $(\mathsf{K}x, x)$ as the total (average) spillovers over the whole domain $\mathcal{O}$. Note that the positivity of the operator does not rule out the possibility of a negative spillover effect locally. In fact the kernel function may also assume negative values locally, but the overall (average over the whole domain) spillover effect will have to be positive. The positivity of the operator K is related to the positivity of its eigenvalues.

3.3 The rational expectations and social optimum equilibria

The RE and SO equilibrium problems, (7) and (8) respectively, can be reformulated into a form which is more convenient to handle, using a generalization of the Fisher separation principle, for this infinite dimensional economy.

The optimization problem (7) can be broken down, by expressing the associated Lagrangian in a separable form, into two distinct but interrelated sub-problems: A problem corresponding to the choice of the agent's consumption, $c(t, z)$, to maximize discounted lifetime utility subject to a lifetime budget constraint; and a problem corresponding to the choice of the agent's investment, $u(s, z) = x^{'}(t, z)$, to maximize the agent's interests in the economy by maximizing the location's present value. This is essentially a generalization of the Fisher separation principle for a single-owner firm which implies that if the optimization problem (7) admits a solution

[7] K is a positive operator if $(\mathsf{K}h, h) \geq 0$ for all $h \in \mathbb{H}$, and strictly positive if furthermore $(\mathsf{K}h, h) = 0$ implies $h = 0$.

(c^*, x^*), then there exists a $\Lambda : \mathcal{O} \to \mathbb{R}_+$ such that the solution of this problem can be split into two separate problems:
(a) A consumption optimization problem which, upon choice of Λ, assumes the form

$$\max_{c(\cdot,z)} \int_0^\infty e^{-\rho t}(U(c(t,z) - \Lambda(z)e^{-rt}c(t,z))dt. \tag{10}$$

(b) An investment optimization problem, independent of the choice of Λ, according to which $x(t,z) = x_0(z) + \int_0^t u(s,z)dz$ is chosen so as to solve

$$\max_{x'(\cdot,z)} \int_0^\infty e^{-rt}\{f(x(t,z), X^e(t,x)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt, \tag{11}$$

with $\lambda = r + \eta$. As the following remark shows, it is reasonable to assume that $r = \rho$.

Remark 1. The first order necessary condition for problem (10) is $U'\left(c\left(t,z\right)\right) = e^{(\rho-r)t}\Lambda\left(z\right)$. Since $\Lambda(z)$ is positive and independent of time, we see that marginal utility goes to infinity, i.e. $c\left(t,z\right)$ goes to zero, if the individual discounts the future higher than r and vice versa if the consumer discounts the future less than r. Thus, if we want to study a steady state for $c(t,z)$, we can assume that the consumer discounts at the same rate as r.

Similarly an application of the Fisher separation to the social planner problem (8) shows that the solution of this problem may be obtained by the solution of two separate problems:
(a) A consumption optimization problem which upon choice of $\Lambda^\Diamond \in \mathbb{R}_+$ independent of z assumes the form

$$\max_c \int_0^\infty \int_\mathcal{O} e^{-r\,t}\left(\psi(z)U(c(t,z)) - \Lambda^\Diamond c(t,z)\right) dz\, dt. \tag{12}$$

(b) An investment optimization problem, independent of the choice of $\Lambda^\Diamond$, where u is chosen so that $x(t,z) = x_0(z) + \int_0^t u(s,z)ds$ solves

$$\max_{x'} \int\limits_0^\infty \int\limits_\mathcal{O} e^{-r\,t}\left(f(x(t,z), (\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\, dt. \tag{13}$$

The above results can be easily shown, e.g., for the RE problem, by using the Lagrangian $\mathcal{L}\left(z\right) = \int_0^\infty e^{-\rho t}U\left(c\right)dt + \Lambda\left(z\right)\mathcal{C}\left(z\right).$

We note that in both cases only the solution of the second problems (11) and (13) respectively, which are independent of the choice of the Lagrange multiplier Λ, is required to characterize the spatial structure of the capital stock. This problem is essentially equivalent to a calculus of variations problem. At this point the following definition is required:

Definition 2 (The RE and SO problems)**.**

(i) RE problem:

$$\max_{x(\cdot,z)} \int_0^\infty e^{-rt}\{f(x(t,z),X^e(t,x) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt, \ \ \forall z \in \mathcal{O}. \tag{14}$$

(ii) SO problem:

$$\max_x \int_0^\infty \int_{\mathcal{O}} e^{-r\,t}\left(f(x(t,z),(\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\, dt. \tag{15}$$

Note that the RE problem is a calculus of variations problem where for each fixed $z \in \mathcal{O}$ we find a function $x(\cdot,z) : \mathbb{R}_+ \to \mathbb{R}$ that maximizes the functional

$$J_{RE}(x(\cdot,z);z) := \int_0^\infty e^{-rt}\{f(x(t,z),X^e(t,x)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt.$$

On the other hand the SO problem is a calculus of variations problem where we find a vector valued function $x : \mathbb{R}_+ \to \mathbb{H}$ that maximizes the functional

$$J_{SO}(x(\cdot)) := \int_0^\infty \int_{\mathcal{O}} e^{-r\,t}\left(f(x(t,z),(\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\, dt.$$

3.4 Existence of equilibria and first order conditions

We now discuss the existence of RE and SO equilibria. The following operators will be needed.

Definition 3. Define the nonlinear operators $A_\nu : \mathbb{H} \to \mathbb{H}$, $\mathsf{A}_\nu : \mathbb{H} \to \mathbb{H}$,

$\nu = RE, SO$, by

$$A_{RE}x := -\alpha^{-1}(f_x(x,\check{X}) - \lambda), \;\; \check{X} = X^e,$$
$$A_{SO}x := -\alpha^{-1}(f_x(x,\check{X}) + \mathsf{K}f_X(x,\check{X}) - \lambda), \;\; \check{X} = \mathsf{K}x$$

and

$$\mathsf{A}_{RE}x := -\alpha^{-1}(f_x(x,\check{X}) - \lambda), \;\; \check{X} = \mathsf{K}x,$$
$$\mathsf{A}_{SO}x := -\alpha^{-1}(f_x(x,\check{X}) + \mathsf{K}f_X(x,\check{X}) - \lambda), \;\; \check{X} = \mathsf{K}x.$$

Note that the operators A_{SO} and A_{SO} coincide, but we include both for notational consistency.

Theorem 1.

(a) *The optimization problems (14) and* (15) *admit a solution.*

(b) *The first order necessary condition for problems (14) and* (15) *is of the form*

$$x'' - rx' - A_\nu x = 0, \quad \nu = RE, SO \tag{16}$$

where A_ν are the nonlinear operators of Definition 3. The first order necessary conditions have to be complemented with the transversality condition

$$\lim_{t\to\infty} e^{-rt}xx' = \lim_{t\to\infty} \frac{1}{2}e^{-rt}(x^2)' = 0. \tag{17}$$

Remark 2. This theorem does not require the positivity Assumption 4 on K. The existence part of the theorem (claim (a)) requires only Assumption 2 (and Assumption 1 as well for the case $\nu = RE$) along with a mild growth condition on the production function to ensure that the supremum is finite. The extra smoothness of the data imposed by Assumption 3 is required for the first order conditions (Euler-Lagrange) to hold (claim (b)). Furthermore, if the maximization is performed on a close convex subspace of $\mathbb{H}$ then the first order condition (16) must be replaced by a variational inequality.

Remark 3. An alternative would be to use the maximum principle, in terms of the current value Hamiltonian $\mathcal{H}_\nu$, $\nu = RE, SO$ where

$$\mathcal{H}_{RE} := f(x, \check{X}) - \lambda x - \frac{\alpha}{2}u^2 + p\,u, \ \ \check{X} = X^e, \tag{18}$$

$$\mathcal{H}_{SO} := \int_{\mathcal{O}} (f(x, \check{X}) - \lambda x - \frac{\alpha}{2}u^2 + p\,u) dz, \ \ \check{X} = \mathsf{K}x. \tag{19}$$

Applying the Pontryagin maximum principle formally and maximizing over u, the Hamiltonian equations are easily seen to be equivalent to (16).

The form of the first order conditions (16) motivates the following definition:

Definition 4 (RE and SO equilibrium)**.** A solution $x : I \to \mathbb{H}$, if it exists, of the nonlinear integro-differential equation

$$x'' - rx' - \mathsf{A}_\nu x = 0 \tag{20}$$

is called an RE equilibrium if $\nu = RE$ and an SO equilibrium if $\nu = SO$.

Remark 4. Note that in the RE equilibrium we use the operator A_{RE} rather than the operator A_{RE}. This means that the agent makes her decision locally using $\check{X} = X^e$ but her decision changes the background spillovers to $\check{X} = \mathsf{K}x$.

4 Optimal Agglomerations in the Long Run: Global Analysis

Having defined the RE equilibrium and the SO equilibrium in the context of geographical spillovers, we turn to the study of the long-run characteristics of these equilibria. These characteristics will provide information about the potential emergence of optimal agglomerations as long-run equilibria, as well as information about potential differences in the long run between the RE equilibrium and the SO. Analyzing these issues requires global analysis to study existence, uniqueness and stability regarding both types of equilibria.

The long-term behavior of the system will be provided by the steady-state solutions of the equilibrium equations given in Definition 4. These are solutions without any temporal variability, and are given as solutions of the nonlinear operator equations $\mathsf{A}_\nu x = 0$, $\nu = RE, SO$. In general, the solution of this operator equation presents spatial variability, i.e., $x = x(z)$, but if operator K has the following property (which we will henceforth refer to as Property $\mathcal{P}$),

$$K\bar{x} \text{ is independent of } z, \text{ if } \bar{x} \text{ is independent of } z, \qquad (\mathcal{P})$$

then the solution of $\mathsf{A}_\nu x = 0$ may be uniform in space. We will call such an equilibrium a flat equilibrium. Property $\mathcal{P}$ holds in the case of periodic boundary conditions (see Proposition 4) so that a flat steady state always exists when periodic boundary conditions are taken into account. The flat steady state is the solution of an algebraic nonlinear equation.

If a flat steady state exists, it is globally stable and if, furthermore, the steady states of the system are unique, then the dynamics of the system preclude the emergence of spatially varying solutions in the long run; therefore, they preclude the emergence of spatial pattern formation in the economy. We will refer to such patterns as potential "agglomeration patterns" to be in accordance with the terminology of economic geography. Furthermore, since such spatial patterns may occur as a result of optimizing behaviour, we will henceforth refer to them as "optimal agglomeration".

The following theorem provides important information on the long-run dynamics.

Theorem 2.

(a) The operator equations $\mathsf{A}_\nu x = 0$, $\nu = RE, SO$, have unique solutions.

(b) All bounded solutions of $x'' - rx' - \mathsf{A}_\nu x = 0$ have as weak limit the solution of $\mathsf{A}_\nu x = 0$, $\nu = RE, SO$.

Remark 5. The results of Theorem 2 hold for SO without Assumption 4. Assumption 4 is a sufficient condition for Theorem 2 to hold in the RE case. Therefore, convergence to the RE steady state depends on the strength of diminishing returns with respect to spatial spillovers (f_{XX}), the strength of the complementarity between the capital stock and spatial spillovers in the production function (f_{xX}), and the structure of the spatial domain as reflected in the largest eigenvalue of K. Furthermore, relaxing the monotonicity assumption on operator K, there may exist multiple solutions for the RE steady-state equation for appropriate values of λ.

Remark 6. We only consider solutions of (20) that satisfy the condition that $\sup_{t\in\mathbb{R}_+} ||x(t)|| < \infty$. This is a very reasonable restriction; since (20) arises from an optimal control problem, it exhibits a saddle-point-like structure and the boundedness condition restricts us on the bounded manifold, i.e. the manifold of the bounded solutions, which is the class of solutions of interest to economic theory.

Example 1. To provide an illustration of what these operator equations look like, let us consider as an example the case $\nu = RE$ under the simplified assumption that the production function is separable $f(x, X) = F_1(x)F_2(X)$. We also assume that operator K (as well as the production function f) satisfies the standing assumptions employed in this work.

In this special case this operator equation can be transformed to a Hammerstein type nonlinear integral equation. By straightforward algebra we see that $\mathsf{A}_{RE}x = 0$ is equivalent to the integral equation $\mathsf{K}x = \Phi_2(\frac{\lambda}{F_{1,x}(x)})$ where Φ_2 is the inverse function of F_2. In terms of the new variable $y = \Phi_2(\frac{\lambda}{F_{1,x}(x)})$, this assumes the Hammerstein form $\mathsf{K}N(y) = y$, in terms of the nonlinear function N, defined by $N(y) := \phi_1(\frac{\lambda}{F_2(y)})$, where ϕ_1 is the inverse of the function $F_{1,x}$. By the properties of the production function it can be seen that N is non-decreasing; therefore by Li et al. (2006) it admits a unique solution.

An application of Theorem 2 in the periodic boundary conditions case allows us to rule out the emergence of optimal agglomeration in the SO case or for certain parameter values in the RE case. In particular,

Proposition 1. *Assume periodic boundary conditions, and assume that a flat steady state $\bar{x}$ exists and define*

$$s_{11} := \alpha^{-1}\partial^2_{xx} f(\bar{x}, \mathsf{K}\bar{x}), \;\; s_{22} := \alpha^{-1}\partial^2_{XX} f(\bar{x}, \mathsf{K}\bar{x}), \;\; s_{12} := \alpha^{-1}\partial^2_{xX} f(\bar{x}, \mathsf{K}\bar{x}).$$

(a) If $\mu/\xi < \mu_1 < |s_{11}|/s_{12}$ holds, then no agglomeration patterns will arise in the fully nonlinear RE equilibrium.

(b) No agglomeration will arise in the fully nonlinear SO equilibrium.

At this point we summarize and comment upon our global results as stated in the above theorem and proposition, focusing on their economic meaning.

For strictly concave production functions f, if the steady state equation $\mathsf{A}_{SO}x = 0$ admits a flat solution then all bounded solutions of the time dependent system will finally tend weakly to that flat solution as $t \to \infty$. Thus agglomeration is not a socially optimal outcome in this case. The uniqueness of the solution of $\mathsf{A}_{SO}x = 0$ precludes the existence of any steady state other than the flat steady state as long as total spillover effects are the same across all sites of the spatial domain. Then the socially optimal spatial distribution of economic activity is the uniform distribution in space. This is always true in the case of periodic boundary conditions, when α is independent of z. This result is a generalization of classical turnpike theory to infinite dimensional spatial models.[8]

When the long-run behavior of the RE and the SO are compared, we note that:

(i) Convergence to the RE steady state is not guaranteed by the strict concavity of the production function, as in the SO case, but depends, according to Theorem 2(a), on the relation between diminishing returns, complementarities, and the spatial geometry;

(ii) If a unique globally stable RE steady state exists it will be flat. Hence for both the RE and SO the unique steady state is the flat steady state.[9]

(iii) If the conditions of Theorem 2(a) are not satisfied, a more complex behavior is expected in the RE equilibrium. In this case, multiple RE steady states cannot be eliminated, and a potential agglomeration at the RE equilibrium takes the form of instability of the flat steady state.[10]

Therefore to study the emergence of agglomeration at the RE in terms of the stability of a RE equilibrium flat steady state, we turn to local analysis.

[8] It should be noted that we obtain global asymptotic stability results for the SO, which are independent of r, whereas some results in classical turnpike theory obtain global asymptotic stability when r is close enough to zero.

[9] The RE and SO steady states will in general be different from each other, which calls for spatially dependent economic policy if the SO steady state is to be attained.

[10] Note that we work on a symmetric space throughout so that any agglomeration that arises is due to endogenous forces, not to any kind of exogenous imposition of structure on the problem.

5 Agglomeration Emergence and Local Spillover Induced Instability

In the event that a flat steady state $\bar{x}$ exists, and the conditions of Theorem 2(a) are not satisfied at the RE equilibrium (in particular Assumption 4 is waived), the emergence of potential agglomeration may be studied in terms of the linear stability of $\bar{x}$ with respect to spatially inhomogeneous perturbations. The question we seek to answer is what will happen to an initial condition of the form $x(0,z) = \bar{x} + \epsilon\hat{x}(0,z)$, for ϵ small, under the action of the dynamical system $x'' - rx' - \mathsf{A}_\nu x = 0$? Since we are interested in local results, in this section we do not necessarily assume that Assumption 4 holds, unless explicitly stated.

If the initial spatial inhomogeneity is suppressed in time, then the flat steady state is stable, thus leading to a new flat steady state and no agglomeration. If it grows in time, then this corresponds to a precursor instability, which may lead to pattern formation and agglomeration in the long run. We call this type of instability, which in the context of economic geography may lead to equilibrium agglomeration, *optimal spillover induced instability.*

The linearized stability of the flat equilibrium is determined by the spectral theory of the following linear operators:

Definition 5. For a flat steady state $\bar{x}$, let

$$s_{11} := \alpha^{-1}\partial^2_{xx} f(\bar{x}, \mathsf{K}\bar{x}), \;\; s_{22} := \alpha^{-1}\partial^2_{XX} f(\bar{x}, \mathsf{K}\bar{x}), \;\; s_{12} := \alpha^{-1}\partial^2_{xX} f(\bar{x}, \mathsf{K}\bar{x}) > 0,$$

and define the linear bounded operators $\mathsf{L}_\nu : \mathbb{H} \to \mathbb{H}$ by[11]

$$\begin{aligned} \mathsf{L}_{RE}\hat{x} &:= s_{11}\hat{x} + s_{12}\mathsf{K}\hat{x} \\ \mathsf{L}_{SO}\hat{x} &:= s_{11}\hat{x} + 2s_{12}\mathsf{K}\hat{x} + s_{22}\mathsf{K}^2\hat{x}. \end{aligned}$$

A typical linearization argument shows that these operators govern the behaviour of spatio-temporal perturbations, $\hat{x}$, from the flat steady state $\bar{x}$: Inserting the ansatz $x = \bar{x} + \epsilon\hat{x}$ into (20) and expanding in ϵ, we obtain the linearized equation for the evolution of the perturbation $\hat{x}(t,z)$ as follows:

$$\hat{x}'' - r\hat{x}' + \mathsf{L}_\nu\hat{x} = 0, \quad \nu = RE,\, SO. \tag{21}$$

[11] By the standard theory of integral operators, K^2 is in turn an integral operator.

From the point of view of pattern formation, a spectral decomposition of (21) may provide detailed results concerning the onset and development of instability.

Proposition 2. *Let $\{\mu_j\}$ be the eigenvalues of operator K and $\{\phi_j\}$ the corresponding eigenfunctions. Then,*

(a) An arbitrary initial perturbation of the flat steady state of the form

$$\hat{x}(0,z) = \sum_j a_j \phi_j(z), \;\; \hat{x}'(0,z) = \sum_j b_j \phi_j(z),$$

evolves under the linearized system (21) to

$$\hat{x}_\nu(t,z) = \sum_j c_{\nu,j}(t)\phi_j(z)$$

where $\{c_{\nu,j}(t)\}$ is the solution of the countably infinite system of ordinary differential equations

$$\begin{aligned} & c''_{\nu,j} - r c'_{\nu,j} + \Lambda_{\nu,j} c_{\nu,j} = 0, \quad \nu = RE, SO, \;\; j \in \mathbb{N} \\ & c_{\nu,j}(0) = a_j, \;\; c'_{\nu,j}(0) = b_j \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Lambda_{RE,j} &= s_{11} + s_{12}\mu_j \\ \Lambda_{SO,j} &= s_{11} + 2s_{12}\mu_j + s_{22}\mu_j^2. \end{aligned}$$

(b) There are three possible types of dynamic behaviour, depending on the values of $\Lambda_{\nu,j}$:

1. *If $\Lambda_{\nu,j} < 0$, then $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ where $\sigma_1 < 0 < \frac{r}{2} < \sigma_2$ (saddle path behaviour).*
2. *If $0 < \Lambda_{\nu,j} < \left(\frac{r}{2}\right)^2$, then $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ where $0 < \sigma_1 < \frac{r}{2} < \sigma_2$ (unstable solutions).*
3. *If $\left(\frac{r}{2}\right)^2 < \Lambda_{\nu,j}$, then $c_j(t) = e^{\frac{r}{2}t}\left(\bar{A}_j \cos(\sigma t) + \bar{B}_j \sin(\sigma t)\right)$, $\sigma \in \mathbb{R}$*

and $\bar{A}_j$, $\bar{B}_j$ are constants related to the initial conditions.

The above general form of the linearized solutions clarifies possible pattern formation patterns that may arise from small perturbations of the flat steady state. Assume for example that for every mode ϕ_j we have that $\Lambda_{\nu,j} < 0$. Then, according to the standard saddle path arguments, the control procedure will lead the state of the system to the stable manifold and we will observe exponential decay of the initial perturbation $\hat{x}(0,z)$ to 0. That is, the system will return to the flat steady state. Therefore, in case $\Lambda_{\nu,j} < 0$ for all j, we do not expect spatially varying patterns in the long run. In all other cases, we lose the saddle path property in the linearized system. Such cases may destabilize the system and take it away from the flat equilibrium state $\bar{x}$. How this is done depends on the type of initial perturbation. If the initial perturbation contains modes for which $0 < \Lambda_{\nu,j} < \left(\frac{r}{2}\right)^2$, then for generic initial conditions we will have the linear combination of two (increasing) exponentials, one with rate larger than $r/2$ and one with rate smaller than $r/2$. Clearly such a general combination will not satisfy the transversality condition (17) and its general validity is doubtful. However, for particular initial conditions (such that $\bar{B}_j = 0$) the remaining part satisfies the transversality condition and will lead to pattern formation with a mechanism that resembles Turing instability. Note that for a randomly selected initial perturbation $\hat{x}(0,z)$, it is not expected that $\bar{B}_j = 0$ so this mechanism for pattern formation will lead to patterns as long as the initial perturbations from the flat steady state are carefully selected. One the other hand, if the initial perturbation contains modes j such that $\left(\frac{r}{2}\right)^2 < \Lambda_{\nu,j}$, then we obtain a spatio-temporal pattern which satisfies the transversality conditions (17) for any choice of initial conditions. Therefore, for "generic" initial perturbations from the flat steady state, we obtain patterns which are compatible with the transversality conditions and correspond to temporal growth accompanied with temporal oscillations. This can be compared to a Turing-Hopf-type pattern formation mechanism. We then obtain spatio-temporal growing, oscillatory patterns, which may correspond to the onset of spatio-temporal cyclic economic behaviour.

To summarize:

- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} < 0$ will die out and the system will converge to the flat steady state – no possible agglomeration is expected.

- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} > 0$ will turn unstable and lead to possible potential agglomeration spatial patterns, either monotone in time or oscillatory in time.

Remark 7. This instability can be contrasted with the celebrated Turing instability mechanism (Turing, 1952), which leads to pattern formation in biological and chemical systems. The important differences here are that: (a) in our model the instability is driven not by the action of the diffusion operator (which is a differential operator) but rather by a compact integral operator that models geographical spillovers, and (b) contrary to the spirit of the Turing model, here the instability is driven by optimizing behavior, so it is the outcome of forward-looking optimizing behavior by economic agents and not the result of reaction diffusion in chemical or biological agents. It is the optimizing nature of our model which dictates precisely the type of unstable modes which are "accepted" by the system, in the sense that they are compatible with the long-term behaviour imposed on the system by the policy maker. In some sense local spillover induced instability is a mixture of Turing- and Hopf-type instabilities.

We now turn to the comparison of stability of the RE and the SO flat equilibrium. A relevant question is under what conditions we might expect possible agglomeration to emerge.

Proposition 3.

(a) At the RE equilibrium, agglomeration is possible for the modes for which $\mu_j \geq -\frac{s_{11}}{s_{12}}$.

(b) At the SO equilibrium, since the production function is strictly concave, agglomeration is never possible.

Remark 8. The above proposition confirms and further clarifies the results of global analysis. The SO has a locally-stable flat steady state which by the global analysis is unique. Thus no agglomeration is possible at the SO with strictly concave production function. However, the flat steady state of the RE system can be locally unstable and this locally instability may induce agglomeration. Note that the local instability condition, $\mu_j \geq \frac{|s_{11}|}{s_{12}}$, is directly comparable with the "no agglomeration" condition of Proposition

1 of the global analysis, in the sense that if the global "no agglomeration" condition is satisfied, the local instability condition is not satisfied for all modes, and the flat RE equilibrium is locally stable and unique. On the other hand if the global "no agglomeration" condition is not satisfied, that is $\mu_1 = ||\mathsf{K}|| \geq \frac{|s_{11}|}{s_{12}}$, this means local instability of the flat RE equilibrium steady state by Proposition 3 (a), and the possibility of endogeous agglomeration emergence for the RE. Thus theorem 2 and Proposition 1 allow us to link the global nonlinear picture of the system with the linearized picture we obtained in this section using the Turing-type analysis.

Remark 9. Furthermore concerning the RE equilibrium, if $-\mathsf{K}$ is a positive operator (negative spillovers) then agglomeration is never possible, whereas if K is a positive operator (positive spillovers) then agglomeration is possible only if $\mu_1 = ||\mathsf{K}|| \geq \frac{|s_{11}|}{s_{12}}$. The economic intuition behind this is that agglomeration can occur if the spillover effects (as measured by the eigenvalues of operator K) are strong enough as compared to the ratio $\frac{|s_{11}|}{s_{12}}$ (which is determined by the production function and gives the relative strength of diminishing returns with respect to complementarity effects).

Remark 10. Note the important qualitative difference between the case considered in this paper (where K is a symmetric bounded compact operator) as compared to models commonly used in biology or chemistry - including Turing's own seminal contribution (where K is a symmetric unbounded operator, e.g., $\mathsf{K} = -\Delta$, the Laplacian). In the latter case the spectrum of operator K is unbounded ($|\mu_j| \to \infty$). This means that if K is an unbounded symmetric positive operator, there will always be a mode which turns unstable.

An important special case is that of periodic boundary conditions. In this case the eigenfunctions and the eigenvalues of operator K are obtained very easily in terms of the Fourier basis, thus leading to very general and easy to implement results. The main general results in this case are summarized in Proposition 4 (where the symmetry of the kernel is explicitly used). Proposition 4 is elementary and is only included here as a reminder, and for completeness of the paper.

Proposition 4 (Periodic Boundary Conditions)**.** *Assume periodic boundary conditions, i.e.,* $\mathbb{H} = L_{per}(\mathcal{O})$, $\mathcal{O} = [-L, L]$. *Then,*

(a) The eigenfunctions of operator K *are the Fourier modes* $\phi_n(z) = \cos(n\pi z/L)$*,* $n \in \mathbb{N}$ *with corresponding eigenvalues* $W_n = \int_{-L}^{L} w(z)\, \phi_n(z) dz$*.*
(b) The action of operator K *on a flat state returns a flat state,* $\mathsf{K}\bar{x} = \bar{x} \int_{-L}^{L} w(z) dz$*.*

Therefore, the onset of instability (i.e., the particular modes which are likely to become unstable) is determined by the Fourier expansion of the kernel w. For specific classes of kernels the calculation of the eigenvalues can be made explicit, thus leading to detailed results on the unstable modes and the shape of the patterns created near the onset of the instability (in the linear regime). The general result is that if certain modes are to become unstable, these will be the low modes, since W_n is expected to decay to 0 as $n \to \infty$.

6 An Illustrative Example: Cobb-Douglas-Type Production with Spillover Effects

In this section we provide an illustrative example of the general theory provided in this paper, using a Cobb-Douglas type production function with spillover effects. We assume periodic boundary conditions and for simplicity we assume the spillovers to be described by exponential kernels. The reason for this choice is twofold: First, kernels of this type were employed by Krugman (1996) in his modelling of spillover effects and second, this type of kernels allows for some closed form expressions for the spectrum of the integral operator K.

We illustrate our general theoretical results through a detailed analysis of the modes that may turn unstable, and some results for the evolution of possible agglomeration patterns obtained by numerical analysis of the system.

6.1 The primitives of the economy and the Euler-Lagrange equation

Consider a standard Cobb-Douglas production function

$$f(x, X) = C_0 x^a X^b, \;\; a + b < 1,$$

where C_0 is a constant. This production function takes into account local effects (modelled by the x contribution) and nonlocal effects (modelled by the X contribution) in the production.

The nonlocal effects are modelled by the integral operator K, defined by the composite exponential kernel function

$$w(z) = \sum_{i=1}^{N} C_i \exp(-\gamma_i |z|), \;\; \gamma_i \geq 0, \;\; C_i \in \mathbb{R}. \tag{23}$$

The coefficients γ_i give a measure of the spatial decay of spillover effects. The larger γ_i is, the faster the spillover effects are decaying as distance increases. These effects may be positive or negative depending on the sign of the corresponding coefficient C_i.

The operators A_{RE} and A_{SO} become

$$\begin{aligned}
&\mathsf{A}_{RE} := -\alpha^{-1} \left(a\, C_0\, x^{a-1}\, (\mathsf{K}x)^b - \lambda\right), \\
&\mathsf{A}_{SO} := -\alpha^{-1} \left(a\, C_0\, x^{a-1}\, (\mathsf{K}x)^b + bC_0 \mathsf{K}\left(x^a\, (\mathsf{K}x)^{b-1}\right) - \lambda\right)
\end{aligned}$$

and the steady-state equations are nonlinear integral equations.

Under the assumption of periodic boundary conditions, the action of operator K on a flat state $\bar{x}$ renders a flat state. Since $\mathsf{K}\bar{x} = W\bar{x}$ where W is known (see next section for a closed form expression), we have that the flat steady state for the RE is the solution of the nonlinear algebraic equation

$$aW^b \bar{x}^{a+b-1} - \bar{\lambda} = 0,$$

whereas for the SO it is the solution of

$$aW^b \bar{x}^{a+b-1} + bw^b \bar{x}^{a+b-1} - \bar{\lambda} = 0,$$

where $\bar{\lambda} = \frac{\lambda}{C_0}$. These immediately yield

$$\begin{aligned}
\bar{x}_{RE} &= \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}}, \\
\bar{x}_{SO} &= \left(\frac{\bar{\lambda}}{a+b}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}}.
\end{aligned}$$

Note that $\bar{x}_{SO} \geq \bar{x}_{RE}$ since $b > 0$ and $a + b < 1$. In the case where spillovers play no role ($b = 0$), these steady states coincide.

6.2 The spectrum of operator K

The spectrum of operator K, for periodic boundary conditions, can be calculated analytically, in closed form using Proposition 4.

Consider first the case where $N = 1$ in (23). Then, setting $\gamma_1 = \gamma$ and $C_1 = C$, a straightforward application of Proposition 4 shows that the eigenfunctions of K are the Fourier modes $\phi_n = \cos\left(\frac{n\pi z}{L}\right)$ with eigenvalues

$$\mu_n(\gamma) = W_n = 2C\frac{L^2\gamma}{n^2\pi^2 + \gamma^2 L^2}(1 - \mathsf{e}^{-\gamma L}\cos(n\pi)),$$

for $n \in \mathbb{N}$. Furthermore, $W = \mu_0 = W(0)$.

Consider now a composite kernel as in (23), for $N \neq 1$. Clearly K is a linear combination of operators generated by simple exponential kernels. Using the linearity of the operators it is clear that the Fourier modes ϕ_n, for $n \in \mathbb{N}$, are eigenfunctions of K with corresponding eigenvalue

$$M_n = \sum_{i=1}^{N} C_i \mu_n(\gamma_i), \tag{24}$$

and $W = M_0$.

6.3 Local spillover induced instability

As already mentioned, our general results preclude emergence of agglomerations in the SO case. However, agglomeration emergence, via local spillover induced instability, is possible under certain conditions for the RE problem.

The emergence of this instability is governed by the linearized system

$$x^{''} - rx^{'} + \mathsf{L}_{RE}x = 0, \tag{25}$$

with initial conditions $x(0, z)$, $x^{'}(0, z)$ where x now denotes a small perturbation from the flat steady state $\bar{x} := \bar{x}_{RE}$ and L_{RE} is the linear operator defined by

$$\mathsf{L}_{RE}x := s_{11}x + s_{12}\mathsf{K}x,$$

where s_{11}, s_{12} are the constants

$$s_{11} = a(a-1)C_0 W^b\,\bar{x}^{a+b-2},$$
$$s_{12} = a\,b\,C_0\,W^{b-1}\,\bar{x}^{a+b-2}.$$

The knowledge of the eigenvalues in combination with Proposition 2 allows us to determine unstable modes and draw some general conclusions.

Consider first the case where the kernel consists of a single exponential ($N = 1$) with $\gamma_1 = \gamma$ and $C_1 = C$. Then, the instability condition yields that a mode n is unstable if

$$\frac{a-1}{b} + \frac{L^2\gamma^2}{n^2\pi^2 + \gamma^2 L^2}(1 - e^{-\gamma L}\cos(n\pi)) \geq \frac{1}{ab}\left(\frac{r}{2}\right)^2 W^{-b}\bar{x}^{2-a-b}.$$

The expression $I(n) = \frac{L^2\gamma^2}{n^2\pi^2+\gamma^2L^2}(1 - e^{-\gamma L}\cos(n\pi))$ attains its maximum value for $n = 0$ so that

$$\frac{a-1}{b} + \frac{L^2\gamma^2}{n^2\pi^2 + \gamma^2 L^2}(1 - e^{-\gamma L}\cos(n\pi)) \leq \frac{a-1}{b} + 1 - e^{-\gamma L} \leq \frac{a+b-1}{b} \leq 0,$$

since $a + b < 1$ so that the instability condition is never satisfied and all modes are stable.

The situation is more interesting when composite kernel functions are taken into account. Consider for example a kernel function as in (23), with $N = 2$ and $\gamma_1 = 0.3$, $C_1 = 2$, $\gamma_2 = 0.1$, $C_2 = -0.75$. In this case the influence of neighboring locations on local state is a weighted average of the state at neighboring locations, but the influence from nearby locations is positive, while the influence is negative from relatively more distant locations. The kernel function chosen is similar to the one employed by Krugman (1996) in the modelling of a market potential function. The shape of the composite kernel function is shown in the left panel of Figure 1.

The operator K associated with the specific composite spillover is positive, so Assumption 4(a) is satisfied. Furthermore, the spectrum of this operator is available in closed form (see equation (24)) so that using the spectrum, we can readily check for the existence of unstable modes, for which spillover induced instability is possible. Since the spectrum depends on the parameters a, b and L in the right panel of Figure 1, we plot the number of unstable modes that emerge as a function of the ratio $\frac{1-a}{b}$ and the

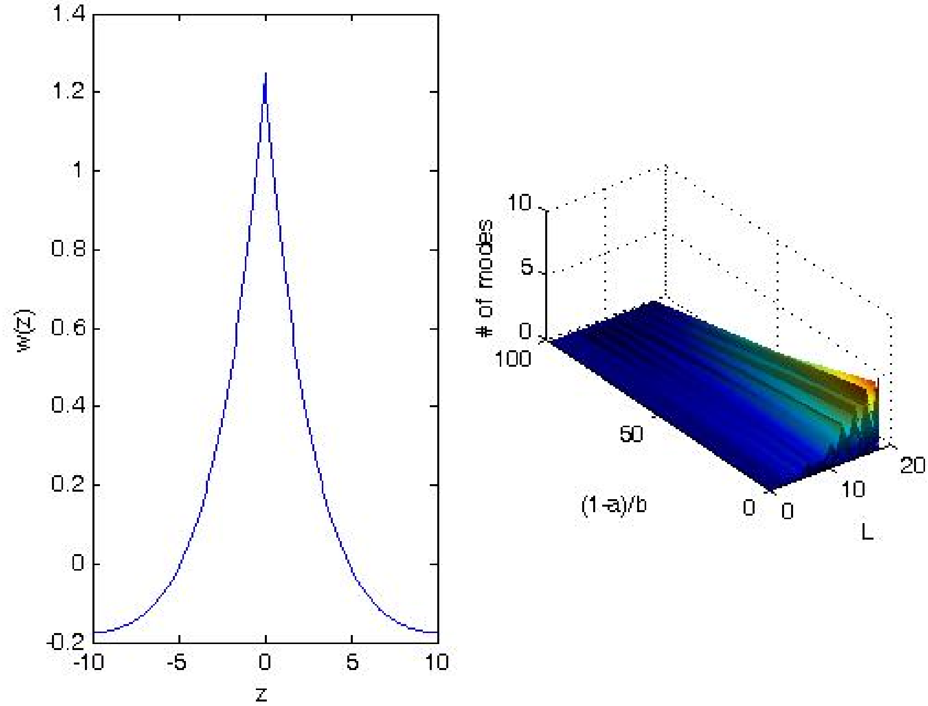

Figure 1: The shape of the composite kernel for $\gamma_1 = 0.3$, $C_1 = 2$, $\gamma_2 = 0.1$, $C_2 = -0.75$ (left panel). The number of unstable modes for this choice of kernel function, as a function of the parameter $\frac{1-a}{b}$ and L (right panel).

length of the domain L. We remark first that low modes may turn unstable (as a result of the monotonicity of the spectrum and its boundness; recall that K is a compact operator). That means that when, e.g., one mode is unstable, this is the first mode corresponding to $n = 1$, when two modes are unstable, this means that these are the modes $n = 1$ and $n = 2$, etc. One observes from our results that as L increases, more modes become unstable, as the increased length of the system can then accomodate more modes. This will lead to different spatial patterns that emerge as the length of the domain increases. The spatial complexity increases with the increase of L, and this is essentially a bifurcation phenomenon.

To display the wealth and variety of possible spatio-temporal agglomeration patterns that may emerge, we solve numerically the linearized equation (25) for the chosen kernel, and for the parameter values $a = 0.8$, $b = 0.1$, for various choices of L. We assume a small random perturbation from the flat steady state and we leave this perturbation evolving by solving (25), in order to determine the $\bar{x} + x(t, z)$ which is an approximation of the optimal path for the full system. The results are shown in Figure 2. In the first panel, where $L = 10$, no modes can become unstable and the perturbation dies out, and we revert to the flat steady state and no agglomeration occurs. In the second panel, $L = 15$ and according to our analysis only the first mode $n = 1$ is unstable. This leads to an agglomeration pattern whose spatial structure resembles that of the first eigenfunction of operator K. In the third panel, $L = 18$ and in this case two modes, $n = 1$ and $n = 2$, are unstable, a fact that leads to agglomeration patterns with spatial structure resembling linear combinations of the first two eigenfunctions of operator K. Finally, in the fourth panel, $L = 20$ and more modes become unstable leading to the occurence of more complex spatio-temporal agglomeration patterns.

6.4 The existence of patterns in the fully nonlinear model

We close this section by showing that the agglomeration patterns predicted by the linear stability analysis exist in the fully nonlinear case, by using a variational argument based on the mountain pass argument. We consider the steady-state equation $\mathsf{A}_{RE}x = 0$, which in this case is a nonlinear integral equation of the form

$$ax^{a-1} (\mathsf{K}x)^b = \bar{\lambda} \tag{26}$$

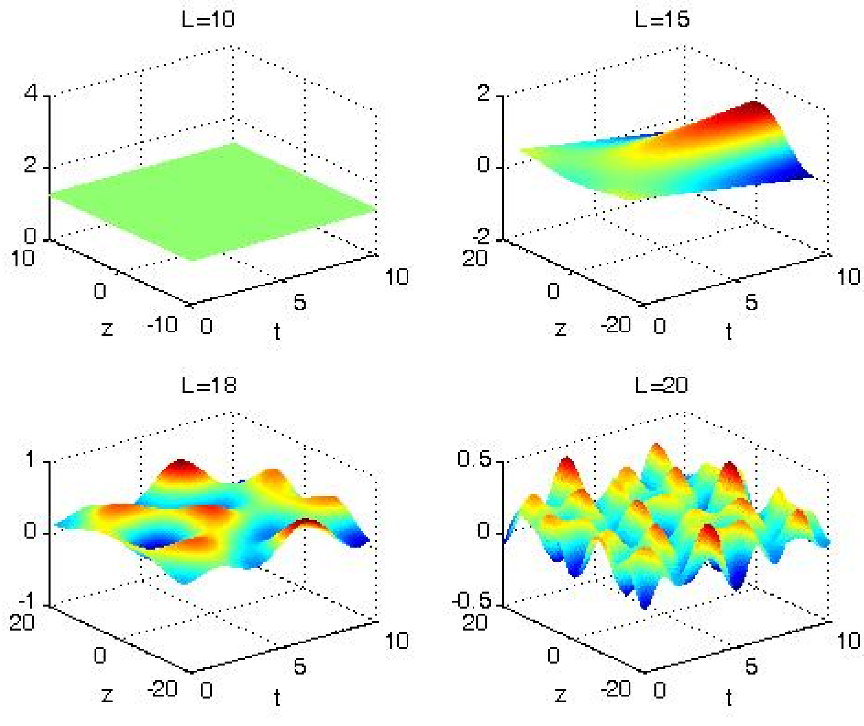

Figure 2: Emerging patterns from the instability for different domain sizes.

which, as we saw earlier on, has a unique flat steady-state solution

$$\bar{x}_{RE} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}}.$$

We solve (26) in terms of $X = \mathsf{K}x$ and rewrite this integral equation as

$$\mathsf{K}x = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} x^{\frac{1-a}{b}}.$$

We write $x = \bar{x}_{RE} + v$ and define the new variable

$$u = (v + \bar{x}_{RE})^{\frac{1-a}{b}}. \tag{27}$$

Using (27), (26)assumes the equivalent form

$$\mathsf{K}u^{\frac{b}{1-a}} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} u \tag{28}$$

which is in the standard form of a Hammerstein nonlinear integral equation. Since $a + b < 1$, we see that $\frac{b}{1-a} < 1$, so this is a sublinear Hammerstein equation. Clearly it admits the trivial solution $u = 0$ which is not acceptable on economic grounds since it leads to a flat steady state which is negative. Furthermore, it also admits a unique nontrivial flat solution $\bar{u}$ such that

$$\bar{u} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1-a}{b(a+b-1)}} W^{-\frac{1-a}{a+b-1}}$$

which by undoing the transformation of variables (27) is easily seen to coincide with $\bar{x}_{RE}$.

The question that arises is whether the Hammerstein equation (28) admits other solutions apart from these two. If it does, then this must be a non-flat solution, which corresponds to agglomeration. This is a fully nonlinear agglomeration pattern. By the linear analysis performed in the previous section, this fully nonlinear pattern must in the appropriate linear limit coincide with the patterns predicted by the linear stability analysis.

There is a well-developed theory concerning the solution of such Hammerstein equations. This theory, which is based on the powerful techniques of critical point theory, allows detailed results on the number and nature of nontrivial solutions. The following proposition provides an answer to the question we have set.

Proposition 5. *There exists a* Λ_* *such that for* $\left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} > \Lambda_*$ *the Hammerstein equation (28) admits at least two nontrivial solutions. Since one of these is the unique flat steady state* $\bar{u}$*, the other corresponds to a nonlinear agglomeration pattern.*

The proof of the proposition follows the proof of Theorem 7 of Faraci and Moroz (2003) with minor modifications and is omitted for the sake of brevity. The critical value Λ_* can be estimated in terms of the primitives of the problem. A full account of the nonlinear bifurcation theory for the steady states is beyond the scope of the present paper and will be addressed in future work.

7 Discussion, Extensions and Concluding Remarks

This paper develops a fairly general approach to the study of infinite dimensional, infinite horizon, intertemporal recursive dynamic optimization models in continuous spatial settings. Using the theory of maximal monotone operators for global analysis and spectral theory of compact operators for local analysis, we studied the spatiotemporal long-run behavior of the rational expectations equilibrium and the social optimum associated with a Ramsey-Fisher-type optimal growth model. We show, in the context of global analysis, that strong concavity of the production function implies convergence of the SO to a unique flat steady state, while similar convergence of the RE equilibrium requires stronger conditions. The possibility of a potential agglomeration at the RE equilibrium induced by instability of the flat steady state led us to local analysis. In the local analysis, we derived conditions for local stability and spillover-induced instability associated with the RE equilibrium which could signal agglomeration emergence. Since both our local and global conditions depend on primitives such as the strength of diminishing returns and complementarities in the production function, the characteristics of the spatial spillovers and the spatial geometry, our results are easily interpretable and potentially testable.

Our global results can be associated with general turnpike theorems in infinite dimensional spaces, which means that they allow the study of the long-run behavior of dynamic systems that include explicit spatial interactions - e.g. spatial spillovers - among economic agents.[12] The global asymptotic stability of the SO, obtained by using monotone operator theory, can be related to global asymptotic stability results obtained by Scheinkman (1978) about the stability of separable Hamiltonians in finite dimensional settings. From (18) and (19) we can write our problem in terms of separable Hamiltonians as:

$$
\begin{aligned}
&\mathcal{H}_\nu(p, x, \check{X}) = \mathcal{H}^1_\nu(p) + \mathcal{H}^2_\nu(x, \check{X}) \ , \nu = RE, SO \\
&\mathcal{H}^1_{RE}(p) := \max_{x'} \left\{ -\frac{\alpha}{2} \left(x'\right)^2 + p\, x' \right\}, \\
&\mathcal{H}^1_{SO}(p) := \max_{x'} \left\{ \int_{\mathcal{O}} \left[-\frac{\alpha}{2} \left(x'\right)^2 + p\, x' \right] dz \right\} \\
&\mathcal{H}^2_{RE}(x, \check{X}) := f(x, X^e) - \lambda x, \\
&\mathcal{H}^2_{SO}(x, \check{X}) := \int_{\mathcal{O}} \left[(f(x, \mathsf{K}x) - \lambda x \right] dz.
\end{aligned}
$$

This implies that the global long-run behavior of infinite dimensional optimization problems that give rise to Hamiltonians which are separable in the above sense, can be analyzed using the theory of monotone operators developed in this paper, while local results can be obtained using spectral theory.

For example, the well-known problem of investment theory of the firm with convex adjustment costs (Lucas 1967a,b), which has been analyzed in terms of separable Hamiltonians by Scheinkman (1978), can be extended using our methods in a spatial setting. In a simplified set-up, consider a large number of firms which occupy a spatial domain $\mathcal{O}$, sell a homogeneous output at an exogenous price, face quadratic adjustment costs with respect to net investment, and experience knowledge spillovers which generate spatial interactions among them. The RE and SO problems can be written as:

[12] An interesting paper by Boucekkine et al. (2010) studies optimal dynamic social welfare, e.g., an analog of our SO problem in an AK type model with a trade balance on a circular space where capital is mobile across space and growth occurs. It would be interesting to model the effect of introducing spillover externalities like ours as well as introducing growth and trends like Boucekkine et al. (2010) and comparing the solutions for SO and RE as we do in our paper. Unfortunately this is beyond the scope of the current paper.

$$\text{RE:} \quad \max_{x'} \int_0^\infty e^{-r\,t} \left(l f(x, X^e) - q\left(x' + \eta x\right) - \frac{\alpha}{2} (x^{'})^2 \right) \, dt$$

$$\text{SO:} \quad \max_{x'} \int_0^\infty \int_{\mathcal{O}} e^{-r\,t} \left(l f(x, \mathsf{K}x) - q\left(x' + \eta x\right) - \frac{\alpha}{2} (x^{'})^2 \right) dz \, dt,$$

where l is the exogenous output price, and q is the unit price of capital, both assumed independent of time. In this case the nonlinear operators of Definition 5 become

$$\mathsf{A}_{RE} x := -\alpha^{-1} (f_x(x, \check{X}) - q\,(\eta - r)), \quad \check{X} = \mathsf{K}x,$$
$$\mathsf{A}_{SO} x := -\alpha^{-1} (f_x(x, \check{X}) + \mathsf{K} f_X(x, \check{X}) - q\,(\eta - r)), \quad \check{X} = \mathsf{K}x.$$

and Theorem 1 suggests that the first order necessary condition for problems RE and SO are, respectively,

$$x^{''} - r x^{'} - \mathsf{A}_\nu x = 0, \quad \nu = RE, SO.$$

Therefore with a strictly concave production function, the flat steady state of SO is globally asyptotically stable, independent of the value of r, and no agglomeration is possible at the SO. On the other hand by Propositions 1 and 3, agglomeration might be possible at the RE.

The factor that could potentially differentiate the spatial structure of the SO and RE in the long run is the way in which the optimizing agent takes into account the spatial spillover. As seen from Propositions 1 and 3, the important quantity for characterizing RE is the ratio $|s_{11}| / s_{12}$. If this quantity is less than the largest eigenvalue of operator K, then agglomeration is possible. If the production function is separable in x and X, then $s_{12} = 0$ and no agglomeration is possible at the RE. For agglomeration to be a possibility, s_{12} should be sufficiently large in relation to the diminishing returns on x. In this case the optimizing agent treats the spillover X as exogenous and takes into account only its complementarity with x which is reflected in s_{12}. At the SO, however, the optimizing agent - e.g., the social planner - by treating X as endogenous, takes into account the diminishing returns of the spillover, in addition to the complementarity. With a strictly concave production function, this diminishing returns implies a maximal monotone operator A_{SO} and equivalently $\Lambda_{SO,j} < 0$ for all modes j in local analysis, and thus no agglomeration at the SO. Therefore it is the full internalization of the spatial externality that prevents the emergence of agglomerations.

We feel that the methods developed in this paper provide insights about the spatial structure of dynamic economic models that could provide a direct link between economic geography and optimal growth. Future research could be directed towards the further study of complexities underlying the RE equilibrium, the impact of increasing returns, and the explicit introduction of capital movement across space in pursuit of higher returns.

8 Appendix: Proofs

8.1 Proof of Theorem 1

We use the notation $F(x, X) = f(x, X) - \lambda x$ to rewrite the functional to be maximized as

$$J(x, x') := J_1(x) + J_2(x') := \int_0^\infty \int_{\mathcal{O}} e^{-rt} F(x, \mathsf{K}x) dz dt - \frac{\alpha}{2} \int_0^\infty \int_{\mathcal{O}} e^{-rt} (x')^2 dz dt, \tag{29}$$

where the explicit (t, z) dependence of x, X is omitted for simplicity.

To be in line with the standard theory of the calculus of variations, we consider the equivalent problem of minimizing the functional $\bar{J} = -J$. We also use the notation $\bar{J}_1(x) = -J_1(x)$, $\bar{J}_2(x') = -J_2(x')$ and

$$\bar{F}(x, X) = -F(x, X) = \lambda x - f(x, X).$$

Clearly, $\bar{F}$ is a strictly convex function.

Finally we use the notation $x_n \to x$ for strong convergence in $\mathbb{H}$ and $x_n \rightharpoonup x$ for weak convergence in $\mathbb{H}$.

The following lemma is needed.

Lemma 1. *The functional $\bar{J}_1 : \mathbb{H} \to \mathbb{R}$ is weakly lower semicontinuous and weakly coercive.*

Proof: Since

$$\bar{J}_1(x) = \int_{\mathcal{O}} (\lambda x - f(x, X)) dz, \quad \lambda > 0$$

it is clear that $(\bar{J}_1(x), x) \to \infty$ as $||x|| \to \infty$, hence $\bar{J}_1$ is weakly coercive.

For the weak lower semicontinuity we use Theorem 7.5 of Fonseca and Leoni (2007, p. 492) (see also Berkovitz, 1974). According to this theorem, let $g : \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and define the functional $\bar{\Phi}(u, v) := \int_{\mathcal{O}} g(z, u(z), v(z))dz$. Also, let $p, q \in [1, \infty)$ and assume that $g(z, u, v) \geq -C(|u|^p + |v|^q) - w(z)$, $C \geq 0$ and $w \in L^1(\mathcal{O})$. The functional $\bar{\Phi}$ is lower semicontinuous with respect to weak convergence of u in $L^p(\mathcal{O})$ and strong convergence[13] of v in $L^q(\mathcal{O})$, if and only if the following three properties hold:

(i) $u \mapsto g(z, u, v)$ is convex for all $z \in \mathcal{O}$ and for all $v \in \mathbb{R}$,

(ii) $g(z, u, v) \geq a(z) + b(x, v)u - c|v|^q$, $c > 0$ and $a \in L^1(\mathcal{O})$.

(iii) For any two sequences $\{v_n\}$ (converging weakly in $L^p(\mathcal{O})$) and $\{u_n\}$ (converging strongly in $L^q(\mathcal{O})$) and such that $\sup_n \Phi(u_n, v_n) < \infty$, then the sequence $|b(\cdot, v_n(\cdot))|^{p'}$ where p' is the conjugate exponent of p, is equi-integrable.

We apply this theorem for $p = q = 2$ and $u = x$, $v = X = \mathsf{K}x$. Then if $g(z, u, v) = -\bar{F}(u, v)$, we observe that $\Phi(u, v) = \bar{J}_1(x)$. Clearly, g satisfies the properties (i)-(iii) by the properties of f (for property (ii) recall that any concave function is bounded above by an affine function).

Consider a sequence $u_n = x_n$ converging weakly in $\mathbb{H} := L^2(\mathcal{O})$; $x_n \rightharpoonup x$. Then, since $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ is a compact operator, there exists a subsequence of $u_n = x_n$ such that $v_n := \mathsf{K}x_n$ converges strongly. Then, an application of the abovementioned theorem yields the weak lower semicontinuity result. *QED*

Proof of Theorem 1: We only provide the proof for $\nu = SO$ as the RE case is similar.

(a) Consider a sequence (x_n, x'_n), $n \in \mathbb{N}$ such that $J(x_n, x'_n) \to M$ where $M = \sup J(x, x')$. Clearly this is a minimizing sequence for $\bar{J}$. The real valued sequence $J(x_n, x'_n)$ is bounded, so that by the properties of F there exists a constant C such that $\int_0^\infty e^{-rt}(x')^2 dz dt < C$, that is x'_n is a bounded sequence in $L^2((0, \infty), e^{-rt}dt; \mathbb{H})$, where $\mathbb{H} = L^2(\mathcal{O})$. Since the measure $\mu = e^{-rt}dt$ is such that $\mu(\mathbb{R}^+) < \infty$ and $\mathbb{H}$ is a separable Hilbert space, there exists a weakly convergent subsequence of x'_n (denoted the same for

[13] Meaning that if $u_n \rightharpoonup u$ in $L^p(\mathcal{O})$ and $v_n \to v$ in $L^q(\mathcal{O})$, then $\bar{\Phi}(u, v) \leq \liminf_n \bar{\Phi}(u_n, v_n)$.

simplicity); there exists a $y \in L^2((0,\infty), e^{-rt}dt;\mathbb{H})$ such that $x_n' \rightharpoonup y$ in $L^2((0,\infty), e^{-rt}dt;\mathbb{H})$. We then set $\bar{x}(t) := x_0 + \int_0^t y(s)ds$ so that $y = \bar{x}'$ and it can be seen that $x_n(t) \rightharpoonup \bar{x}(t)$ in $\mathbb{H}$ uniformly on compact subsets of $(0,\infty)$ and $x_n \rightharpoonup \bar{x}$ in $L^2((0,\infty), e^{-rt}dt;\mathbb{H})$.

Since K is a compact operator, the sequence $\mathsf{K}x_n(t)$ converges strongly to $\mathsf{K}\bar{x}(t)$ in $\mathbb{H}$ uniformly on compact subsets of $(0,\infty)$. An application of Lemma 1 provides the result that $\bar{J}_1(\bar{x}) \leq \lim_n \inf \bar{J}_1(x_n)$. Furthermore, the map $y \mapsto \int_0^\infty \int_{\mathcal{O}} e^{-rt} y(t,z)^2 dz dt$ is convex and lower semicontinuous;[14] it is also weakly lower semicontinuous, $\bar{J}_2(\bar{x}') \leq \lim_n \inf \bar{J}_2(x_n')$. Adding up, and since the minimizing sequence is convergent, we obtain that $\bar{J}(\bar{x},\bar{x}') \leq -M = \inf \bar{J}(x,x')$ which leads to the result that $\bar{x}$ is the minimizer for $\bar{J}$, hence the maximizer for J.

(b) We now consider the functional J, defined in (29), as a functional of $u = x'$ and $x = x_0 + \int_0^t u(s)ds$ (still denoted as J). The first order necessary condition will be of the form $(\nabla J, \phi) = 0$ where ∇ denotes the Gâteaux derivative and ϕ is a test function in $\mathbb{H}$.[15] We proceed to the determination of the Gâteaux derivative. To this end, fix any direction $v \in \mathbb{H}$, define $u_\epsilon = u + \epsilon v$, $V = \int_0^t v(s)ds$ and calculate

$$\frac{d}{d\epsilon} J(u_\epsilon)\bigg|_{\epsilon=0} = \int_0^\infty \int_{\mathcal{O}} e^{-rt} \left(\partial_x f(x,\mathsf{K}x) + \mathsf{K}^* \partial_X f(x,\mathsf{K}x) V - \lambda V - \alpha u v\right) dz\, dt$$

where K^* is the adjoint of operator K, and $\mathsf{K}^* = \mathsf{K}$ by symmetry. In the above calculation we have used the smoothness Assumption 3 that allows the use of the Lebesgue dominated convergence theorem, in order to intechange the limit defining the derivative with integration so as to reach the stated result. Since $v = V'$, by integration by parts over t and using the transversality condition, the first order condition becomes

$$\int_0^\infty \int_{\mathcal{O}} e^{-rt} \big(\partial_x f(x,\mathsf{K}x) + \mathsf{K}^* \partial_X f(x,\mathsf{K}x) - \lambda + \alpha u' - r\alpha u\big) V dz\, dt = 0.$$

This must be true for all v therefore for all V which implies that the first

[14] Mazur's lemma (according to which out of a weakly convergent sequence we may construct a subsequence which is a convex combination of the elements of the original sequence) and convexity play an important role in passing from strong semicontinuity to weak semicontinuity.

[15] This is assuming we treat the problem over the whole of $\mathbb{H}$, otherwise it is replaced by a variational inequality of similar form.

order condition becomes

$$\partial_x f(x, \mathsf{K}x) + \mathsf{K}^* \partial_X f(x, \mathsf{K}x) - \lambda + \alpha u' - r\alpha u = 0,$$

(a.e.) and keeping in mind that $u = x'$, we reach the stated result. *QED*

8.2 Proof of Theorem 2

The following technical result concerning the operators $\mathsf{A}_\nu : \mathbb{H} \to \mathbb{H}$, $\nu = RE, SO$, plays an important role in the Proof of Theorem 2.

Lemma 2.

(i) *The operators $\mathsf{A}_\nu : \mathbb{H} \to \mathbb{H}$, $\nu = RE, SO$ are maximal monotone.*[16]

(ii) *The operators $\mathsf{T}_\nu : \mathbb{H} \to \mathbb{H}$ defined by $\mathsf{T}_\nu := I - \mathsf{A}_\nu : \mathbb{H}$, $\nu = RE, SO$, where I is the identity operator, are pseudocontractive operators, i.e.,*

$$||\mathsf{T}_\nu x_1 - \mathsf{T}_\nu x_2||^2 \leq ||x_1 - x_2||^2 + ||(I - \mathsf{T}_\nu)x_1 - (I - \mathsf{T}_\nu)x_2||^2,$$

for all $x_1, x_2 \in \mathbb{H}$.

(iii) *Let $\mathbb{K} := \{x \in \mathbb{H} : x(z) \geq 0, \text{ a.e. } z \in \mathcal{O}\}$. Then, the operators $\mathsf{T}_\nu := I - \mathsf{A}_\nu : \mathbb{K} \to \mathbb{H}$, $\nu = RE, SO$, are weakly inward, i.e. they have the property $\mathsf{T}_\nu x \in \overline{I_{\mathbb{K}}(x)}$ for all $x \in \mathbb{K}$ where $\overline{I_{\mathbb{K}}(x)}$ is the closure of the inward set of x relative to $\mathbb{K}$, defined as*

$$I_{\mathbb{K}}(x) = \{(1-k)x + k\, y \,:\, y \in \mathbb{K},\ k \geq 0\}.$$

Proof: (i) The result is immediate for $\nu = SO$ since A_{SO} is a (sub)differential of the strictly convex functional $-J_1$, hence it is a maximal monotone operator (see e.g., Barbu, 2010, Ch. 2, Th. 2.8, p. 47).
The case $\nu = RE$ is a bit more involved, since A_{RE} cannot be expressed directly in terms of the (sub)differential of a convex functional but rather as a perturbation of such an entity. Indeed, note that A_{RE} can be expressed as $\mathsf{A}_{RE} = A + B$ where $A = \mathsf{A}_{SO}$ and $B = \mathsf{K} f_X$, where A is maximal monotone.

[16] A possibly nonlinear operator $\mathsf{A} : \mathbb{H} \to \mathbb{H}$ is called monotone if $(\mathsf{A}x - \mathsf{A}y, x - y) \geq 0$ for all $x, y \in \mathbb{H}$ and maximal monotone if its graph is not properly contained in the graph of any other monotone operator. Observe that monotonicity is related to positivity if the operator is linear.

We will address the question of maximal monotonicity of $A+B$. By Corollary 2.1 in Barbu (2010, Ch. 2, p. 35), which states that the sum of a maximal monotone operator with a hemicontinuous and monotone operator retains the maximal monotonicity property, it is enough to prove that B is hemicontinuous and monotone. The hemicontinuity of B is straightforward by the smoothness properties of the function f. To check monotonicity we employ a statement of Kachurovskii (1968, Theorem 1.1c), according to which a necessary and sufficient condition for a nonlinear operator B to be monotone is that B is Gâteaux differentiable, with Gateaux derivative ∇B such that $(h, \nabla Bh)$ is continuous for every $h \in \mathbb{H}$ and $(h, \nabla Bh) \geq 0$ for every $h \in \mathbb{H}$ (at any point in $\mathbb{H}$).

A simple calculation shows that

$$\begin{aligned} I &:= (\nabla Bh, h) = (f_{xX}(x, \mathsf{K}x)h, \mathsf{K}h) + (f_{XX}(x, \mathsf{K}x)\mathsf{K}h, \mathsf{K}h) \\ &= (\mathsf{K}^{1/2} f_{xX}(x, \mathsf{K}x)h, \mathsf{K}^{1/2}h) + (f_{XX}(x, \mathsf{K}x)\mathsf{K}h, \mathsf{K}h) \end{aligned}$$

where we have used the self-adjoint property of K and the definition of the square root of K. By Assumption 3(a),

$$\begin{aligned} &(\mathsf{K}^{1/2} f_{xX}(x, \mathsf{K}x)h, \mathsf{K}^{1/2}h) \geq \xi ||\mathsf{K}^{1/2}h||^2, \\ &(f_{XX}(x, \mathsf{K}x)\mathsf{K}h, \mathsf{K}h) \geq -\mu ||\mathsf{K}h||^2 \end{aligned}$$

so that $I \geq \xi \, ||\mathsf{K}^{1/2}h||^2 - \mu \, ||\mathsf{K}h||^2$. The quantity I will always be positive for all $h \in \mathbb{H}$ if $\inf_{h \in \mathbb{H}}(\xi \, ||\mathsf{K}^{1/2}h||^2 - \mu \, ||\mathsf{K}h||^2) > 0$. Note that $\mathsf{K}h = \mathsf{K}^{1/2}\mathsf{K}^{1/2}h$ and that the eigenvalues of K are such that $\mu_1 \geq \mu_2 \geq \cdots \geq 0$, so that the eigenvalues of $\mathsf{K}^{1/2}$ are such that $\mu_1^{1/2} \geq \mu_2^{1/2} \geq \cdots \geq 0$. This implies that $||\mathsf{K}^{1/2}y|| \leq \mu_1^{1/2} \, ||y||$ for all $y \in \mathbb{H}$; therefore substituting $y = \mathsf{K}^{1/2}h$ in this inequality we obtain the estimate $||\mathsf{K}h|| \leq \mu_1^{1/2} ||\mathsf{K}^{1/2}h||$ so that $-||\mathsf{K}h||^2 \geq -\mu_1 ||\mathsf{K}^{1/2}h||^2$. We therefore obtain the estimate

$$\xi \, ||\mathsf{K}^{1/2}h||^2 - \mu \, ||\mathsf{K}h||^2 \geq (\xi - \mu\mu_1) ||\mathsf{K}^{1/2}h||^2$$

which is always positive as long as $\xi - \mu\mu_1 > 0$. Then, the infimum of the left hand side will always be greater than or equal to 0. We conclude that the nonlinear operator B is monotone if $\mu/\xi < \mu_1$, therefore leading to the stated result.

(ii) Since A_ν are monotone operators for $\nu = RE, SO$, it holds that

$$(\mathsf{A}_\nu x_1 - \mathsf{A}_\nu x_2, x_1 - x_2) \geq 0$$

for every $x_1, x_2 \in \mathbb{H}$. Then, a simple calculation, using the Hilbert space structure of $\mathbb{H}$,

$$\begin{aligned}||\mathsf{T}_\nu x_1 - \mathsf{T}_\nu x_2||^2 &= ((I - \mathsf{A}_\nu)x_1 - (I - \mathsf{A}_\nu)x_2, (I - \mathsf{A}_\nu)x_1 - (I - \mathsf{A}_\nu)x_2 \\ &= ||x_1 - x_2||^2 - 2(\mathsf{A}_\nu x_1 - \mathsf{A}_\nu x_2, x_1 - x_2) + ||\mathsf{A}_\nu x_1 - \mathsf{A}_\nu x_2||^2,\end{aligned}$$

yields the required result.

(iii) We use the following characterization of weakly inward operators (see Deimling, 1985, Lemma 18.2, p. 208 and Section 20.4, p. 245), according to which if $\mathbb{K}$ is a closed and convex cone, then T_ν is weakly inward if and only if $x \in \partial\mathbb{K}$, $x^* \in \mathbb{K}^*$ and $\langle x^*, x\rangle = 0$ implies that $\langle x^*, \mathsf{T}_\nu x\rangle \geq 0$, where $\mathbb{K}^* = \{x^* \in \mathbb{H}^* : \langle x^*, x\rangle \geq 0,\ x \in \mathbb{K}\}$ denotes the dual cone and $\mathbb{H}^*$ is the dual space of $\mathbb{H}$. By $\langle\cdot,\cdot\rangle$ we denote the duality pairing between $\mathbb{H}^*$ and $\mathbb{H}$. Here, since we are in a Hilbert space setting, we consider $\mathbb{H}^* \simeq \mathbb{H}$ (by the Riesz representation) and we identify the duality pairing $\langle\cdot,\cdot\rangle$ with the inner product $(\cdot,\cdot)$ on $\mathbb{H}$.

Since $\mathbb{H} = L^2(\mathcal{O})$, we identify $\mathbb{H}^* = L^2(\mathcal{O})$ so that an element of the dual space x^* is also a square integrable function $x^* : \mathcal{O} \to \mathbb{R}$. Then

$$\langle x^*, x\rangle = (x^*, x) = \int_{\mathcal{O}} x^*(z)x(z)dz$$

so that $x \in \partial\mathbb{K}$, and $x^* \in \mathbb{K}^*$ and $(x^*, x) = 0$ imply that x^* is such that $x^*(z) \geq 0$ on $\mathcal{O}^+ := \{z \in \mathcal{O} : x(z) = 0\}$ (otherwise x^* is identically zero). We now calculate

$$\begin{aligned}(x^*, \mathsf{T}_{SO}x) &= \int_{\mathcal{O}} (x - \lambda + f_x(x, X) + \mathsf{K}f_X(x, X))(z)x^*(z)dz \\ &= \int_{\mathcal{O}^+} (-\lambda + f_x(0, 0) + \mathsf{K}f_X(0, 0))x^*(z)dz \geq 0\end{aligned}$$

by the properties of f (Assumption 3(b)). This shows that T_{SO} is weakly inward. Similarly

$$\begin{aligned}(x^*, \mathsf{T}_{RE}x) &= \int_{\mathcal{O}} (x - \lambda + f_x(x, X))(z)x^*(z)dz \\ &= \int_{\mathcal{O}^+} (-\lambda + f_x(0,0))x^*(z)dz \geq 0\end{aligned}$$

by the properties of f (Assumption 3(b)), therefore T_{RE} is also weakly inward. *Q.E.D.*

Proof of Theorem 2 (a) The result for both cases follows by a fixed point argument. We employ a fixed point theorem (see Joshi and Bose, 1985, Theorem 4.2.18 and Caristi, 1976, Theorem 2.6), according to which a continuous,[17] pseudo contractive and weakly inward mapping $F : \mathbb{K} \subset \mathbb{H} \to \mathbb{H}$ of a nonempty closed and bounded subset $\mathbb{K}$ of a Hilbert space $\mathbb{H}$ has a fixed point. By the results of Lemma 2 we apply this theorem to T_ν, $\nu = SO, RE$ to guarantee the existence of a fixed point for the operator T_ν. But this fixed point is also a solution of the operator equation $\mathsf{A}_\nu x = 0$, $\nu = SO, RE$. Uniqueness follows by strict monotonicity.
(b) By Lemma 2, operator A_ν is maximal monotone. We now use Theorem 3.3. of Rouhani and Khatibzadeh (2009) to obtain the stated result. According to a special case of this theorem a bounded solution of

$$x^{''} - rx^{'} = \mathsf{A}_\nu x$$

for any initial condition x_0, converges weakly as $t \to \infty$ to an element of $\mathsf{A}_\nu^{-1}(0)$, if A_ν is a maximally monotone. *Q.E.D.*

8.3 Proof of Proposition 1

Proof of Proposition 1: Follows from a combination of Theorem 2 and Propositions 3 and 4. *Q.E.D.*

8.4 Proof of Proposition 2

Proof of Proposition 2: (a) Since $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ is a compact operator, by Fredholm theory we know that the spectrum of K consists only of the point

[17] Or Lipschitz, if we employ Caristi's result.

spectrum (i.e. only of the eigenvalues $\{\mu_j\}$ of operator K). Furthermore, the spectrum is at most a countable set, and if it is not finite the only accumulation point for the sequence $\{|\mu_j|\}$ is 0. Since K is a bounded self-adjoint operator, its spectrum is also bounded and real, and the eigenfunctions $\{\phi_j\}$ corresponding to the eigenvalues $\{\mu_j\}$ may be chosen so as to form an orthonormal set in $\mathbb{H}$. This set is complete in $\overline{Ran(\mathsf{K})} \subset \mathbb{H}$.[18] If additionally K has the property of strict positivity then the spectrum is contained in a bounded subset of $\mathbb{R}_+$ and $\{\phi_j\}$ is complete in $\mathbb{H}$.

Using the basis of $\overline{Ran(\mathsf{K})} \subseteq \mathbb{H}$ defined by the eigenfunctions $\{\phi_j\}$ of operator K, we can obtain a spectral decomposition of (21) as follows: We perform a Galerkin approximation of the solution of (21) using the set $\{\phi_j\}$. We consider the sequence of functions $\hat{x}_n(t,z) = \sum_{i=1}^{n} c_{\nu,i}(t)\phi_i(z)$ and we insert this into (21). Projecting along ϕ_j, $j = 1, \dots, n$ we obtain the system of second order ODEs

$$c''_{\nu,j} - rc'_{\nu,j} + \Lambda_{\nu,j}c_{\nu,j} = 0, \quad \nu = RE, SO, \quad j = 1, \dots, n \tag{30}$$

with $\Lambda_{\nu,j}$ as given in the statement of the proposition. Assume that the initial conditions $\hat{x}(0), \hat{x}'(0) \in \overline{Ran(\mathsf{K})} \subset \mathbb{H}$. By the completeness of the orthonormal basis $\{\phi_n\}$ there exists an expansion $\hat{x}(0,z) = \sum_j a_j\phi_j(z)$, $\hat{x}'(0,z) = \sum_n b_n\phi_n(z)$ where the series converge in $\mathbb{H}$. Therefore, solving system (30) with initial conditions $c_{\nu,j}(0) = a_j$, $c'_{\nu,j}(0) = b_j$, we obtain an approximation of the solution in terms of the Galerkin expansion. The Galerkin expansion transforms the infinite dimensional systems (21), which characterize RE and SO equilibria respectively, into a countable set of finite dimensional problems (22), each problem corresponding to a mode $j = 1, ..., n$. Using a priori estimates and weak convergence arguments, we may pass to the limit as $n \to \infty$ in a standard fashion. The conditions for agglomeration emergence can be determined by looking at the exact solution of (22) for each mode j.

(b) The solutions are characterized by the roots of the characteristic polynomial $\sigma^2 - r\sigma + \Lambda_{\nu,j} = 0$. The roots are easily found to be

$$\sigma_{1,2} = \frac{r}{2} \pm \left(\left(\frac{r}{2}\right)^2 - \Lambda_{\nu,j}\right)^{1/2}.$$

[18] By *Ran* we denote the range of the operator K, while the overline denotes closure.

A quick inspection shows that if $\Lambda_{\nu,j} < 0$, then $\sigma_1 < 0$ and $\sigma_2 > \frac{r}{2}$ which is the usual saddle point stability. Furthermore, if $0 < \Lambda_{\nu,j} < (\frac{r}{2})^2$ then we obtain two real eigenvalues $0 < \sigma_1 < \frac{r}{2} < \sigma_2$. Finally in the case $\Lambda_{\nu,j} \geq (\frac{r}{2})^2$,

$$\sigma_{1,2} = \frac{r}{2} \pm i \left(\left| \left(\frac{r}{2}\right)^2 - \Lambda_{\nu,j} \right| \right)^{1/2}$$

so that we have modes growing with exponential growth rate $\frac{r}{2}$.[19] *Q.E.D.*

8.5 Proof of Proposition 3

PROOF OF PROPOSITION 3: (a) By Assumption 2(c) on the production function $s_{11} < 0$, $s_{22} < 0$ and $s_{12} > 0$. This leads to the observation that $\Lambda_{RE,j} > 0$ if $\mu_j > -\frac{s_{11}}{s_{12}}$ and $\Lambda_{RE,j} \geq \left(\frac{r}{2}\right)^2$ if $\mu_j \geq \frac{1}{s_{12}} \left(\frac{r}{2}\right)^2 - \frac{s_{11}}{s_{12}}$.
(b) $\Lambda_{SO,j} \geq 0$ implies $-|s_{11}| + 2s_{12}\mu_j - |s_{22}|\mu_j^2 > 0$, i.e., μ_j must lie between the two real roots of this quadratic polynomial. However, strong concavity of the production function implies $s_{11}s_{22} - s_{12}^2 > 0$, therefore $\Lambda_{SO,j}$ keeps the sign of $-|s_{22}|$ for all values of μ_j so that $\Lambda_{SO,j} < 0$ for all j, which implies stability. *Q.E.D.*

8.6 Proof of Proposition 4

PROOF OF PROPOSITION 4: (a) For every $x \in \mathbb{H}$ there exists a Fourier expansion in terms of Fourier series as $x(z) = \sum_{\ell=-\infty}^{\infty} x_\ell \exp(i\ell\pi z/L)$ with x_ℓ given by $x_\ell = \frac{1}{2L}\int_{-L}^{L} x(z)\exp(i\ell\pi z/L)dz$, where the convergence is in the $L^2(\mathcal{O})$ sense. A similar expansion exists for the kernel function w, $w(z) = \sum_{m=-\infty}^{\infty} w_m \exp(im\pi z/L)$. The condition for x to be real is $x_\ell^* = -x_{-\ell}$ where $*$ denotes the complex conjugate. To verify that the eigenfunctions are the Fourier modes, it suffices to observe that

$$\begin{aligned}(\mathsf{K}\phi_n)(z) &= \frac{1}{\sqrt{2L}} \sum_\ell w_\ell \exp(i\ell\pi z/L) \int_{-L}^{L} \exp(i(n-\ell)\pi s/L)ds \\ &= \sqrt{2L}\sum_\ell w_\ell \exp(i\ell\pi z/L)\delta_{n,\ell} = \sqrt{2L}w_n \exp(in\pi s/L) = W_n\phi_n(z)\end{aligned}$$

where $W_n = 2Lw_n$. This calculation shows that the Fourier basis are eigenfunctions of K with eigenvalue $\mu_n = W_n$ at mode n. Note that this set

[19] This is compatible with the well posedness of the functional J, and/or with the transversality conditions.

of eigenfunctions forms a complete basis of $\mathbb{H}$. The symmetry of the kernel shows that only the cosine part of the eigenfunctions corresponds to nontrivial eigenvalues.

(b) The action of K on the flat state $\bar{x}$ is as follows:

$$\mathsf{K}\bar{x} = \bar{x} \int_{-L}^{L} \sum_{\ell} w_\ell \exp(i\ell\pi(z-s)/L) ds = 2Lw_0\bar{x},$$

therefore the flat state generates spillovers which remain uniformly distributed in space. *QED*

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