AGGREGATIONAL GAUSSIANITY AND BARELY INFINITE VARIANCE IN CROP PRICES

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SUMMARY

This paper aims at reconciling two apparently contradictory empirical regularities of financial returns, namely the fact that the empirical distribution of returns tends to normality as the frequency of observation decreases (aggregational Gaussianity) combined with the fact that the conditional variance of high frequency returns seems to have a unit root, in which case the unconditional variance is infinite. We show that aggregational Gaussianity and infinite variance can coexist, provided that all the moments of the unconditional distribution whose order is less than two exist. The latter characterises the case of Integrated GARCH (IGARCH) processes. Finally, we discuss testing for aggregational Gaussianity under barely infinite variance.

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1 INTRODUCTION

One of the most important questions in the financial literature concerns the distribution of financial prices. The interest for this question originated in the early 1950s with the detailed empirical study of Kendall (1953) on the statistical properties of a set of economic time series including commodity prices such as the Chicago wheat and New York cotton prices. This study was the first to notice that the empirical distributions of successive price changes deviate from normality mainly because they exhibit excess kurtosis. Then, the issue of leptokurtosis was taken up by Mandelbrot (1963) who put forward the idea that the observed leptokurtosis reflects the fact that the variance of commodity or stock price changes is infinite. More specifically, Mandelbrot observed that the logarithmic price changes within a specific period of time, say a day, is the sum of elementary logarithmic price changes, ξ_i, between transactions that occur in that day. He then assumed that the variance of these elementary price changes is infinite, which in turn implies that the Central Limit Theorem is not applicable. As a result, the sum of ξ_i's converges not to the normal distribution, but instead, to a Stable Paretian distribution. The latter is leptokurtic and has infinite variance.

An alternative explanation for the observed leptokurtosis in the empirical distributions of price changes was offered by, among others, Clark (1973), and Blattberg and Gonedes (1974). These studies attempt to explain leptokurtosis without sacrificing the finite-variance assumption. In particular, they put forward the idea that the transactions are not spread uniformly across time, which in turn implies that the underlying distribution of price changes is a mixture of normals.

The two competing explanations for leptokurtosis mentioned above bare different implications about the behavior of the distribution of logarithmic price changes as we move from higher (say daily) to lower (say monthly) frequencies of observations. In particular, it has been observed that as we move from higher to lower frequencies the degree of leptokurtosis diminishes and the empirical distributions tend to approximate normality. This stylized fact, refered to as "Aggregational Gaussianity", can be accounted for only by the mixture of normals explanation of leptokurtosis and not by the infinite-variance alternative. Indeed, the stable-Paretian explanation is characterised by the property of "stability under addition" according to which if the daily price changes follow a stable Paretian ditribution with characteristic exponent equal to a, then the monthly price changes also have to follow the same distribution. This in turn implies that the property of infinite variance cannot coincide with that of Aggregational Gaussianity.

In late 1980's, when a new class of models, namely the GARCH models, was put forward, the issue of the parallell existence of infinite variance and Aggregational Gaussianity re-emerged in the context of the estimates of the GARCH parameters. In particular, the estimation of GARCH models for commodity or stock price changes seemed to suggest (i) the presence of a unit root (or near-to-unit root) in the conditional variance, which gave rise to the so-called Integrated GARCH (IGARCH) models and (ii) the gradual declining of conditional heteroskedasticity and the associated leptokurtosis of the unconditional distribution as we move from higher to lower frequencies of observation (see Diebold 1988, Drost and Nijman 1993). In view of the fact that the presence of a unit root in the conditional variance implies that the unconditional distribution has infinite variance, a case in which the classical Central Limit Theorem (CLT) does not apply, the empirical studies seemed to suggest the simultaneous presence of two seemingly contradictory facts: aggregational Gaussianity and infinite variance.

In this paper we aim at reconciling the above mentioned paradox. We show that infinite variance and aggregational Gaussianity can coexist, provided that all the moments of the unconditional distribtion whose order is less than two exist. This moment condition is satisfied in the case of IGARCH processes, or put it differently, an IGARCH process is indeed a process with barely infinite variance (see Kourogenis and Pittis 2008). In other words, what we show in this paper is that aggregational Gaussianity can coexist with infinite variance, once the latter arises from a unit root in the conditional variance.

The paper is organised as follows: In Section II we present evidence indicating that the price changes of six major crops, namely cocoa, coffee, corn, soybean, sugar and wheat, observed at high frequencies, seem to be characterised by both leptokurtosis and unit root in the conditional variance. We also show that both these effects tend to diminish as we move to lower frequencies. In Section III we explain why there is no paradox in admitting the simultaneous existence of aggregational Gaussianity and IGARCH, by means of some limit theorems for mixing processes with barely infinite variance, developed in the probability theory over the last twenty years or so. In this Section we also discuss whether the mixing properties of an IGARCH process, obtained so far in the literature, conforms to those assumed in the relevant limit theorems. In Section IV we discuss some issues that arise in testing for aggregational Gaussianity under infinite variance and present some additional empirical evidence supporting the coexistence of infinite variance and aggregational Gaussianity. The last Section concludes the paper.

2 EMPIRICAL MOTIVATION: DISTRIBUTIONAL CHARACTERISTICS OF CROP PRICE CHANGES

The motivation for this paper derives from analyzing the dataset of spot crop prices obtained from S&P Goldman Sachs Commodity Indices for cocoa, coffee, corn, soybean, sugar and wheat. In this dataset, the inception date of each crop price index ranges from 12/31/1969 to 1/6/1984. Figure 1 reports the empirical distributions of logarithmic price changes for sugar at daily, weekly, monthly, quarterly semi-annual and annual frequencies (similar results are obtained for all the crops considered here).

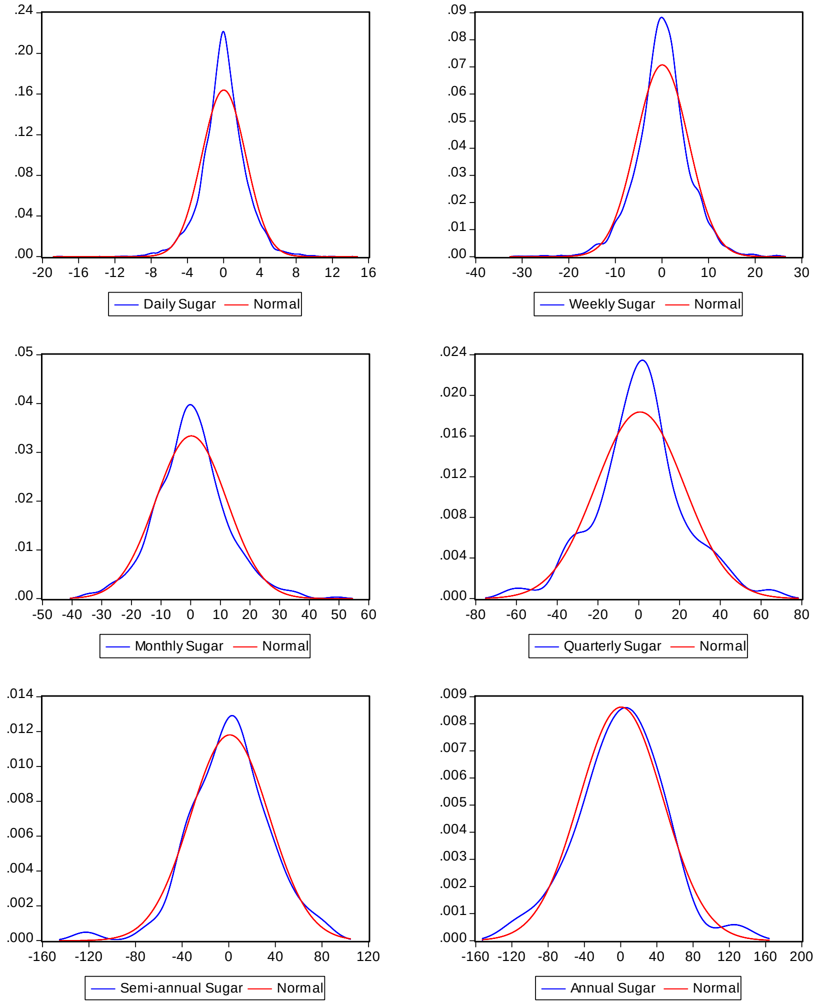

Figure 1. Sugar Returns

We also estimate a GARCH(1,1) model for the daily logarithmic price changes of all the six crops under consideration (see Model (2)) of Section III). The results may be summarized as follows:

(i) The sum of the maximum likelihood estimates of the GARCH(1,1) parameters is 0.994, 0999, 0.997, 0.993, 0.995 and 0.994 for cocoa, coffee, corn, soybean, sugar and wheat daily price changes, respectively. These results suggest the presence of a near-to-unit root in the conditional variance of the daily series. Note that this sum decreases with the frequency of observation. For example, the sum of the GARCH parameters is 0.219, 0.4516, 0.752, 0.658, 0.885 and 0.776 for cocoa, coffee, corn, soybean, sugar and wheat semi-annual price changes, respectively. These results suggest that, on average, the GARCH effects in semi–annual frequency are much weaker than the corresponding ones for daily frequency.

(ii) Visual inspection of the empirical distributions of the crop price changes under consideration suggests that these distributions are leptokurtic for daily, weekly and monthly frequencies. Overall, the degree of leptokurtosis seems to decrease as we move from daily to annual frequency at a slow rate. More specifically, the leptokurtosis does not seem to decrease substantially before we reach at least the quarterly frequency.

(iii) Overall, the combined evidence from (i) and (ii) above, suggests the simultaneous presence of a unit root in the conditional variance together with aggregational Gaussianity for all the six series under consideration.

3 AGGREGATIONAL GAUSSIANITY UNDER BARELY INFINITE VARIANCE

Let R_t be the one-period (say daily) continuously compounded return on a crop, defined as $R_t = p_t - p_{t-1}$, where p_t is the natural logarithm of the price of the particular crop. In a similar fashion we define the k-period (say weekly or monthly) return $R_\tau(k)$ as:

$$R_\tau(k) = p_t - p_{t-k} = \sum_{i=1}^{k} R_{t-k+i}. \tag{1}$$

The new index, τ, is introduced for notational simplicity, representing the k-period interval, in terms of t. More specifically, since we consider non-overlapping returns, the series of k-period returns, produced by taking non-overlapping sums of the original one-period return series, will be of the form $\{\ldots, p_{t-k} - p_{t-2k}, p_t - p_{t-k}, p_{t+k} - p_t, \ldots\}$. This means that one unit in terms of τ will correspond to k units in terms of t.

Next, let us assume that the one-period returns, follow an Integrated GARCH(1,1) (IGARCH(1,1)) process:

$$\begin{aligned} R_t &= h_t \nu_t \qquad (2) \\ \nu_t &\sim NIID(0, \sigma_\nu^2) \\ h_t^2 &= c + b h_{t-1}^2 + \gamma \nu_{t-1}^2, \text{ with} \\ c &> 0,\ 0 \leq b < 1,\ 0 \leq \gamma < 1 \text{ and } b + \gamma = 1. \end{aligned}$$

We shall attempt to answer the following question: Given that R_t follows an IGARCH process with infinite variance, how does the distribution of $R_\tau(k)$ behave as the returns horizon k increases? To answer this question, we must examine whether the probabilistic properties of R_t are such that enable the application of a relevant limit theorem. To this end, let us first briefly discuss the case of a stable GARCH process, that is when $b + \gamma < 1$. It is well known that under the restriction $b + \gamma < 1$, R_t is a second-order stationary process whose unconditional variance is equal to $\sigma_{R_t} = c/(1 - (b + \gamma))$. This process is also $\beta-$mixing with exponential decay (see Carrasco and Chen 2002 and Francq and Zakian 2006). Since a $\beta-$mixing process is also $\alpha-$mixing, we can appeal to the central limit theorem of Ibragimov (1962) and conclude that as $k \to \infty$, the sequence $R_\tau(k)$ converges in law to the normal distribution. Alternatively we may say that the distribution function of R_t belongs to the domain of attraction of the normal law. Moreover, in this case, the standardizing sequence is given by $\sigma_{R_t}\sqrt{k}$, which enables us to say that the distribution function of R_t belongs to the domain of *normal* attraction (DNA) of the normal law (see Ibragimov and Linnik 1971). A similar result in a different context was obtained by Diebold (1988) who showed that the GARCH effects tend to disappear under temporal aggregation.

Let us now focus attention on the case under study, that is when $b + \gamma = 1$ in which case, the variance of R_t is infinite. In this case we cannot apply the central limit theorem mentioned above. Moreover, the results of Diebold (1988) are derived under the assumption $b + \gamma < 1$ which means that they do not cover the IGARCH case. Therefore, we cannot say anything about the temporal aggregation properties of IGARCH processes. The presence of infinite variance seems to suggest that we must move away from the central limit theorem into limit theorems developed for the case of random variables with infinite variances. Historically, the problem described above was first dealt with by Lévy (1935) in the context of independent and identically distributed (*iid*) random variables and later by Ibragimov and Linnik (1971) for the case of mixing random variables (see Kourogenis and Pittis 2009 for an extentive discussion). Given the infinite variance of R_t, it seems reasonable to assume that R_t belongs to the domain of non-normal attraction of a stable law with exponent a. If this were the case, it would have implied two things: (a) the limiting distribution of $R_\tau(k)$ is a stable distribution (but *not* the normal distribution);(b) the sequence by which the partial sum process, $R_\tau(k)$, is standardised cannot be $\sigma_{R_t}\sqrt{k}$.

However, the case of IGARCH is different: An IGARCH process exhibits barely infinite variance meaning that all the moments $E\left|u_1\right|^\delta$ for every δ, $0 \leq \delta < 2$ are finite (see Corollary 1 in Kourogenis and Pittis 2008). In such a case, despite having infinite variance, the R_t's belong to the domain of non-normal attraction of the normal law. In other words, there exists a sequence $\{\delta_k\}$, which necessarily has the form $\delta_k = L(k)\sqrt{k}$, such that:

$$\frac{R_\tau(k)}{\delta_k}$$

weakly converges to the normal distribution. The function $L(k)$ is of particular interest: it is usually referred to as "slowly varying (at infinity)" meaning that $\frac{L(tx)}{L(x)} \to 1$ as $x \to \infty$ for every $t > 0$. The limit theorems that ensure this result are produced by Bradley (1988) or Peligrad (1990) for $\rho-$mixing and $\phi-$mixing sequences, respectively (see Kourogenis and Pittis 2008, 2009). These results show that the finite variance assumption is not necessary for the central limit theorem. More specifically, for strictly stationary sequences, (as is the IGARCH case considered here) the central limit theorem amounts to the truncated moment function, defined by:

$$H(x) = ER_1^2 \mathbf{I}_{|R_1| \leq x} \ ,$$

being slowly varying as $x \to \infty$, that is:

$$H(x) \text{ is slowly varying as } x \to \infty \tag{3}$$

In fact, the condition of slow variation of $H(x)$ is both necessary and sufficient for R_t to lie in the domain of attraction of the normal distribution (see Ibragimov and Linnik 1971). The requirement that $H(x)$ is a slowly varying function is equivalent to the condition:

$$E\left|R_1\right|^{\delta} < \infty, \ 0 \leq \delta < 2. \tag{4}$$

The latter condition amounts to saying that the R_t's have *just barely* infinite variance (see Bradley 1988). This implies that the central limit theorem may hold even in cases that the variance of the R_t's is infinite, provided that all the moments of order $\delta < 2$ are finite.

The preceding discussion suggests that the empirical features of Aggregational Gaussianity and Infinite Variance in crop price changes can coincide due to the limit theorems for mixing sequences with barely infinite variance mentioned above. However, one word of caution is in order. In order to apply the central limit theorem of Bradley (1988) or that of Peligrad (1990) we must ensure that an IGARCH process is either $\rho-$mixing or $\phi-$mixing, respectively. As far as we know, the relevant literature is yet to produce such a result. Having said this, it is worth mentioning

Francq and Zakian's (2006) relevant result, which proves that an IGARCH process is $\beta-$mixing with exponential decay. However since there is no proof to date that $\beta-$mixing implies either $\rho-$mixing or $\phi-$mixing, the use of the above mentioned theorems should be exercised with caution.

4 TESTING FOR AGGREGATIONAL GAUSSIANITY UNDER IGARCH

The preceding discussion must have made clear that aggregational Gaussianity is allowed to coincide with the assumption that the returns over the shortest horizon (say daily) follow an IGARCH process with barely infinite variance. However to establish this fact empirically, using formal statistical methods is rather tricky. The usual procedure for evaluating whether a given empirical distribution is normal involves estimating the sample skewness and kurtosis coefficients, α_3 and α_4, respectively. To this end, establishing aggregational Gaussianity would imply to estimate these coefficients over various frequencies, and observe that α_3 and α_4 tend to 0 and 3, respectively, as the freqency of observation (returns horizon) decreases (increases). However, this strategy does not work in the case under study, because the returns over the shortest horizon (one-period), R_t, are assumed to follow an IGARCH process. In this case, the population skewness and kurtosis coefficients are infinite, which in turn implies that the corresponding sample estimates, $\widehat{\alpha}_3$ and $\widehat{\alpha}_4$ will diverge to infinity as the sample size (of daily observations) increases.

Let us examine more closely the behavior of the estimated kurtosis coefficient, $\widehat{\alpha}_4$, of $R_\tau(k)$ as k increases under the assumption that the one-period returns, R_t, is an IGARCH process. To this end, we conduct a small Monte Carlo experiment. Specifically, we generate 1000 near-to- IGARCH(1,1) series of length equal to 10056 which is the number of daily observations in our sample. The conditional variance parameters were set equal to $b = 0.059299$ and $\gamma = 0.935634$, which are the average values of the estimated parameters across the three crops under consideration. For each of these 1000 replications, we generate five more series, $R_\tau(k)$, $k = 5$, 20, 60, 120, and 240 according to (1), corresponding to weekly, monthly, quarterly, semi-annual and annual frequencies. Note that the number of observations decreases with k; in particular we end up with 2011, 503, 168, 84 and 42 observations for $k = 5$, 20, 60, 120, and 240, respectively. Then, for each replication, we estimate the kurtosis coefficient for all the available frequencies, namely $k = 0$, 5, 20, 60, 120, and 240 and take the average (referred to as $\widehat{\alpha}_{4,MC}$) across the 1000 replications for each frequency. The results are reported in Figure 2, together with the corresponding average estimated kurtosis coefficients (referred to as $\widehat{\alpha}_{4,D}$) across the six crops under consideration.

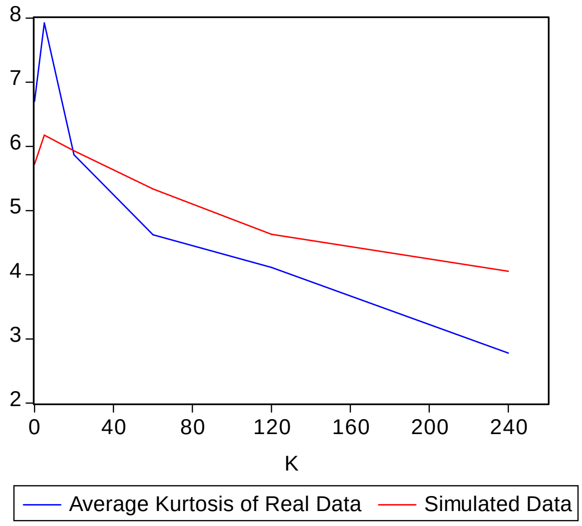

Figure 2

The results may be summarised as follows:

(i) The Monte Carlo kurtosis coefficent, $\widehat{\alpha}_{4,MC}$ appears to exhibit a pattern similar to that observed for the kurtosis coefficient, $\widehat{\alpha}_{4,D}$, of the real data. In particular, $\widehat{\alpha}_{4,MC}$ increases temporarily as we move from $k = 0$ to $k = 5$ and then decreases with k.

(ii) The behaviour of $\widehat{\alpha}_{4,MC}$ reported above is typical for IGARCH (or near-to-IGARCH) processes. On the contrary for GARCH parameters safely inside the stationarity region the behaviour of $\widehat{\alpha}_{4,MC}$ is exactly that predicted by CLT, namely $\widehat{\alpha}_{4,MC}$ converges monotonically to 3 as the returns horizon increases.

(iii) The behaviour of $\widehat{\alpha}_{4,MC}$ reported above may be due to the following reasons: First, as k increases there are two opposite forces at work: The first one stems from the fact that R_t does belong to the domain of attraction of the normal law, which means that as k increases, the corresponding processes $R_\tau(k)$ become "more normal". This force creates a tendency for the estimates of the kurtosis coefficient to approach the value of 3. However, as k increases, the number of observations on the corresponding $k-$horizon returns, available in a given time period, decreases. For example, for the time period 29/12/1969 to 12/11/2009 we have 10056 daily observations but only 2081 weekly, 480 monthly, 161 quarterly, 81 semi annual and 41 annual observations.

The smaller number of observations makes it harder for CLT to take effect, thus creating a tendency for $\widehat{\alpha}_{4,MC}$ to deviate from 3. The second reason, which may explain the non-monotonicity in the behavior of $\widehat{\alpha}_{4,MC}$ is related to the rate of convergence of $R_\tau(k)$ to normal. In the absence of a finite second moment, the rate of convergence to the normal distribution is expected to be much slower than the corresponding one for the finite-variance case. This property combined with the fact that the number of observations decreases with k may explain the slow and non-monotonic way by which $\widehat{\alpha}_{4,MC}$ approaches the value of 3 as k increases. To this end, it is also interesting to note that the rate of convergence to normality in the presence of a barely infinite variance as suggested by the normalizing sequence, δ_k, is $L(k)\sqrt{k}$ with $L(k)$ being a slowly-varying and possibly non-monotonic function.

5 CONCLUSIONS

Motivated by empirical evidence indicating that the price changes of six major crops, when observed at high frequencies, seem to be characterised by both leptokurtosis and unit root in the conditional variance, while both of these effects tend to diminish as one moves to lower frequencies, we explain why there is no paradox in admitting the simultaneous existence of aggregational Gaussianity and infinite variance. In particular, we show that aggregational Gaussianity and infinite variance can coexist, provided that all the moments of the unconditional distribution whose order is less than two exist. Our theoretical explanation derives from limit theorems for mixing processes with barely infinite variance, developed in the probability theory literature. More specifically, we suggest that the limit theorems of Bradley (1988) or that of Peligrad (1990) for mixing sequences with barely infinite variance, for $\rho-$mixing and

$\phi-$mixing sequences respectively, ensure the coincidence of the empirical features of Aggregational Gaussianity and Infinite Variance in crop price changes. Finally, we discuss some issues that arise in testing for aggregational Gaussianity under infinite variance and present some additional empirical evidence supporting the coexistence of IGARCH effects in high frequency data and aggregational Gaussianity.

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