DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

DIFFUSION AND SPATIAL ASPECTS

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Working Paper Series

12-32

November 2012

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1. Introduction

An important element of most ecological theories seeking to understand ecosystems is the spatial and temporal structure of ecosystems. Spatial heterogeneity involving spatial structures such as patches or gradients seems to prevail in nature, in contrast to spatial homogeneity, and has a central role in the analysis of issues such as theories of competition, succession, evolution and adaptation, maintenance of species diversity, parasitism, population genetics, population growth, and predator-prey interactions (Legendre and Fortin 1989).

The study of the emergence and the properties of regular spatial or spatiotemporal patterns which can be found in abundance in nature, such as for example stripes or spots on animal coats, ripples in sandy deserts, vegetation patterns in arid grazing systems or spatial patterns of fish species, has drawn much attention in natural sciences. Furthermore, empirical evidence suggests that disturbances in an ecosystem caused by human actions can either increase or decrease landscape heterogeneity depending on the parameter and spatial scale examined (e.g. Mladenoff et al. 1993).

In economics the importance of space has long been recognized in the context of location theory,[1] although as noted by Krugman (1998) a systematic analysis of spatial economics has been neglected. This neglect is associated mainly with difficulties in developing tractable models of imperfect competition which are essential in the analysis of location patterns. After the early 1990s there was a renewed interest in spatial economics mainly in the context of new economic geography. Krugman (1998) attributes this new research to: the ability to model monopolistic competition using the well known model of Dixit and Stiglitz (1977); the proper modeling of transaction costs; the use of evolutionary game theory; and the use of computers for numerical examples.

[1] See for example Weber (1909), Hotelling (1929), Christaller (1933), and Löcsh (1940) for early analysis, or Krugman (1993, 1996), Fujita et al. (1999) for the more recent literature.

In environmental and resource management problems the majority of the analysis has been focused on taking into account the temporal variation of the phenomena, and has concentrated on issues such as the transition dynamics towards a steady state, or the steady-state stability characteristics. However, it is clear that when renewable and especially biological resources are analyzed, the spatial variation of the phenomenon is also important. Biological resources tend to disperse in space under forces promoting 'spreading', or 'concentrating' (Okubo and Levin 2001); these processes along with intra and inter species interactions induce the formation of spatial patterns for species.

In the management of economic-ecological problems, the importance of introducing the spatial dimension can be associated with attempts to incorporate spatial issues, such as resource management in patchy environments or reserve creation (e.g. Sanchirico and Wilen 1999, 2001, 2005; Brock and Xepapadeas 2002, 2005; Smith and Wilen 2003; Sanchirico 2005; Wilen 2007; Costello and Polasky 2008), the study of control models for interacting species (Lenhart and Bhat 1992; Lenhart et al. 1999), the control of surface contamination in water bodies (Bhat et al. 1999), or the exploration of the conditions under which interacting processes characterizing movements of biological resources, and economic variables which reflect human effects on the resource (e.g. harvesting effort), could generate steady-state spatial patterns for the resource and the economic variables. That is, conditions which could generate a steady-state concentration of the resource and the economic variable, which varies across locations in a given spatial domain (Brock and Xepapadeas 2008, 2010). We will call this formation of steady spatial patterns persistent 'spatial heterogeneity', in contrast to 'spatial homogeneity' which implies that the steady state concentration of the resource and the economic variable is the same at all points in a given spatial domain.

A central concept in modeling the dispersal of biological resources is that of *diffusion*. Diffusion is defined as a process where the microscopic irregular movement of particles such as cells, bacteria, chemicals, or animals results in some macroscopic regular motion of the group (Levin and Segel 1985; Okubo and Levin 2001; Murray 2003). Biological diffusion is based on random walk models which, when coupled with population growth equations of interacting species, lead to general reaction-diffusion systems.[2] In general a diffusion process in an ecosystem tends to produce a uniform population density, that is, spatial homogeneity. Thus it might be expected that diffusion would 'stabilize' ecosystems where species disperse and humans intervene through harvesting. There is however one exception known as 'diffusion-induced instability', or 'diffusive instability' (Okubo and Levin 2001). It was Turing (1952) who suggested that under certain conditions, reaction-diffusion systems can generate spatially heterogeneous patterns. This is the so-called 'Turing mechanism' for generating diffusion instability. Turing's diffusion-induced instability signals the emergence of spatial patterns as a result of spatial diffusion. These emerging patterns might lead to persistent spatial heterogeneity, depending on the features of the dynamical system.

Biological reaction-diffusion systems are descriptive non-optimizing systems, in the sense that the biological agents cannot be regarded as fully forward looking optimizing agents. Therefore, to study spatial diffusion in the context of economic models, the reaction-diffusion systems need to be coupled with an economic model. This modeling typically involves control functions which are chosen by economic agents and which affect the evolution of state functions of the reaction-diffusion system, and an objective that depends on the controls and the states. The objective should be maximized by the choice of the controls, subject to the constraint of the reaction diffusion system. Brock and Xepapadeas (2008, 2010) have studied these systems in the context of resource management problems and have identified conditions for the emergence of an 'optimal diffusion-induced instability'.[3] This instability signals the emergence of the spatial patterns resulting from forward-looking optimizing behavior under spatial diffusion. Persistent spatial heterogeneity resulting from optimal diffusion-induced instability can be regarded as describing optimal agglomerations or optimal clustering of the state variables of the system in the long run.

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[2] When only one species is examined, the coupling of classical diffusion with a logistic growth function leads to the so-called Fisher-Kolmogorov equation.

[3] Boucekkine et al. (2009) has studied a similar problem in the context of a Ramsey growth model. For a recent short survey, see Xepapadeas (2010).

The purpose of this chapter is to present methods for studying environmental and resource economics models in a spatial-dynamic framework, along with current results regarding the optimization of such models and the emergence of optimal diffusion-induced instabilities and optimal agglomerations. The emergence of optimal agglomerations and clustering in this context results from the interactions between forward-looking optimizing economic agents whose actions - either in the form of harvesting or in the form of regulation - affect environmental systems which generate useful services, and the natural processes which govern the movements of environmental resources in time and space.

The approach presented in this chapter is placed in the general context of bioeconomic models governed by spatiotemporal dynamics, but differs from the main body of the existing literature, which uses metapopulation and discrete spatial-dynamic models, by using continuous spatial-dynamics. This adds on the one hand the mathematical complication of employing partial differential equations in modeling, but on the other hand it significantly reduces the dimensionality of the optimal control problems resulting from metapopulation models. This is because metapopulation models require one state variable for each patch. Thus the study of models with multiple interacting state variables is facilitated.

The interpretation of the costate variable in these models as showing the spatiotemporal evolution of the state's shadow value is similar to that of discrete spatial-dynamic models (e.g. Sanchirico and Wilen 2005). Thus the costate variables can be used as a basis for spatially dependent regulation.

2. Modeling Spatial Movements

2.1 Short-Range Effects

Let $x(t,z)$ denote the concentration of a biological or economic entity at time $t \geq 0$ at the spatial point $z \in Z$, where space is assumed to be one-dimensional and modeled by a line segment.[4] The real function $x(t,z)$ describes the state of

[4] The use of two- or three-dimensional space does not change the basic analysis, however it complicates the mathematical presentation.

the system.[5] The classic approach for modeling spatial movements of this state function is through diffusion. Under diffusion the microscopic irregular motion of an assemblance of particles results in a macroscopic regular motion of the group. This classical approach to diffusion implies that diffusion has local or short-range effects. This means that economic or ecological activity at point *z* is only affected by the economic activity at nearby spatial points, i.e. points $z \pm dz$ and for *dz* tending to 0. In general short-range effects are modeled by linear, or nonlinear in more general cases, differential operators. The most common example is the use of the Laplace operator $Ax(t,z) = \nabla^2 x(t,z) = \frac{\partial^2 x(t,z)}{\partial z^2}$ which leads to the well known one-dimensional heat equation $\frac{\partial x(t,z)}{\partial t} - D_x Ax(t,z) = 0$.

A measure of diffusion is the *diffusion coefficient*, or *diffusivity,* D_x which measures how efficiently particles move from high to low density. Let $f(x(t,z),u(t,z))$ be a growth function or a source for the state function that depends on the density of the state of the system, where $u(t,z)$ is a control function, defined in the same way as the state function. A control function could be, for example, harvesting by economic agents. Let $\phi(t,z)$ denote the flow of 'material' such as animals or commodities past *z* at time *t*. The classic assumption is that this flux is proportional to the gradient of the concentration of material or $\phi(t,z) = -D_x \frac{\partial x(t,z)}{\partial z}$, where D_x is the diffusion coefficient and the minus sign indicates that material moves from high levels of concentration to low levels of concentration. Under this assumption the evolution of the material's stock in a small interval Δz is defined as:

$$\frac{d}{dt}\int_z^{z+\Delta z} x(t,s)ds = \phi(t,z) - \phi(t,z+\Delta z) + \int_z^{z+\Delta z} f(x(t,s),u(t,s))ds.$$

[5] In mathematical terms, problems involving space and time are distributed parameter problems and $x(t,z)$ is a function that takes values in a separable Hilbert space of square integrable functions which can be written more precisely as $x(t,z) = x(t)(z)$.

If we divide the equation above by Δz and take limits as $\Delta z \to 0$, then the spatiotemporal evolution of our state will be determined by the partial differential equation:[6]

$$\frac{\partial x(t,z)}{\partial t} = f\left(x(t,z), u(t,z)\right) + D_x \nabla^2 x(t,z), x(0,z) = x_0(t), \nabla^2 x(t,z) = \frac{\partial^2 x(t,z)}{\partial z^2}. \quad (1)$$

In most applications it is assumed that the spatial domain is finite with $z \in [-Z, Z]$. Spatial boundary conditions for (1) could imply: (i) that the spatial domain is a circle or $x(t,-Z) = x(t,Z)$ for all *t,* (ii) hostile boundaries or $x(t,-Z) = x(t,Z) = 0$ for all *t,* or (iii) zero flux at the boundaries $\frac{\partial x(t,-Z)}{\partial z} = \frac{\partial x(t,Z)}{\partial z} = 0$ for all *t.* If the source term represents logistic population growth and the control function $u(t,z)$ represents harvesting at spatial point *z* and time *t* or $f(x,u) = x(t,z)(s - rx(t,z)) - u(t,z)$, then we obtain the *Fisher* equation:

$$\frac{\partial x(t,z)}{\partial t} = sx(t,z)\left(1 - \frac{rx(t,z)}{s}\right) - u(t,z) + D_x \nabla^2 x(t,z), x(0,z) = x_0(t). \quad (2)$$

The Fisher equation can be generalized to several interacting species or activities. With two interacting species $\left(x(t,z), y(t,z)\right)$ which are both harvested at rates $\left(u^x(t,z), u^y(t,z)\right)$ and diffuse in space with constant diffusivities $\left(D_x, D_y\right)$ respectively, we obtain:

$$\frac{\partial x}{\partial t} = f_1\left(x, y, u^x\right) + D_x \nabla^2 x \quad (3)$$

$$\frac{\partial y}{\partial t} = f_2\left(x, y, u^y\right) + D_y \nabla^2 y \quad (4)$$

.

System (3-4) is referred to as a *reaction-diffusion system* or as an *interacting population diffusion system*.[7] If species *x* promotes the growth of *y*, then *x* is an *activator*, while if *y* reduces the growth of *x*, then *y* is an *inhibitor*. In

[6]For details, see Murray (2003). In more general diffusion models the diffusion coefficient could be density dependent or $D_x = D_x\left(x(t,z)\right)$.

[7]Generalization to *n* species is straightforward.

this case the system (3)-(4) is an activator-inhibitor system. In systems like (3)-(4) patterns may emerge as the result of Turing diffusion-induced instability.

Diffusivity can be also nonlinear. In energy balance climate models for example (see North 1975a, 1975b; North et al. 1981), outgoing radiation is described by the following partial differential equation:

$$\frac{\partial I(,zt)}{\partial t} = QS(z)\alpha(z,z_s(t)) - \left[I(z,t) - h(z,t)\right] + D\frac{\partial}{\partial x}\left[(1-z^2)\frac{\partial I(z,t)}{\partial z}\right] \qquad (5)$$

where units of z are chosen so that $z=0$ denotes the Equator, $z=1$ denotes the North Pole, and $z=-1$ denotes the South Pole; Q is the solar constant[8] divided by 4; $S(z)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from -1 to 1 is unity; $\alpha(z,z_s(t))$ is the absorption coefficient which is one minus the albedo of the earth-atmosphere system, with $z_s(t)$ being the latitude of the ice line at time t; and D is a thermal diffusion coefficient that has been computed as $D = 0.649Wm^{-2}$.

Equation (5) states that the rate of change of outgoing radiation is determined by the difference between the incoming absorbed radiant heat $QS(z)\alpha(z,z_s(t))$ and the outgoing radiation $\left[I(z,t)-h(z,t)\right]$. Note that the outgoing radiation is reduced by the human input $h(z,t)$. Thus the human input at time t and latitude z, can be interpreted as the impact of the accumulated carbon dioxide that reduces outgoing radiation.

2.2 Long-Range Effects

In many cases, however, it is necessary to model *nonlocal* or *long range* spatial interactions. This is done by using integral operators which model the long-range spatial interactions. These operators are of the general form

$$(Ax)(z,t) = \int \mathrm{K}\left(z,z',x(t,z')\right)x(t,z')dz' \qquad (6)$$

and the integration takes place over the whole spatial domain where economic or ecological activity is assumed to happen. The function K is called the kernel function and models the effect that economic or ecological activity has as a possible distant point *y* to the activity at point *z*. This integral operator has many

[8]The solar constant includes all types of solar radiation, not just the visible light. It is measured by satellite to be roughly 1.366 kilowatts per square meter (kW/m²).

important differences from the differential operator, from both the mathematical and the modeling point of view.[9] From the modeling point of view, the integral operator formulation allows us to model long-range spatial effects, since the point *y* may be as distant as possible from *x*, and the strength of the interaction is provided by the size of the kernel function K.

In the presence of nonlocal effects, the temporal change of the state variable at spatial point *z* depends on the influence of neighboring state variables in all other locations *z'*. In this case the spatiotemporal evolution of the system's state which is analogous to (1) is:

$$\frac{\partial x(z,t)}{\partial t} = f\left(x(z,t),u(z,t)\right) + \int_{-Z}^{Z} w(z-z')x(z',t)dz' \,,\; x(0,z) = x_0(z)\forall z \tag{7}$$

where again the spatial domain is finite with $z \in [-Z, Z]$, and spatial boundary conditions could be similar to (1). In (7), $w(z-z')$ is the *kernel function* which quantifies the effects on the state $x(t,z)$ at z from states in other locations $z' \in [-Z, Z]$. Typical kernel functions are presented in figures 1 and 2.

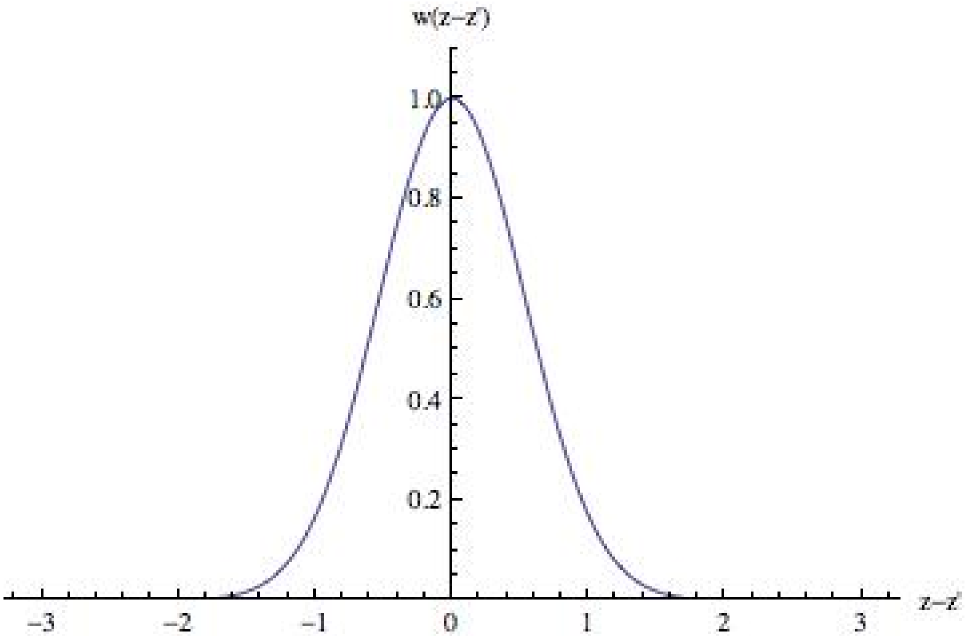

Figure 1: Positive long-range effects

[9] From the mathematical point of view, for a wide range of kernel functions, the operator *A* is a compact operator, thus leading to the most natural generalization of finite dimensional continuous and bounded operators to the infinite dimensional case. This is in contrast to the case where *A* is a differential operator, which leads to unbounded and non-compact operators. Therefore, equation (1) for the case where *A* is an integral operator enjoys some nice properties with respect to its solvability and the qualitative properties of the solution.

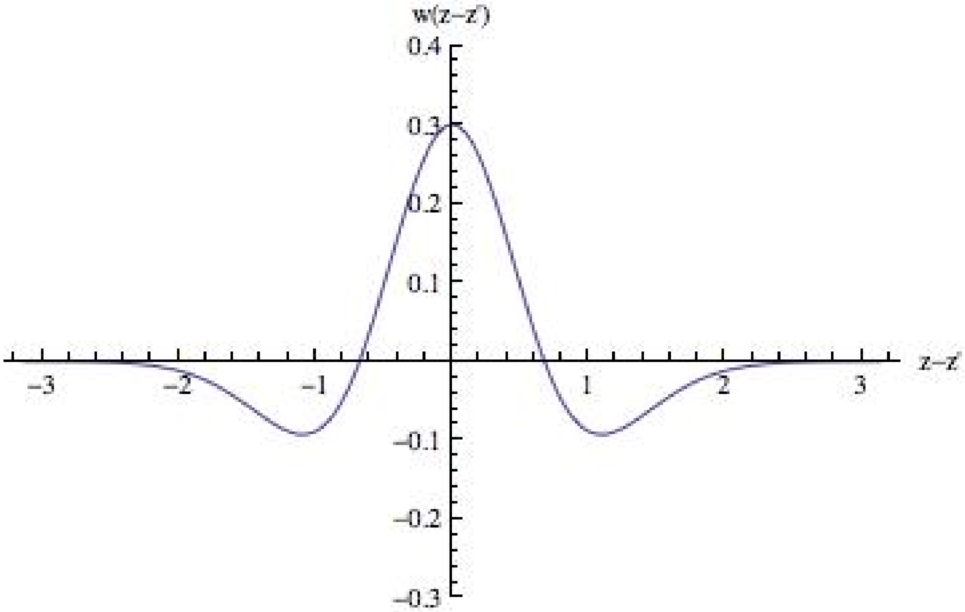

Figure 2: Positive and negative long-range effects

It should be noted that in figure 1 long-range effects are always positive, while in figure 2 positive and negative spatial effects are present.

Nonlocal effects and the integral equation formulation are widely used in economics to model knowledge or productivity spillovers affecting the production function (e.g., Lucas 2001; Lucas and Rossi-Hansberg 2002) or to model long-range effects of knowledge accumulation (e.g., Quah 2002). For example a constant returns to scale production function with spatial knowledge spillovers can be written as:

$$Q(z) = \exp(\gamma V(z)) L(z)^a K(z)^b X(z)^{1-a-b}$$

where Q is the output, L is the labor input, K is physical capital, X is land, and V is the productivity spillover which depends on how many workers are employed at all other locations. The spatial productivity externality is defined as $\exp(\gamma V(z))$, with $V(z) = \delta \int_{-Z}^{Z} e^{-\delta|z-z'|} L(z') dz'$. The function $e^{-\delta|z-z'|}$ is the kernel. The productivity externality is a positive function of labor employed in all areas and is assumed to be linear and to decay exponentially at a rate δ with the distance between z and z'. The idea is that workers at a spatial point benefit from labor in nearby areas, and thus the distance between firms determines the production of ideas and the productivity of firms in a given region. A high δ indicates that only labor in nearby areas affects production positively. In terms of agglomeration economics, the production externality is a *centripetal* force, i.e., a force that promotes the spatial concentration of economic activity.

Nonlocal effects are regarded as more appropriate for the analysis of problems where only spatial spillovers associated with economic variables are involved. In these models, like for example models with knowledge spillovers, the externality is assumed instantaneous and not emerging through the state dynamics. Local effects could be more appropriate for the analysis of environmental and resource management problems where there is explicit spatial movement of state variables, and the movement has local characteristics.

It is also possible to combine local and nonlocal effects. For example Genieys et al. (2006) study a reaction-diffusion equation with an integral term describing nonlocal effects or

$$\frac{\partial x(t,z)}{\partial t} = D_x \frac{\partial^2 x(t,z)}{\partial z^2} + x(t,z)\left(\sigma(t,z) - \int_{-\infty}^{\infty} w(z-z')x(t,z')dz'\right) \qquad (7a)$$

where $x(t,z)$ describes the density of a biological population. The first term of the right hand side describes local diffusion, while the second term describes reproduction, which is density dependent and is proportional to available resources $\sigma(t,z)$. The integral term relates to the impact of nonlocal consumption of resources. Models with integrodifferential equations can be associated with the case of many marine resources where after spawning the larvae are transported by currents and wind over long distances.[10]

The rest of this chapter will focus on local effects.

3. Optimal Control Under Diffusion: The Maximum Principle

Dynamic problems of coupled economic and ecological systems are usually modeled as optimal control problems with system dynamics acting as a constraint to the optimization problem. When system dynamics are characterized by spatial diffusion, the problem becomes a problem of optimal control of a distributed parameter system, which for the case of one state $x(t,z)$ and one control $u(t,z)$ can be stated as:

[10] I am grateful to a reviewer for pointing out this example.

$$\max_{\{u(t,z)\}} \int_0^\infty \int_{-Z}^{Z} e^{-\rho t} \left[U\left(x(t,z), u(t,z)\right)\right] dz dt \tag{8}$$

subject to

$$\frac{\partial x(t,z)}{\partial t} = f\left(x(t,z), u(t,z)\right) + D_x \frac{\partial x^2 (t,z)}{\partial z^2},\ x(0,z) = x_0(t), x(t,-Z) = x(t,Z). \tag{8a}$$

A maximum principle for this problem has been derived by Derzko et al. (1984) (see also Brock and Xepapadeas 2008). To use this maximum principle we need to introduce the Hamiltonian function

$$\tilde{\mathrm{H}}(x,u,p) = U\left(x(t,z), u(t,z)\right) + p\left[f\left(x(t,z), u(t,z)\right) + D_x \frac{\partial x^2 (t,z)}{\partial z^2}\right] \tag{9}$$

where $p(t,z)$ is the costate variable. The Hamiltonian function (9) is a generalization of the 'flat Hamiltonian'

$$\mathrm{H} = U\left(x(t), u(t)\right) + p(t)\left[f\left(x(t), u(t)\right)\right] \tag{10}$$

for $D_x = 0$. The first-order conditions for the optimal control $u^*(t,z)$ imply $u^*(t,z) = \arg\max_u \tilde{\mathrm{H}}\left(x(t,z), u(t,z), p(t,z)\right)$. Assuming that the Hamiltonian function satisfies appropriate concavity assumptions, $u^*(t,z)$ is defined, for interior solutions, by:

$$\frac{\partial \tilde{\mathrm{H}}\left(x(t,z), u(t,z), p(t,z)\right)}{\partial u} = 0. \tag{11}$$

Optimal controls are then defined in terms of the state and the costate variables as:

$$u^*(t,z) = g^*\left(x(t,x), u(t,z)\right). \tag{12}$$

The costate variable satisfies:

$$\frac{\partial p(t,z)}{\partial t} = \rho p - \frac{\partial \tilde{\mathrm{H}}\left(x(t,z), p(t,z), g^*(t,z)\right)}{\partial x} - D_X \frac{\partial^2 p(t,z)}{\partial z^2} \tag{13}$$

where $g^*(x,p)$ is the optimal control function defined by (12). Note that the costate is interpreted as the shadow price of the stock (or the state) and that the diffusion term in (13) has a negative sign, while the diffusion term for the state dynamics of the system (8a), which reflect stock quantities, has a positive sign according to classic diffusion. This change in the sign of the diffusion coefficient means that prices and quantities move in the opposite directions in the spatial domain, for the optimally-controlled system. This result is in agreement with the economic intuition.

Finally the following temporal and spatial transversality conditions should be satisfied at the optimum:

$$\lim_{T\to\infty} e^{-\rho T}\int_{-Z}^{Z} p(T,z)x(T,z)dz = 0$$
$$p(t,-Z) = p(t,Z). \qquad (14)$$

The first condition states that at an optimal solution the aggregate value of the state variable over the whole spatial domain, where the state variable at each site is valued at the local shadow price $p(t,z)$, remains finite as $t \to \infty$. The second condition states that the costate variable, or local shadow price, should – like the state – satisfy circle boundary conditions for all times. The transversality conditions provide boundary conditions for the solution of the problem.

The transition equation (8a) with $u(t,z)$ replaced by the optimal control $u^*(t,z) = g^*(x(t,x),u(t,z))$, along with (13) constitute a system of two partial differential equations. This is the Hamiltonian system which, along with the initial conditions and the transversality conditions (14), determines the spatiotemporal evolution of the state and costate functions along the optimal path.

4. Pattern Formation

In reaction-diffusion systems without optimization the emergence of spatial patterns is analyzed through the 'Turing mechanism.' We briefly present the Turing mechanism below since this mechanism will be used to study optimal diffusion-induced instability.

4.1 Turing mechanism and economic behavior

The idea behind spatial pattern formation through the Turing mechanism can be presented as follows. It is expected that spatial local diffusion will eventually smooth out spatial patterns and produce a homogeneous landscape, or a flat landscape. Turing suggested that in reaction-diffusion, inhibitor-activator systems, where states move in space at different speeds, local diffusion might, under certain parameter values and contrary to what might have been expected, trigger the emergence of spatial patterns. To examine conditions that would generate spatial patterns, Turing suggested that it would be sufficient to study conditions under which a system which is at a stable spatially homogeneous steady state could be destabilized by the introduction of spatial diffusion of the system's states. The idea is that if the stable flat state is destabilized when the state starts moving in space due to the diffusion perturbation, then with the passage of time the system will not return to its original spatially homogeneous state and spatial patterns will start emerging.

The mechanism can be presented using system (3)-(4), by introducing, in addition to the Turing setup, economic behavior which is formulated by economic agents choosing the controls. The economic agents are located on the spatial domain and decide about harvesting at each site. That is, they choose controls at each site *z*.[11] Assume that economic agents choose the controls in (3)-(4) in a certain feedback form $u^x = g_1(x,y,\mathbf{b}), u^y = g_2(x,y,\mathbf{b})$, where **b** is a vector of economic parameters (e.g. prices, unit costs). The feedback controls could be the result of behavior such as optimization, imitation, rule of thumb, or open access competition. Then the system (3)-(4) can be written as:

$$\frac{\partial x}{\partial t} = F_1(x,y,\mathbf{b}) + D_x \nabla^2 x \qquad (15)$$

$$\frac{\partial y}{\partial t} = F_2(x,y,\mathbf{b}) + D_y \nabla^2 y \qquad (16)$$

where, $F_i(x,y,\mathbf{b}) \equiv f_i(x,y,g_i(x,y,\mathbf{b})), i = 1,2$.

To define a spatially homogeneous steady state or flat steady state (FSS), set $D_x = D_y = 0$ and then define the FSS as $(x^*, y^*) : F_i(x^*, y^*, \mathbf{b}) = 0, i = 1,2$. The FSS will be locally stable to temporal perturbation if the eigenvalues of the Jacobian matrix of the linearization of (15)-(16) evaluated at the FSS (x^*, y^*) are negative or have negative real parts. Let this Jacobian be

$$J(x^*, y^*, \mathbf{b}) = \begin{pmatrix} \frac{\partial F_1(x^*, y^*, \mathbf{b})}{\partial x} & \frac{\partial F_1(x^*, y^*, \mathbf{b})}{\partial y} \\ \frac{\partial F_2(x^*, y^*, \mathbf{b})}{\partial x} & \frac{\partial F_1(x^*, y^*, \mathbf{b})}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \qquad (17)$$

Therefore the linearization of (7)-(8) at the FSS will be:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (18)$$

[11] To simplify we can assume that each agent is located at each spatial point.

Assume that $\mathrm{tr}(J) = a_{11} + a_{22} < 0$, $\det(J) = a_{11}a_{22} - a_{12}a_{21} > 0$. This implies that the Jacobian matrix of the linearization has two real negative eigenvalues, thus the FSS is locally stable to spatially homogeneous perturbations. Turing's method is based on studying the stability of the FSS to spatially heterogeneous perturbations off the FSS.

This is obtained by transforming the infinite dimensional system (15)-(16) into a countable sequence of linear systems of ordinary differential equations so that linear stability analysis can be used. To obtain this the usual approach is to consider pairs of square integrable solutions $(x(t)(z), y(t)(z)) = (x(t,z), y(t,z))$ and construct trial solutions using an orthogonal basis of a Hilbert space of square integrable functions. This basis is created in terms of functions $\cos(kz), \sin(kz), z \in [-\pi, \pi]$, for mode *k* = 0, 1, 2, ... which form a complete orthogonal basis over $[-\pi, \pi]$. Our assumptions about functions $f_i, i = 1, 2$ suggest that the solutions $(x(t,z), y(t,z))$ of the system (15)-(16) will be smooth enough to be expressed in terms of a Fourier basis. In view of this, the approach is to introduce now spatial perturbations and consider spatial dependent solutions of the form:

$$x(t,z) = \sum_k c_{xk} e^{\sigma t} \cos(kz),\ y(t,z) = \sum_k c_{yk} e^{\sigma t} \cos(kz),\ k = \frac{2n\pi}{L},\ n = \pm 1, \pm 2, \ldots \quad (19)$$

where k is called the *wavenumber* and $1/k$, which is a measure of the wave-like pattern, is proportional to the wavelength ω : $\omega = 2\pi/k = L/n$ at *mode n*, where L is the length of the spatial domain. σ is the eigenvalue which determines temporal growth and c_{xk}, c_{yk} are constants determined by initial conditions and the eigenspace of σ. These trial solutions should be understood as deviations from the FSS (x^*, y^*).

In (19) the cosine terms express the spatial deviation from the FSS and the $\exp(\sigma t)$ term the temporal deviation. The spatial deviation, which is a wave like pattern, is the sum of sinusoidal components with each one having a wavelength L/n, with the wavelength being the distance between two sequential crests of the wave. In this case the mode corresponds to the wavelength of each component that forms the 'total' spatial deviation.

Assume that $-Z=0, Z=L$ so that the spatial domain has length L and that furthermore the spatial domain is a circle. Substituting (11) into (9)-(10) and noting that they satisfy circle boundary conditions at $z=0$ and $z=L$, we obtain the following result:

Behavior of economic agents as implied by choosing controls according to feedback rules $g_i(x,y,\mathbf{b}), i=1,2$ *in the management of a reaction-diffusion system, generates spatial patterns around a flat steady state if*

$$\frac{a_{22}D_x + a_{11}D_y}{2D_xD_y} > 0$$
$$\frac{(a_{22}D_x + a_{11}D_y)^2}{4D_xD_y} + \det J(x^*, y^*, \mathbf{b}) < 0. \tag{20}$$

For the proof, see Brock and Xepapadeas (2010, theorem 1).

If the above conditions are satisfied, then when the spatially *heterogeneous* perturbations are introduced, one of the eigenvalues of the linearization matrix of (17) is positive and therefore the steady state FSS (x^*, y^*) is locally unstable. This result means that once the state starts moving within the spatial domain with different speeds, then a spatial pattern starts emerging. This pattern will not die out but it will continue growing with the passage of time along the positive eigenvalue. Since the Jacobian matrix depends on the vector of economic parameters **b,** the economics of the problem contribute to the emergence or not of spatial patterns.

The local instability analysis around the steady state suggests that a spatial pattern starts emerging, but does not provide firm indications about the structure of the spatial pattern at which the system will eventually settle at the steady state, since the eigenvalues analysis of the linearized system is valid only in the neighborhood of the FSS.

The steady-state spatial pattern can be determined by solving the system (15)-(16) at a steady state where $\frac{\partial x}{\partial t} = \frac{\partial y}{\partial t} = 0$. Then the system becomes

$$\frac{\partial^2 x}{\partial z^2} = \frac{1}{D_x} F_1(x, y, \mathbf{b}) \tag{21}$$

$$\frac{\partial^2 y}{\partial z^2} = \frac{1}{D_y} F_2(x, y, \mathbf{b}) . \tag{22}$$

System (21)-(22) is a second order system of ordinary differential equations in the spatial domain. Solution of this system with appropriate spatial boundary conditions will provide the steady-state spatial pattern for the stocks of the system. This pattern is determined numerically most of the times so additional care should be taken when the results are interpreted, especially regarding the temporal stability of the steady-state spatial pattern.

A graph of emerging spatial patterns, which eventually converge to a spatially heterogeneous steady state, is shown in figure 3 for a state variable denoted by *P*.

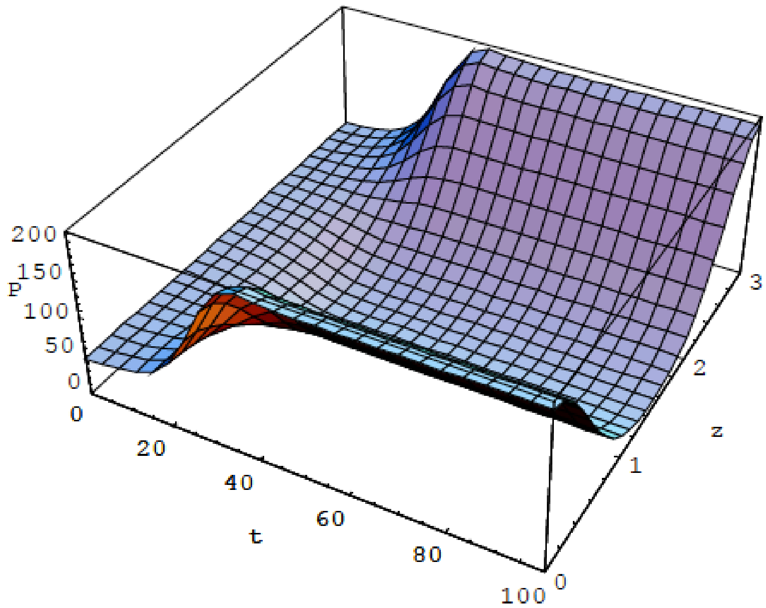

Figure 3: Emergence of pattern and spatially heterogeneous steady state

4.2 Optimal diffusion-induced agglomeration

In the analysis in the previous section, economic agents act myopically in both the temporal and the spatial dimensions, and do not take into account the spatial externality since they ignore the impact of their own harvesting on the stocks located on the sites of other agents. This impact emerges through the movement of stocks under the influence of diffusion. The spatial externality, as well as the temporal externality, can be taken into account by a social planner or a regulator that chooses the optimal control by solving problem (8)-(8a). The emergence of 'optimal spatial patterns' can be studied by applying Turing's approach to the Hamiltonian system of problem (8)-(8a), which can be written as:

$$\frac{\partial x(z,t)}{\partial t} = f\left(x(t,z), g^*(x,p)\right) + D_x \frac{\partial^2 x(z,t)}{\partial z^2}, x(0,z) = x_0(t), x(t,-Z) = x(t,z)$$
$$\frac{\partial p(z,t)}{\partial t} = \rho p - \frac{\partial \tilde{\mathrm{H}}\left(x(t,z), p(t,z), g^*(t,z)\right)}{\partial x} - D_X \frac{\partial^2 p(t,z)}{\partial z^2}$$
$$\lim_{T\to\infty} e^{-\rho T} \int_0^L p(T,z)x(T,z)dz = 0 \tag{23}$$
$$p(t,-Z) = p(t,Z).$$

In order to analyze pattern formation at the social optimum we examine the stability of a flat optimal steady state (FOSS) of the Hamiltonian system to spatially heterogeneous perturbations. A FOSS is a steady state where the state and the costate are spatially homogeneous. To ease notation the Hamiltonian system can be written in a compact way, where subscripts *t,z* denote partial derivatives with respect to *t* and *z* respectively, as

$$x_t = H_{p_x} + D_x x_{zz}$$
$$p_t = \rho p - H_x - D_x p_{zz}. \quad \ldots(24)$$

A FOSS is defined, from the Hamiltonian system (24), as a pair $(x^*, p^*,)$: $x_t = p_t = 0$ for $D_x = 0$. It is known from the work of Kurz (1968) that such a FOSS will either be unstable or will have the saddle point property. Assume that the FOSS (x^*, p^*) has the local saddle point property, which means that the Jacobian matrix of the linearization of (24) has one positive and one negative eigenvalue. To study pattern formation due to spatial diffusion around the FOSS, we linearize (24) at the FOSS and we introduce again spatially dependent solutions for the state and the costate which are expressed in terms of a Fourier basis, or

$$x(t,z) = \sum_k c_k^x e^{\sigma t} \cos(kz),\ p(t,z) = \sum_k c_k^p e^{\sigma t} \cos(kz),\ k = \frac{2n\pi}{L},\ n = \pm 1, \pm 2, \ldots$$

These trial solutions should be understood as deviations from the FOSS (x^*, p^*). The spatial pattern of the deviations is wave like, as discussed in section 4.1, and is determined by the sum of *k* components, each one corresponding to mode *n*. If for example $L = 2\pi$ then $k = n.$ Brock and Xepapadeas (2008) derive conditions under which the Jacobian matrix, which is defined for each mode of the linearized spatially perturbed system, may have two positive eigenvalues or complex eigenvalues with positive real parts at a certain mode *n*. This mode will be called an unstable mode.

This result implies that a spatial pattern is emerging for the optimally-controlled price quantity system at the unstable mode, where prices are represented by the costate variable and quantities by the state variable. This is because the component of the spatial pattern that corresponds to the unstable mode does not die out with the passage of time since it grows according to the positive eigenvalues. Thus the total patterned deviation from the FOSS does not die out as time goes by and therefore the system will not return to the FOSS.

This is the optimal diffusion-induced instability since it emerges in the context of a dynamic optimization problem, where the classic Turing diffusion-induced instability is not the result of dynamic optimization.

The intuition behind this result can be described as follows. Controlling the system to a FOSS along the stable manifold is costly for the social planner, especially when the uncontrolled system is close to instability. The optimal diffusion-induced instability can be regarded as the result of comparing the current costs of stabilizing the system versus the future benefits from stabilization. Depending on the relative costs and benefits of controlling a system to a spatially homogeneous steady state and the discount rate, it might be desirable in economic terms to have a spatially heterogeneous system. For sufficiently high discount rate the present costs of controlling the system to a flat optimal steady state might exceed the present value of future benefits. In this case it might be optimal to let the system become unstable and thus to cause the emergence of spatial patterns. On the other hand when the discount rate is close to zero, spatial patterns will tend to be ‘‘smoothed out’’.

The type of instability described here depends upon the trade-off between the current control cost and the present value of future gains from stabilization and can emerge even in systems with one state variable. Thus there is a difference between optimal diffusion induced instability and the standard Turing type diffusion induced instability which requires at least two state variables in the dynamics.

The local optimal instability analysis around the FOSS suggests that an optimal spatial pattern starts emerging but again does not provide firm indications about the structure of the spatial pattern regarding the state and the costate (shadow prices) in the long run. Some insights about the optimal long-run spatial pattern, if it exists, can be gained by solving the system (23) for $x_t = p_x = 0$. Then:

$$x_{zz} = -\frac{1}{D_z} H_{p_x}$$
$$p_{zz} = \frac{1}{D_x}\left(\rho p_x - H_x\right). \qquad (25)$$

System (25) is a second order system of ordinary differential equations for the state and costate, or the quantity-price system, in the spatial domain. Solution of this system with appropriate spatial boundary conditions will provide the optimal steady-state spatial pattern for the stock and its shadow price. As before, additional care should be taken when the results are interpreted, especially regarding the temporal stability of the steady-state spatial pattern.

Figure 4 shows a typical long-run steady-state spatial pattern in a space domain $Z = [0,4]$, for the state variable $x(z)$ and the corresponding costate $p(z)$ emerging from the problems studied in Brock and Xepapadeas (2008). The state variable (solid line) shows higher concentration in the middle of the spatial domain, while its shadow price (dashed line) shows a symmetrically opposite pattern. Both state and costate satisfy circle boundary conditions, that is, $x(0) = x(4)$, $p(0) = p(4)$.

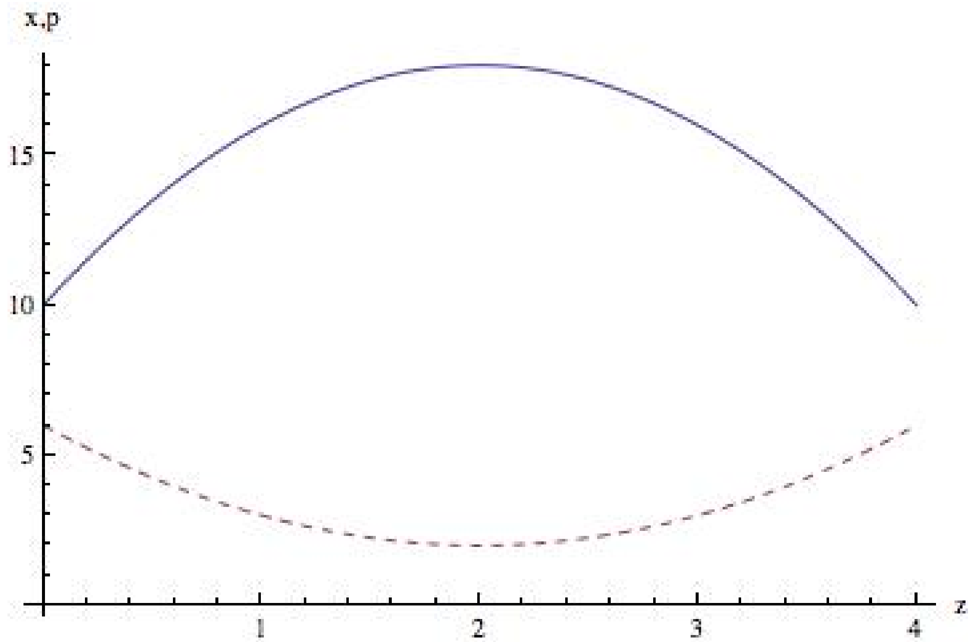

Figure 4: Spatially heterogeneous steady state for a state and a costate

4.2.1 Optimal agglomerations with reaction-diffusion systems

The results obtained in the previous section are not based on a reaction-diffusion system, which is the basis for studying pattern formation through the Turing mechanism, since they were derived from a dynamical system with one state, while a reaction-diffusion system requires at least two states.

The mechanism for generating spatial pattern in an optimizing model with one state variable is qualitatively different from Turing's original result, since it is not founded on the diffusion of two state variables at different speeds, but on the diffusion of one state variable and the diffusion in the opposite direction of its shadow price. The shadow price emerges as the costate variable of the optimal control problem.

The optimal agglomeration mechanism can be extended to a reaction diffusion system with two or more state variables. Thus we consider the reaction diffusion system (3)-(4) where a social planner or a regulator chooses optimally the controls $u^x = g_1(x, y, \mathbf{b}), u^y = g_2(x, y, \mathbf{b})$. [12]

The purpose is to maximize the present value of an objective over the entire spatial domain subject to the spatiotemporal evolution of the state variables. The planner's problem can be written as:

$$\max_{\{\mathbf{u}(t,z)\}} \int_0^{\infty} \int_0^{L} e^{-\rho t} \left[U\left(x(t,z), y(t,z), u^x(t,z), u^y(t,z)\right)\right] dzdt \tag{26}$$

subject to (3) – (4).

To use this maximum principle described above we introduce the Hamiltonian function:

$$\begin{aligned} H&\left(x, y, u^x, u^y, p_x, p_y\right) = U\left(x(t,z), y(t,z), u^x(t,z), u^y(t,z)\right) + \\ &p_x(t,z)\left[f_1\left(x(t,z), y(t,z), u^x(t,z), u^y(t,z)\right) + D_x \frac{\partial^2 y(t,z)}{\partial z^2}\right] + \\ &p_y(t,z)\left[f_2\left(x(t,z), y(t,z), u^x(t,z), u^y(t,z)\right) + D_y \frac{\partial^2 x(t,z)}{\partial z^2}\right] \end{aligned} \tag{27}$$

where $\mathbf{p} = (p_x, p_y)$ is the vector of the costate variables. The Hamiltonian function (27) is a generalization of the 'flat Hamiltonian'

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[12] A similar system has been analyzed by Brock and Xepapadeas (2010).

$$\mathrm{H} = U\left(x(t), y(t), u^x(t), u^y(t)\right) + p_x(t,)\left[f_1\left(x(t), y(t), u^x(t), u^y(t)\right)\right] + p_y(t,)\left[f_2\left(x(t,), y(t), u^x(t,), u^y(t)\right)\right] \quad (28)$$

for $D_x = D_y = 0$. The first-order conditions for the optimal control vector $\mathbf{u}^*(t,z) = \left(u^{x*}(t,z), u^{y*}(t,z)\right)$, assuming that the Hamiltonian function satisfies appropriate concavity assumptions, are defined, for interior solutions, by

$$\frac{\partial H}{\partial u^j} = 0\,,\ j = x, y. \quad (29)$$

Then the costate variables satisfy:

$$\frac{\partial p_j(t,z)}{\partial t} = \rho p_j - H_j\left(x(t,z), y(t,z), \mathbf{p}(t,z), \mathbf{g}^*(x,y,\mathbf{p})\right) - D_j \frac{\partial^2 p_j(t,z)}{\partial z^2}\ j = x, y \quad (30)$$

where $\mathbf{g}^*\left(x(t,z), y(t,z), \mathbf{p}(t,z)\right)$ is the vector of the optimal control functions defined by (29). The costates are interpreted as the shadow prices of the stocks and the negative sign of the diffusion coefficient means that prices and quantities move in the opposite directions in the spatial domain, for the optimally-controlled system. Finally the following temporal and spatial transversality conditions should be satisfied at the optimum:

$$\lim_{T\to\infty} e^{-\rho T}\int_0^L p_j(T,z)\, j(T,z)\, dz = 0\,,\ j = x, y$$
$$p_j(t,-Z) = p_j(t,Z). \quad (31)$$

The reaction-diffusion system of (3) and (4) with $\mathbf{u} = \left(u^x, u^y\right)$ replaced by the optimal controls $\mathbf{u}^* = \mathbf{g}^*\left(x(t,z), y(t,z), \mathbf{p}(t,z)\right)$ and the system of (30) constitute a system of four partial differential equations. This is the Hamiltonian system, which along with the initial conditions and the transversality conditions (31), determine the spatiotemporal evolution of the state and costate variables along the socially optimal path.

Writing the Hamiltonian in a more compact way we have:

$$\begin{aligned} x_t &= H_{p_x} + D_x x_{zz} \\ y_t &= H_{p_y} + D_y y_{zz} \\ p_{xt} &= \rho p_x - H_x - D_x p_{xzz} \\ p_{yt} &= \rho p_y - H_y - D_y p_{yzz}. \end{aligned} \quad \ldots(32)$$

A FOSS is defined, from the Hamiltonian system (32), as a quadruple $\left(x^*, y^*, p_x^*, p_y^*\right)$: $x_t = y_t = p_{xt} = p_{yt} = 0$ for $D_{x_1} = D_{x_2} = 0$. It is known from the work of Kurz (1968) that such a FOSS will either be unstable or will have the saddle point property. Assume that the FOSS (x^*, y^*) has the local saddle point property which means that the Jacobian matrix of the linearization of (20) has two positive and two negative eigenvalues. To study pattern formation due to spatial diffusion around the FOSS, we linearize (32) at the FOSS and we introduce again spatially dependent solutions for the state and the costate which are expressed in terms of a Fourier basis, or:

$$j(t,z) = \sum_k c_{jk} e^{\sigma t} \cos(kz),\ p_j(t,z) = \sum_k c^p_{jk} e^{\sigma t} \cos(kz)\ , k = \frac{2n\pi}{L},\ n = \pm 1, \pm 2, .., j = x, y.$$

These trial solutions should be understood as deviations from the FOSS (x^*, y^*). Brock and Xepapadeas (2010) derive conditions under which the Jacobian matrix of the linearized spatially perturbed system has three or four real positive eigenvalues or complex eigenvalues with positive real parts at some mode. When this happens the unstable mode grows with time and spatial patterns start emerging.

Insights regarding the optimal long-run spatial pattern, if it exists, can be determined by solving the system (32) for $x_t = y_t = p_{xt} = p_{yt} = 0$. In this case we have the following system in the spatial domain:

$$\begin{aligned} x_{zz} &= -\frac{1}{D_z} H_{p_x} \\ y_{zz} &= -\frac{1}{D_y} H_{p_y} \\ p_{xzz} &= \frac{1}{D_x}(\rho p_x - H_x) \\ p_{yzz} &= \frac{1}{D_y}(\rho p_y - H_y). \end{aligned} \qquad (33)$$

The results of this section imply that a spatial pattern may emerge for the optimally controlled price quantity system, where prices are represented by the costate variables and quantities by the state variables. This is again the optimal diffusion-induced instability since it emerges in the context of a dynamic optimization problem.

The analysis in this section also suggests how problems with many state variables can be analyzed, although increasing the state variables reduces the ability to obtain analytical results since the dimension of the Hamiltonian system at each mode will be high.

There is an interesting distinction between the one and the two state variable problems. When the system has one state variable, Turing instability cannot emerge in the uncontrolled system. Turing instability requires two state variables with different speed of moving across space. However diffusion-induced instability can emerge in the one state optimally controlled system. In this case the interaction between states and costates through the optimization process may, given appropriate discount rates, induce spatial clustering. This is because state and costate variables move in opposite directions within the spatial domain, as stated in section 3, and this type of movement 'mimics' the differential speed required for Turing instability.

The pattern, in the case of more than one state variable, emerges as a result of diffusion of the state variables and the spatial interactions of the price quantity system. Thus there is the possibility that the unoptimized reaction-diffusion system will provide a spatial pattern as a result of Turing diffusion-induced instability, while the optimized system will provide a different spatial pattern as a result of the diffusion-induced instability. This deviation can be regarded as a basis for studying spatially dependent regulation.

5. Summary and Conclusions

This chapter presented methods for analyzing coupled economic and ecological systems, which evolve in both the temporal and the spatial dimension. These methods could be useful in understanding the mechanisms that create spatial patterns and the design of spatial regulation. The approach presented in this chapter, which uses continuous spatial dynamics, differs from the main body of the existing literature, which uses metapopulations and discrete spatial-dynamic models (e.g Smith et al. 2009). This increases the mathematical complication since it introduces distributed parameter systems and optimal control of partial differential equations in modeling. On the other hand it significantly reduces the number of state variables involved in the optimal control problems resulting from metapopulation models, since these problems require one state variable for each patch. Thus the use of continuous spatial dynamics might make it easier to study models with multiple state variables with interaction among themselves.

This chapter presents ways to model short- and long-range spatial movements. Short-range movements, which relate more to ecological systems, are modeled through diffusion, linear or nonlinear, and partial differential equations. On the other hand, long-range movements are modeled through integral operators and integrodifferential equations, which can be regarded as a more appropriate method for analyzing economic phenomena such as spatial productivity or knowledge spillovers.

Whether the modeling is associated with short- or long-range spatial effects, the appropriate analytical framework is the framework of infinite dimensional systems. Thus for the case of short-range effects and diffusion, which is the main focus of the chapter, we present an extension of the maximum principle which can be used for the optimal control of partial differential equations with classic diffusion.

This maximum principle leads, however, to an infinite dimensional Hamiltonian system, which is very difficult to handle analytically. It is shown that by using a Fourier basis the infinite dimensional Hamiltonian system can be decomposed into a countable set of finite dimensional Hamiltonian systems which are indexed by mode $n = 0.1, 2, \ldots$. This decomposition allows us to study the stability of a spatially homogeneous - or flat - steady state to spatial perturbations. If there are unstable modes, then a spatial pattern starts emerging. This mechanism, which is essentially the Turing mechanism for diffusion-induced instability, is extended to optimizing systems. It is shown in this chapter how unstable modes can contribute to the emergence of optimal diffusion- induced instability or optimal agglomerations. Also shown is how, given the emergence of spatial patterns, insights - although no conclusive evidence - regarding the existence and the structure of spatially heterogeneous steady states can be obtained by studying a temporal steady state in the spatial domain.

An area of further research in this analytical framework is the combination of local and nonlocal effects. This approach might produce interesting results regarding the emergence of spatial patterns and agglomerations on combined ecological and economic models where short- and long-range effects coexist.

The introduction of spatial dynamics using differential or integral operators in the optimizing models studied in economics is still relatively new. It may, however, provide new insights regarding mechanisms generating spatial patterns and inequalities, which are issues of ongoing interest in economics.

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