DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Spatially Structured Deep Uncertainty, Robust Control, and Climate Change Policies

Anastasios Xepapadeas

Athanasios Yannacopoulos

Working Paper Series

18-07

March 2018

Spatially Structured Deep Uncertainty, Robust Control, and Climate Change Policies

Anastasios Xepapadeas* and Athanasios Yannacopoulos**
*Athens University of Economics and Business and University of Bologna
**Athens University of Economics and Business

March 27, 2018

Abstract

In view of the ambiguities and the deep uncertainty associated with climate change, we study the features of climate change policies that account for spatially structured ambiguity. Ambiguity related to the evolution of the damages from climate change is introduced into a coupled economy-climate model with explicit spatial structure due to heat transport across the globe. We seek to answer questions about how spatial robust regulation regarding climate policies can be formulated; what the potential links of this regulation to the weak and strong version of the precautionary principle (PP) are; and how insights about whether it is costly to follow a PP can be obtained. We also study the emergence of hot spots, which are locations where local deep uncertainty may cause robust regulation to break down for the whole spatial domain.

Keywords: Climate change, ambiguity, robust control, spatial regulation.

JEL Classification: Q54,R11,D81,C61

1 Introduction

Climate change as a global phenomenon has important spatial aspects which are related to both natural and economic forces. These forces shape the spatial distribution of impacts which eventually determine temperature and damages across locations. Thus, temperature and associated damage differentials are expected to be different across locations, with these differences implying important implications for policy design.

One of the important drivers of temperature spatial differentials is the so called polar amplification (PA) which relates to the well-established fact in the science of climate change is that when the climate cools or warms, high latitude regions tend to exaggerate the changes seen at lower latitudes.[1] The main PA causing mechanisms are: (i) the surface-albedo feedback (SAF) which can be traced back to Arrhenius (1897). The SAF mechanism suggests that initial warming in the North Pole will melt some of the Arctic's highly reflective (high albedo) snow and ice cover. This will expose darker surfaces which will absorb solar energy, leading to further warming and further retreat of snow and ice cover. (ii) an increase in the meridional heat transport (Langen and Alexeev 2007) which makes PA an inherent dynamical property of the system. PA makes therefore the temperature anomaly to be higher at the Poles relative to the Equator (See figure 2).

Damage differentials stem from economic-related forces which determine the damages that a regional (or local) economy is expected to suffer from a given increase in the local temperature. These damages depend primarily on the production characteristics (e.g., agriculture vs services) or local natural characteristics (e.g., proximity to the sea and elevation) and could affect output, utility or total factor productivity. The interactions between the spatially non-uniform temperature distribution and the spatially non-uniform economic characteristics ultimately shape the spatial distribution of temperature and damages.

The main driving force in the economics of climate change, during the recent decades has been the integrated assessment models (IAMs), such as the DICE, RICE models (see, for example, Nordhaus 2007, 2010). Some of these models (e.g., the RICE model) provide a spatial distribution of damages across the regions of the RICE model. However this model, as well as other IAMs, does not account for the natural mechanism - heat transfer or SAF - which induces PA and a non uniform temperature distribution across the globe.

In the terminology of climate science, IAMs with no spatial dimension are zero-dimensional models. Energy balance climate models (EBCMs) on

[1] Bekryaev et al. (2010), documents a high-latitude (> 60 N) warming rate of 1.36 C/century for 1875–2008, with the trend being almost two times stronger than the average Northern Hemisphere trend of 0.79 C/century.

the other hand - one- or two-dimensional - include heat transport and SAF across latitudes or across latitudes and longitudes (e.g., North 1975, North et al. 1981, Wu and North 2007) and induce a spatial structure which conforms to reality. One- and two-dimensional coupled economic-climate models have recently been developed (Brock et al. 2014, 2013). Among their most striking results are the generation of distributions of temperature, fossil fuel use, and damages across latitudes and time, which are derived from a social planner's optimization problem, as well as the characterization of spatially differentiated climate policy in the form of optimal carbon taxes.

Deep uncertainty in the context of climate change is mainly associated with the natural system, and characterizes an environment where ambiguity and concerns about model misspecification are present and significant. As Weitzman (2009) points out, the high structural uncertainty over the physics of environmental phenomena makes the assignment of precise probabilistic model structure untenable, while there is high sensitivity of model outputs to alternative modeling assumptions such as the functional form of the chosen damage function and the value of the social discount rate (e.g., Stern 2006, Weitzman 2010). High structural uncertainty implies inability, for a decision maker or regulator, to assign a unique probability distribution to stochastic factors affecting the dynamics of climate change and the damages that climate change may cause.

In particular, deep uncertainty or ambiguity can be regarded as a situation where a decision maker does not formulate decisions based on a single probability model but rather on a set of probability models. Gilboa and Schmeilder (1989) extended decision making under uncertainty by incorporating ambiguity and by moving away from the framework of *expected utility* maximization. They adopted a *maxmin* expected utility framework by arguing that when the underlying uncertainty of the system is not well understood and the decision maker faces a set of prior probability density functions associated with the phenomenon, it is sensible - and axiomatically compelling - to optimize over the worst-case outcome (i.e., the worst-case prior) that may conceivably come to pass. Doing so guards against potentially devastating losses in any possible state of the world, and thus adds an element of robustness to the decision-making process. Thus in situations characterized by deep uncertainty decision making should not rely on expected utility but, given that preferences exhibit ambiguity aversion, on *maxmin expected utility.*

Motivated by concerns about model misspecification in macroeconomics, Hansen and Sargent (2001a,b, 2008, 2012) and Hansen et al. (2006) extended Gilboa and Schmeidler's insights into dynamic optimization problems, thus introducing the concept of robust control to economic environments. A decision maker characterized by robust preferences takes into account the possibility that the model used to design regulation, call it benchmark or approximating model $\mathbb{P}$, may not be the correct one but only an approximation of the correct one. Other possible models, say $\mathbb{Q}_1, \ldots, \mathbb{Q}_J$, which surround $\mathbb{P}$, should also be taken into account with the relative differences among these models measured by an entropy measure. Hansen and Sargent (2003) characterize robust control as a theory "... [that] instructs decision makers to investigate the fragility of decision rules by conducting worst-case analyses," and suggest that this type of model uncertainty can be related to ambiguity or deep uncertainty so that robust control can be interpreted as a recursive version of maxmin expected utility theory.

It is clear from the discussion above that robust control approaches fit very well with climate change problems, as well as with more general environmental and resource economics problems, given the deep uncertainties associated with these issues.[2] For example a specific density function for climate sensitivity from the set of densities reported by Meinshausen et al. (2009) can be regarded as the benchmark model, but other possible densities should be taken into account when designing regulation. One of these densities that corresponds to the least favorable outcome regarding climate change impacts can be associated with the concept of the worst case.

The situation where a single model - or a unique prior - is sufficient for analyzing the phenomenon and formulating decision rules can be identified as the case of pure risk or measurable uncertainty where the decision maker is able to assign probabilities to outcomes. On the other hand the situation where the decision maker operates in the realm of many models - or multiple

[2] Issues of regulation under ambiguity have been studied using two main approaches: smooth ambiguity and robust control. Smooth ambiguity (Klibanoff, Marinacci and Mukerji 2005), parameterizes uncertainty or ambiguity aversion in terms of preferences and nests the worst-case, corresponding to robust control, as a limit of absolute ambiguity aversion. The approach has been used in climate change issues (e.g., Millner, Dietz, and Heal 2010), but questions regarding the calibration of the regulator's ambiguity aversion remain open. Robust control methods have been applied to climate change by Athanassoglou and Xepapadeas (2012).

priors - is the case of ambiguity or deep uncertainty. Under ambiguity the decision maker does not have the ability to determine a precise probability structure for the physical or the economic model, or to put it differently, to measure uncertainty using a single probability model.[3]

The inability to measure uncertainty can be viewed as associating decision making and regulation under ambiguity with the concept of a precautionary principle (PP).[4] Different formulations and versions of the PP can be found in the literature. Sunstein (2002-2003, 2007) discusses two versions of the PP: the weak PP where "lack of decisive evidence of harm should not be a ground for refusing to regulate"; and the strong PP, suggesting that when "potential adverse effects are not fully understood, the activities should not proceed." Sunstein regards the weak PP as sensible but the strong PP as a paralyzing principle.

When the spatial dimension of the climate change problem is brought into the picture, deep uncertainty acquires a spatial structure. In this case ambiguity about the natural system, the effectiveness of policies and potential damages are associated local characteristics. The emergence of the spatial structure for ambiguity can be associated with the fact that even if the approximating model of the regulator is the same for each location, locations could differ in terms of the worst-case model due to differences in the climate change physics, or economic characteristics across these locations. These differences cause the regulator to have different misspecification concerns for different locations and thus cause ambiguity to acquire a spatial structure. For the approximating model $\mathbb{P}$ and models $\mathbb{Q}$ surrounding it, this means that the local entropy balls containing the local $\mathbb{P}$s differ from location to location reflecting spatial differences in misspecification concerns and in worst-case priors.

Since there are spatial interactions across locations in terms of both the natural and the economic systems, regulation under localized ambiguity in a specific location will affect regulation in other locations operating under their own localized ambiguity conditions. If, as it is most likely, the "size" of ambiguity and ambiguity aversion is different across locations, then a spatial asymmetry is introduced into the regulatory process. This asymmetry is

[3]When the decision maker lacks adequate information to assign probabilities to events, we are in the realm of uncertainty as introduced by Frank Knight (1921).

[4]For example Weisbach (2012) studies whether environmental taxes should be precautionary.

induced by spatial differences in the “deepness” of uncertainty.

What kind of impacts could be expected from these interactions that might be of interest to regulators? Recent results on the robust control of spatiotemporal economic systems (Brock, Xepapadeas and Yannacopoulos 2013, 2014) suggest that deep uncertainty in certain locations might have a very important impact on the regulation for the whole spatial domain. This is because the regulator should design the robust rules not only with respect to the spatial characteristics of the problem in a specific location or the average characteristics of the whole spatial domain, but also with respect to the degree of the regulator's ambiguity for each specific location. This means that if deep uncertainty has a spatial structure across locations a spatially dependent robust rule is required in order to capture these differences.

This observation allows us to identify locations, referred to as *spatial hot spots,* which could be classified into two types:

> **Type I**: *Locations where robust control breaks down for the whole spatial domain because of extreme ambiguity aversion in these specific locations.*
>
> **Type II**: *Locations where robust control is very costly as a function of the degree of the regulator's ambiguities across all sites, relative to standard regulation under risk (measurable uncertainty).*[5]

In this analytical framework the purpose of the present paper is to present a one-dimensional spatial climate-economic model characterized by spatially structured deep uncertainty. Our objective is to explore the application of robust control methods to derive spatially structured climate policies and identify conditions for the emergence of hot spots as defined above. Furthermore, we seek to obtain insights regarding spatial robust control regulation which can be associated with a spatially structured concept of PP.

In the rest of the paper we present climate change policies with a focus on their spatial aspects, along with Along with the survey existing results we provide some new results regarding the solution of spatially structured robust control problems emerging form maxmin expected utility, and we discuss extensions and areas for further research.

[5] A type II hot spot can also be compatible with the case where robust control regulation results in a larger value for the system relative to regulation under risk.

2 Climate Change Policies

A general framework for climate change policies should consider three main types of policies that can affect climate change and its impacts:

(1) Mitigation that involves reduction in the flow of emissions of greenhouse gases (GHGs) and consequently the stock of accumulated GHGs in the atmosphere. A reduced stock of GHGs allows a larger flux of outgoing infrared radiation and thus less radiation is "trapped." This is expected to reduce pressures for temperature to increase.

(2) Adaptation that involves policies to cope with the detrimental impacts of climate change which cannot be avoided. The aim is to anticipate and adapt to the impacts in order to minimize their costs which may extend from the local to the international level. Adaptation is both a matter of need, as climate change is most likely unavoidable, and a matter of equity, as its impacts falls disproportionately on those least able to bear them. Therefore activities that use scarce resources to prevent damages from climate change can be considered adaptation.

(3) Geoengineering that involves methods that prevent GHGs from entering the atmosphere through carbon capture and storage (CCS) or carbon capture and sequestration, or methods of solar radiation management (SLR) that block incoming solar radiation by shading for example the earth from the sun through the spreading of reflective particles (e.g., Schelling 1996, Robock 2008, Shepherd 2009).

These policies can be considered as defining the foundation of a regulatory framework which affects the evolution of temperature and, through adaptation, reduces damages when increases in temperature become inevitable even after mitigation or geoengineering methods are applied.

The basic structure of the coupled economic climate system which includes climate change policies is presented in figure 1, which describes a climate module modelled by: an EBCM; an economic module, which is based on a standard neoclassical growth model; and their interactions. In this model climate change (i.e. increase in temperature) damages aggregate output and possibly reduces utility from consumption, while the economy generates emissions that increase the stock of GHGs and temperature.

[Figure 1]

Regulation can affect climate change and associated damages through a possible combination of mitigation, adaptation, and geoengineering. These policies are, however, costly in terms of output and may lead to further damages as for example in the case of SLR that involves pumping sulphur dioxide into the stratosphere. (see for example Barret et al. 2014, Manoussi and Xepapadeas 2015) In this paper we do not study all the aspects and interrelations of the coupled system but we discuss two important and interrelated aspects of the coupled system: spatial aspects and deep uncertainty in relation to regulatory policies.

The spatial aspects of climate change are related to natural or economic forces that shape the distribution of temperature and damages across locations. As shown in figure 1, the spatial structure of our model is induced by two main factors: the SAF and the transport of heat from the equator toward the Poles which produces a spatially non-uniform distribution of the surface temperature across the globe; and (ii) economic-related forces which determine the damages that a regional (or local) economy is expected to suffer from a given increase in the local temperature in terms of individuals' utility and global production, and productivity. These damages depend primarily on the production characteristics (e.g., agriculture vs services) or local natural characteristics (e.g., proximity to the sea and elevation) and have been estimated at a regional level by IAMs (e.g., the RICE model).

Deep uncertainty can be related to many aspects of the model. In the continuous debate about the climate change and its consequences the modelling and the description of damages seems to include significant and deep uncertainties, and IAMs have attracted lots of criticisms about the way the estimate damages and treat uncertainties associated with damages. Thus although deep uncertainty can be associated with the evolution of the natural system and its impacts, such as for example the effectiveness of climate change policies (mitigation, geoengineering) in affecting the rate of change of temperature across the globe, or the effectiveness of adaptation in restricting damages due to climate change, we choose to associate uncertainty with damages. In particular we assume that damages affect output, following the RICE, DICE approach, but we treat damages as a state variable modelled by a stochastic process with drift and volatility. The drift depends on the evolution of local temperature, as well as the impacts of local adaptation and potential local effects from SLR methods.[6] The regulator (or a social planner) has concerns about model misspecification related to the evolution of damages and is not able to assign a unique probability model to stochastic factors affecting the dynamics damages that climate change may cause to output, as shown in figure 1. Given the spatial structure of the model it is clear that misspecification concerns acquire local characteristics and may differ from location to location.

Since climate change policies are expected to affect the rate of change and the distribution of temperature, as well as the distribution of damages across the globe, a regulatory framework based on global averages might not be efficient relative to regulation that depends explicitly on local characteristics. Potentially important questions in this context could be how spatial robust regulation regarding climate policies can be formulated, what the potential links of this regulation to the PP are, and how some insights about whether it is costly to follow a PP can be obtained.

3 Spatially Structured Ambiguity, Precaution and Climate Change

Ambiguity and concerns about model misspecification underlying natural systems can be manifested in many probability models. The decision maker cannot choose one of them to define expected utility, but the emergence of the worst-case model could lead to severe damages or irreversible change. To prevent these damages, which are not clearly demonstrated since the decision maker does not know that the worst-case model will prevail, precaution might be desirable in designing specific policy rules, which implies that the decision rule should take into account the worst-case scenario. The maxmin expected utility could be used as a conceptual framework for designing good or robust management rules which will work reasonably well given the multiplicity of the possible models.[7] The worst case which is one of many possible models that may prevail, cannot be demonstrated clearly; therefore robust control can be regarded as adhering to a precautionary

[6] Geoengineering in the form of SLR is very likely to create environmental damages such as ocean acidification or acid depositions.

[7] Approaches such as minimax regret, or H_∞ regulation, can also be considered. We follow the maxmin criterion suggested by robust control, since it clearly defines a regulatory framework that can incorporate spatially structured ambiguity.

behavior under conditions of deep uncertainty and ambiguity aversion.

Being robust and precautious in policy design under ambiguity can be relevant and potentially desirable, for example in the current discussion about whether to take strong action now or have a gradual response regarding policies to address climate change, given the uncertainties associated with the issue. However, being robust and precautious could also be costly in the sense of Sunstein's (2002-2003) paralyzing situation where potential benefits are foregone due to inaction, or costly stringent regulation is called for. In such a case, a policy maker should address the relation between deep uncertainty and the structure or the limits of regulation, given a measure of the "severity" of deep uncertainty.

Assume that the regulator has a benchmark or approximate model $\mathbb{P}$ surrounded by other possible models, say $\mathbb{Q}_1, \ldots, \mathbb{Q}_J$, with the difference between $\mathbb{P}$ and $\mathbb{Q}$s measured by relative entropy. The worst-case model that the decision maker is willing to consider, given the existing knowledge and information, is the one differing the most from $\mathbb{P}$ in terms of entropy. Thus the size of ambiguity can be regarded as the length of the radius H of the entropy ball that surrounds $\mathbb{P}$.

A fundamental parameter in robust control problems is the weight, or the penalty parameter or the robustness parameter, assigned by the regulator to the possibility that the chosen probability model might not be the correct one. Equivalently the penalty parameter can be related to a measure of "how far" the worst-case climate sensitivity density can be from the benchmark sensitivity. Given a benchmark probability model in the climate change problem, the regulator can in principle approximate - given the existing knowledge - the deviation between the benchmark and the worst-case model, and determine the extra constraint that deep uncertainty imposes on the regulatory processes. The impact of this extra constraint on regulation can be associated with both the weak and the strong versions of the PP.

Regulation designed subject to the constraint that an appropriately defined worst case may emerge can be associated with the weak version of the PP in the sense. Although there is not decisive evidence that damages associated with the benchmark model will not be exceeded, since the worst case model may emerge; even if the worst case emerges robust regulation can provide an acceptable outcome. So the possibility of having a worst case does not prevent regulation and does not prevent regulated activities to continue. On the other hand, if the deviation between the benchmark and the worst-case distribution exceeds a threshold, then robust control regulation is not possible because the impact of the worst-case distribution is so large that regulation using the maxmin expected utility criterion is meaningless. This is because the worst case is so far from the benchmark case, i.e. H is so large, that maximization over the worst case is not possible. This breakdown can be viewed as a situation where an adversarial agent who is trying to minimize the regulator's objective can choose a worst case which is so “bad” that it will create a very large loss for the regulator or, put differently, it will push the regulator's objective to minus infinity. In such a case, any maximization on the regulator's part would be meaningless. Such a regulation breakdown due to deep uncertainty could be associated with the strong version of the PP in the sense that the possibility of very large losses implies effects are not fully understood and since regulation is not possible activities should stop. This "stalemate" suggests the need for actions such as acquiring more information that might reduce the entropy ball, thus allowing regulation in the spirit of the weak PP, or completely changing the regulatory model.

When the spatial dimension of the climate change problem is introduced, deep uncertainty acquires a spatial structure. In such a case, concerns and ambiguity about climate change damages and their distribution across the globe, introduce deviations between the local benchmark model and the worst case for the specific location. In this case there will be a benchmark model P_n of damage evolution at each $n = 1, ..., N$ location and a set of possible models $\mathbb{Q}_n = (Q_{1n}, \ldots, Q_{Jn})$. It will be reasonable to assume that even if the benchmark model is the same across locations, the entropy ball surrounding each benchmark model need not be the same. More precisely, the radius of the entropy ball will be different across locations, i.e. $H_n \neq H_m$, $n \neq m$, $n, m = 1, ..., N$. This observation suggests a spatial structure to ambiguity, which is induced by differences in the “deepness” of uncertainty across locations and by spatial interactions of the natural and the economic systems. The spatially structured ambiguity is shown in figure 2.

[Figure 2]

Due to the local interactions, regulation under the local worst-case constraint in a specific location will affect regulation in other locations operating under their own local worst-case constraint. Thus spatially structured ambiguity is expected to induce spatially dependent robust regulation in terms of mitigation, adaptation and geoengineering, which will reflect the structure of deep uncertainty regarding local damages. This type of regulation can be associated with a localized weak version of the PP. A spatially dependent climate policy emerges therefore in the context of an economic EBCM with spatially structured ambiguity. In terms of policy design this is a departure from the spatially uniform policies mostly suggested by the IAMs. The rigorous formulation of optimal spatially robust climate policies is not an easy task given the complexity of the model. A first attempt to address this issue is presented in the appendix.

3.1 How Costly is Robust Control Regulation and the Weak PP?

Because the constraint imposed by the worst-case model should be accounted for, robust control regulation (or regulation by following a weak PP) is different from regulation under risk, which is the case of measurable uncertainty where it is accepted that the regulator trusts the benchmark model.[8] Therefore, one way of answering the question of how costly it is to follow a weak PP, is to compare robust control regulation in climate change under deep uncertainty, with regulation under risk which might be regarded as the "benchmark regulation." A way of performing this comparison is by comparing the optimized value of the regulator's objective under robust control regulation with the corresponding optimized value of the objective under benchmark regulation (see appendix, section 5.4 for details). Optimized objective in the case of climate change means the maximized global discounted value of utility less damages from climate change under the regulatory scheme (see figure 1). If the optimized objective under robust control is less than the optimized objective under benchmark regulation, the difference between the two maximized objectives can be interpreted as the cost of following the weak PP.

[8] Confidence in the benchmark model means that $H_n = 0$ for all n.

3.2 Spatially Structured Ambiguity and Hot Spots

What are the features of spatially robust climate policies that might be of interest to regulators? Recent results on the robust control of spatiotemporal economic systems (Brock, Xepapadeas and Yannacopoulos 2013, 2014) suggest that deep uncertainty in certain locations might have a very important impact on the regulation for the whole spatial domain. This is because, given the spatial structure of ambiguity in terms of worst-case models, the regulator designs the robust rules, not only with respect to the spatial characteristics of the problem in a specific location or the average characteristics of the whole spatial domain, but also with respect to the degree of the regulator's ambiguity - the radius of the entropy ball - for each specific location.

This observation allows us to identify locations, referred to as *spatial hot spots,* which are classified into two types (see appendix for details):

> **Hot Spot Type I** : *Locations where robust control breaks down for the whole spatial domain.*
>
> **Hot Spot Type II** : *Locations where robust control is very costly as a function of the degree of the regulator's ambiguities across all sites, relative to standard regulation under pure risk.*

A type I hot spot is a location where the deviation between the benchmark and the worst-case model exceeds a threshold, which causes the regulation for the whole spatial domain to break down. This is because mistrust of the benchmark model for this specific location is so large that it makes regulation meaningless in the sense that the worst case for this specific location will push the regulator's objective to minus infinity. Since this location is linked to the rest of the locations in the spatial domain, and regulation should be designed for the whole domain, the severe ambiguities of the hot spot are "transmitted" to the rest of the domain, thus making regulation impossible. Thus an I hot spot can be associated with the strong PP.

A type II hot spot is a location where, because of ambiguity, the maximized value of the regulator's objective under robust control is substantially lower, relative to regulation under pure risk. This means that for a given level of precaution, defined by the worst-case choice in each location, regulation for the whole spatial domain is costly due to deep uncertainties in a specific location. This could happen if precaution induces costly robust policies relative to benchmark regulation, while the expected savings in terms of damages are not sufficiently large. The emergence of a II hot spot implies that the mistrust of the benchmark model and worst-case considerations in a specific location create an interesting trade-off between the weak PP and the cost it implies.

The two types of hot spots and the associated domains for the weak and strong PP are shown in figure 3.

[Figure 3]

A reversal of the type II may also be possible. This occurs if the value of the regulator's maximized objective under robust control is relatively higher than the corresponding value when the regulator is using the benchmark model. This could happen if the policies adopted are more costly under robust control relative to benchmark regulation, but they generate relatively larger benefits in terms of expected damage savings. A reversal of the type II may be associated with the concept of optimal precaution which is the level of precaution defined by the worst-case choice that maximizes the regulator's objective.

4 Concluding Remarks

This paper discusses issues which arise in the process of regulating a coupled economic climate system when (i) there is ambiguity and concerns about model misspecification (or deep uncertainty) associated with the mechanisms of the natural system, and (ii) there are spatial interactions between the natural and the economic systems. We explore the implications of deep uncertainty and spatial interactions on climate change policies and link them to the PP. Under deep uncertainty climate change policies can be regarded as precautionary.

The main result is that the combination of deep uncertainty and spatial interactions induces spatially structured ambiguity which is an important characteristic for designing climate change policies. This spatial structure may cause certain locations to emerge as spatial hot spots. The existence of hot spots introduces a potentially important relationship between local interactions and global regulation. It has recently been argued (e.g., Haldane

2009) that increased interconnectedness among networks has made various networks - such as ecological networks, power grid networks, transportation networks, and financial networks - more unstable. This interconnectedness and the potential instabilities induced can be associated with the hot spots discussed in this paper and the impact of local properties on global regulation.

In terms of climate change policies, given the existence of deep uncertainties associated with various components of the system and the spatial interrelations between the natural and the economic systems, these observations give rise to a large number of questions. For example, how large a cost are we willing to incur in order to be precautious? Should we advocate uniform or spatially differentiated carbon taxes or other mitigation policies? How will deep uncertainties associated with the impact of solar radiation management methods affect policies based on solar radiation management? Is it likely that deep uncertainty in a specific location will cause regulation using a specific instrument to break down globally? What is the proper response in this case: do we immunize the whole system with respect to the specific location - if this is feasible - or do we look for a qualitatively different policy framework?

The general framework of spatial robust control regulation described here could provide insights and ways of formulating informed answers to these questions.

5 Appendix

This appendix provides a technical description of the coupled economy-climate model presented in figure 1 and the associated robust control problem. It extends Brock, Engström and Xepapadeas (2014) by introducing: (i) spatially structured ambiguity, and (ii) geoengineering and adaptation expenses as additional climate change policy instruments.

5.1 Temperature and GHGs Dynamics

We develop a one-dimensional EBCM model with human inputs. We assume that the surface temperature)T depends upon location ϕ, $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and time $t \geq 0$. We use $x = \sin(\phi) \in [-1, 1]$, with $x = \pm 1$ corresponding to the North and South Pole respectively, while $x = 0$ corresponds to the Equator.

We refer to x as latitude, and denote by $T(x,t)$ the surface temperature in $^{\circ}$C at latitude x at time t. The temperature is affected by heat transfer due to thermal diffusion and the solar energy input in the atmosphere, and by human actions (GHGs emissions and geoengineering). Following Wu and North (2007) the basic energy balance equation with human input added can be written in terms of a partial differential equation (PDE) connecting the temporal and spatial rates of change of the temperature, $\partial T(x,t)/\partial t$, with the various processes, which is of the form

$$C_c \frac{\partial T(x,t)}{\partial t} = D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right] - [A + B\,T(x,t)] + Q\,S(x)\alpha(x,T(x,t)) \\ - \psi(Z) + g(M(t)), \tag{1}$$

with initial condition $T(x,0) = T_0(x)$, and a boundary condition stating that the flux at the boundary vanishes, i.e., T must be such that:

$$\sqrt{1-x^2}\frac{\partial T(x,t)}{\partial x} = 0 \text{ for } x = \pm 1 \text{ for all } t \geq 0.$$

The terms on the first line of the right hand side of (1) correspond to non-human sources that affect the temperature dynamics, while the second line collects all the human-related sources. The term $D\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial T(x,t)}{\partial x}\right]$ is the effect of thermal diffusion effects with D a heat transport coefficient; $-[A + B\,T(x,t)]$ is the rate of outgoing infrared radiation to space with A and B empirical coefficients; $Q\,S(x)\alpha(x)$ models absorption effects from solar energy; $S(x)$ is the mean annual distribution of solar radiation energy; $\alpha(x, T(x,t)$ is the co-albedo which is one minus the albedo of the earth–atmosphere system, with $\frac{\partial \alpha}{\partial T} > 0$ indicating that an increase in local temperature increases the absorption of solar energy; and C_c is the effective heat capacity per unit area of earth atmospheric system.

The term $-\psi(Z(t))$ models the reduction in incoming solar radiation due to geoengineering activities of total scale $Z(t) = \int_{-1}^{1} z(x,t)\,dx$, where $z(x,t)$ denotes local geoengineering. The global concentration $M(t)$ of GHGs at time t reduces outgoing radiation thus increasing temperature. The term $g(M)$ models the effect that accumulated GHGs have on the reduction of the outgoing radiation. We assume that $g(M(t)) = \xi \ln\left(\frac{M(t)}{M_0}\right)$ where M_0 denotes the preindustrial concentration of GHGs, and ξ is a temperature-forcing parameter. GHGs emissions are assumed proportional to the amount

$q(x,t)$ of fossil fuel used in production process and $M(t)$ evolves according to:

$$\frac{d}{dt}M(t) = \beta Q(t) - \delta_m M(t), \tag{2}$$

where $Q(t) := \int_{-1}^{1} q(x,t)\, dx$ is the global quantity of fossil fuel used, β is independent of x and t, and δ_m is a natural decay rate for the GHGs.

To facilitate the exposition and numerical analysis we obtain a finite dimensional model by discretization of the infinite dimensional dynamical system described above. This is done by approximating the continuous space $[-1,1]$ by a one-dimensional discrete finite lattice with N points $x_n \in X = \{x_1, ..., x_N\}$, $n = 1, ...N$, with $x_1 = -1$ and $x_N = 1$. We approximate the function $T(x)$ by a vector $T = (T_1, \cdots, T_N) \in \mathbb{R}^N$, where $T_i \simeq T(x_i)$, $i = 1, \cdots, N$. By approximating the spatial derivatives with finite differences and choosing appropriate boundary conditions, the PDE (1) is transformed to a system of coupled ODEs in $\mathbb{R}^N$, of the form

$$C_e T_i' = D(a_{i,i+1}T_{i+1} + a_{i,i}T_i + a_{i,i-1}T_{i-1}) - [A + BT_i] + QS_i a_i - \psi(Z) + g(M),$$

$\forall\ i = 1, \cdots, N$ where $T_i' = \frac{dT_i}{dt}, a_{i,j}$ $j = i, i \pm 1$ (the nearest neighbors of the site i) are real numbers chosen so as to obtain the best possible approximation for the second derivative, and

$$Z = \sum_{i=1}^{N} z_i,$$
$$M' = \beta \sum_{i=1}^{N} q_i - \delta_M M,$$

where $z_i(t) = z(x_i, t)$ and similarly for all the other functions.

The discretized system can then be written in compact form, using $\mathbb{I}_N$, the identity matrix in $\mathbb{R}^N$, and the vector $\mathbf{1}_N = (1, \cdots, 1) \in \mathbb{R}^{1\times N}$, as

$$T' = \mathsf{A}_d T - \mathsf{A}_r T - \mathsf{B}_z(Z) + \mathsf{B}_e(M) + F, \tag{3}$$

where $T = (T_1, \cdots, T_N)^{tr}$, A_d is the diffusion matrix and corresponds to the discretization of the diffusion operator, $\mathsf{A}_r = B\mathbb{I}_N T$, $\mathsf{B}_z(Z) = \psi(Z)\mathbf{1}_N^{tr}$ is the geoengineering term which models the effects of global geoengineering on temperature, $\mathsf{B}_e(M) = g(M)\mathbf{1}_N^{tr}$ is the term modeling the effect of GHGs on climate and $F = (-A + QS_1 a_1, \cdots, -A + QS_N a_N)^{tr}$, all properly scaled by C_e. We use the vectors $z = (z_1, \cdots, z_N)^{tr}$ and $q = (q_1, \cdots, q_N)^{tr}$ and $\mathbf{1}_N$, to express $Z(t) = \sum_{i=1}^{N} z_i(t) = \mathbf{1}_N z = \mathbf{1}_N^{tr} \cdot z$ and similarly,

$$M' = \beta \mathbf{1}_N q - \delta_m M. \tag{4}$$

We end up with the controlled dynamical system (3), (4) in $\mathbb{R}^N \times \mathbb{R}$; $(T, M) \in \mathbb{R}^N \times \mathbb{R}$ being the state variable and $(z, q) \in \mathbb{R}^N \times \mathbb{R}^N$ being the control variables.

5.2 The Global Economy

We assume a representative household at location i having preferences described by the utility function

$$U(c_i(t)/\ell_i) = \frac{[c_i(t)/\ell_i]^{1-\gamma} - 1}{1-\gamma} \tag{5}$$

where $c_i(t)$ and $\ell_i(t)$ are consumption and the size of the representative household (equal to population) at time t for location i, respectively. Labor, supplied inelastically, is equal to population which is assumed constant to simplify the model.

Production takes place at each location i, according to a production function

$$y_i(t) = \Omega(T_i(t), \phi_i(t), Z(t)) F(k_i(t), \ell_i, q_i(t)) \tag{6}$$

where $k_i(t), \ell_i(t), q_i(t)$ denote capital, labor and fossil fuels respectively used at point i, time t. F is a standard Cobb-Douglas, which is multiplied by a damage function Ω modelling the effects of climate change on the economy. Local damages depend on the local temperature T_i, local adaptation expenses ϕ_i that mitigate damages, and global geoengineering activities $Z(t)$. It is assumed that:[9]

$$\frac{\partial \Omega}{\partial T_i} < 0, \quad \frac{\partial \Omega}{\partial \phi_i} > 0, \quad \frac{\partial \Omega}{\partial Z} \geq 0.$$

As shown in Brock Engström and Xepapadeas (2014), for the global economy, the potential world GDP can be defined as the maximum output

[9] The local damage function depends on global geoengineering activities to allow for negative externalities at i due to other regions' geoengineering activities.

that can be produced with fixed and immobile labor, given total capital $K(t) = \sum_{i=1}^{N} k_i(t)$ available and total fossil fuel $Q(t) = \sum_{i=1}^{N} q_i(t)$ used, for a given distribution of temperature $T(t) = (T_1(t), \cdots, T_N(t))$, or

$$\hat{Y}(t) = \Omega\left(\{T_i(t), \phi_i(t)\}_{i=1}^{N}, Z(t)\right) F(K, Q). \tag{7}$$

The global budget constraint is:

$$K'(t) = \hat{Y}(t) - [C(t) + Z(t) + \Phi(t) + \delta K(t)] \tag{8}$$

where $C(t) = \sum_{i=1}^{N} c_i(t), c = (c_1, ..., c_N), Z(t)$ denote global consumption and geoengineering expenses respectively, δ is the depreciation rate, and $\Phi(t) = \sum_{i=1}^{N} \phi_i(t), \phi = (\phi_1, ..., \phi_N)$ denote global adaptation expenses.

Under certainty a social planner will choose paths for (c, q, z, ϕ) to maximize global discounted utility

$$J = \sum_{i=1}^{N} \int_0^\infty e^{-\rho t} \omega_i \ell_i U(c_i(t)/\ell_i)\, dt \tag{9}$$

subject to (3),(4),(8), and initial and boundary conditions where $\omega = (\omega_1, \cdots, \omega_N)$ are time independent welfare weights associated with the utility of consumption of each location.

Remark: *From optimal growth models with heterogenous agents (e.g. Lucas and Stokey 1984) we know when global utility is time additive like (9), then if all agents, that is locations in our case, have the same utility discount rate, ρ, the welfare weights are fixed and equal to the initial weight vector ω_0. With recursive preferences which are non time additive, the assumption of equal discount rates can be relaxed, but in this case welfare weights are time dependent and need to be determined by the solution of the problem. At this stage of the problem we assume that the social planner fixes the welfare weights for the whole planning horizon. This assumption can be justified that the planner is not willing through climate policy the welfare weights which are determined no more general distributional grounds among countries.*

5.3 Spatially Structured Uncertainty

Uncertainty is associated with damage dynamics. We assume that the social planner does not formulate decisions regarding the paths for (c, q, z, ϕ)

based on a single probability model but on a set of probability models regarding climate change damages for each location $i = 1, ..., N$. The planner formulates a benchmark model for climate damage dynamics defined as:

$$d\Omega_i(t) = \zeta_i \left(T_i(t), \phi_i(t), Z_t\right) \Omega_i(t) dt + \Omega_i(t) \sum_{j=1}^{N} \sigma_{ij} dw_j(t). \tag{10}$$

where now $w_1, \cdots, w_j$ are independent Wiener processes and the σ_{ij} introduce the spatial correlation.

In this formulation the drift of the damage process depends on local temperature, local adaptation and global SLR. It is reasonable to assume that $\frac{\partial \zeta}{\partial T} > 0$, $\frac{\partial \zeta}{\partial \phi} < 0$ and $\frac{\partial \zeta}{\partial Z} \geq 0$. The last derivative is based on the idea that global SLR activities may reduce temperature but may create collateral damages to specific locations. The planner has concerns about model misspecification and is willing, for each location, to consider local damage dynamics models $\mathbb{Q}_i$ which are within an appropriately defined entropy ball centered at the benchmark model $\mathbb{P}_i$. The entropy constraints can be written as:

$$\mathcal{Q} = \{\mathbb{Q} : \mathcal{H}(\mathbb{Q}_i \mid \mathbb{P}_i) \leq H_i, \;\; i = 1, \cdots, N\} \tag{11}$$

where H_i is the radius of the entropy ball indicating the planner's mistrust in the benchmark model, and correspond to the global entropy constraints for GHGs and capital stock dynamics. If $H_i = 0$ the planner trusts the i-th benchmark model and has no concerns about model misspecification. Since in general $H_i \neq H_j$ for $i \neq j$ $i, j = 1, ...N$, ambiguity acquires a spatial structure since the planner has different degrees of model mistrust across locations. If $H_i > H_j$ the planner trusts the benchmark model in location i less relative to location j.

Following Hansen and Sargent, misspecification concerns can be modelled as drift distortions of a multivariate Wiener process associated with the stochastic factors affecting damage dynamics, or

$$d\Omega_i(t) = \left[\zeta_i \left(T_i(t), \phi_i(t), Z(t)\right) + \sum_{j=1}^{N} \sigma_{ij} \upsilon_j(t)\right] \Omega_i(t) dt + \Omega_i(t) \sum_{j=1}^{N} \sigma_{ij} dw_j(t) \tag{12}$$

where $\upsilon = (\upsilon_1, \cdots, \upsilon_N)^{tr}$ is the vector of distortions of the benchmark models, w is a vector Wiener process $w = (w_1, \cdots, w_N)^{tr}$ associated with *local* sources of uncertainty with joint distribution $\mathsf{N}(0, \mathbb{I}_N\, t)$ where $\mathbb{I}_J$ is the $N \times N$ identity matrix, and $\sigma = [\sigma_{ij}] \in \mathbb{R}^{N\times N}$. The stochastic shock affecting the damage dynamics in the i-th location can written as $\Omega_i \sum_{j=1}^N \sigma_{ij} dw_j$ and σ can be interpreted as a spatial autocorrelation matrix characterizing interrelations among local damages, while$\sum_{j=1}^N \sigma_{ij} \upsilon_j$ is the corresponding drift distortion for damages in location i.

In this set-up the multiplier robust control problem can be written as

$$\max_{(c,q,z,\phi)} \min_{\upsilon} \mathbb{E}_Q \left[\int_0^\infty \mathrm{e}^{-\rho t} \sum_{i=1}^N \omega_i \left[\ell_i U \left(\frac{c_i(t)}{\ell_i} \right) + \frac{\theta_i}{2} \left(\sum_{j=1}^N \sigma_{ij} \upsilon_j \right)^2 \right] dt \right]$$
$$\text{subject to temperature dynamics, GHG dynamics, and (12).} \tag{13}$$

where the adversarial agent chooses distortions υ to minimize the planner's objective. The parameters θ, the robustness parameters, can be regarded as the Lagrangean multipliers associated with the entropic constraints (11). The N multipliers $\theta_1, \cdots, \theta_N$ correspond to the local entropic constraints for damages.

Remark *The exact value of the Lagrange multipliers $\theta_i \geq 0$ depends on the radius of the local entropy balls, i.e., on the value of H_i. Since $\theta_i^2 > C\,\frac{1}{H_i}$, the limit $\theta_i \to \infty$ corresponds to $H_i \to 0$, which is the case where the planner trusts the benchmark model and has no concerns about model misspecification. We call this limit the risk limit, in the sense that there is noise present but the benchmark model P is trusted. The opposite limit $\theta_i \to 0$ corresponds to the case where $H_i \to \infty$, therefore the planner has very little trust in the benchmark model and allows for very large model misspecification. We call this limit the deep uncertainty limit.*

The stochastic differential game (13) can be solved by using the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. This is expressed in terms of the generator operator which is a second order differential operator $\mathcal{L}$, acting on the value function $V = V(T, M, K, \Omega)$, $V : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \to \mathbb{R}$ of the game. It can be expressed as

$$\mathcal{L} = \mathcal{L}^{(T)} + \mathcal{L}^{(M)} + \mathcal{L}^{(K)} + \mathcal{L}^{(\Omega)} + \mathcal{L}_n, \tag{14}$$

where

$$
\begin{aligned}
\mathcal{L}^{(T)}V &= (\mathsf{A}_d T - \mathsf{A}_r T - \mathsf{B}_z(\mathbf{1}_N z) + \mathsf{B}_e(M) + F) \cdot \mathsf{D}_T V \\
\mathcal{L}^{(M)}V &= (\beta \mathbf{1}_N q - \delta_m M)\mathsf{D}_M V \\
\mathcal{L}^{(K)}V &= (\Omega(T, \phi, \mathbf{1}_N z)\, F(K, \mathbf{1}_N q) - \delta K - [\mathbf{1}_N c + \mathbf{1}_N z + \mathbf{1}_N \phi])\, \mathsf{D}_K V \\
\mathcal{L}^{(\Omega)}V &= \sum_{i=1}^{N}(\zeta_i + \sum_{j=1}^{N}\sigma_{ij}\upsilon_j)\Omega_i \frac{\partial V}{\partial \Omega_i} = diag(\Omega_i)(\zeta + \sigma\upsilon)\mathsf{D}_\Omega V, \\
\mathcal{L}_n V &= \frac{1}{2} Tr\left(\sigma\sigma^{(tr)}\mathsf{D}^2_\Omega V\right),
\end{aligned}
$$

$\mathsf{D}V^{tr} = (\mathsf{D}_T V, \mathsf{D}_M V, \mathsf{D}_K V, \mathsf{D}_\Omega V)$ is the gradient of V with respect to (T, M, K, Ω) (e.g., $\mathsf{D}_M V = \frac{\partial V}{\partial M}$ and similarly for $\mathsf{D}_K V$), and $\mathsf{D}^2_\Omega V \in \mathbb{R}^{(N)\times(N)}$ is the Hessian matrix, consisting of all the second derivatives of V with respect to Ω. Since the variance of the system dynamics does not depend on the controls, and the decisions regarding (c, q, z, ϕ) and υ separate, the time protocol regarding maximization and minimization decisions does not matter, so the min, max operators can be interchanged. This means that the robust control static game has a Nash equilibrium, which is provided by the solution of the HJBI equation which is of the form (Fleming and Souganidis 1989)

$$
\rho V - H(V, DV, D^2 V) = 0 \tag{15}
$$

$$
H(V, DV, D^2V) = \max_{(c,q,z,\phi)} \min_{\upsilon} \left[\sum_{i=1}^{N} \omega_i \ell_i U(\frac{c_i}{\ell_i}) + \sum_{i=1}^{N} \frac{\theta_i}{2} \left(\sum_{j=1}^{J} \sigma_{ij}\upsilon_j \right)^2 + \right.
$$
$$
\left. \mathcal{L}^{(T)}V + \mathcal{L}^{(M)}V + \mathcal{L}^{(K)}V + \mathcal{L}^{(\Omega)}V + \mathcal{L}_n V \right].
$$

Feedback controls for (c, q, z, ϕ) and υ are obtained as functions of DV by performing the optimization in (15). Substituting the feedback controls into (15) we obtain the relevant HJBI equation as:

$$
\mathsf{F}(V, DV, D^2V) := \rho V - H_d(DV) - \frac{1}{2} Tr\left(\mathsf{C}\mathsf{C}^{tr} D^2 V\right) = 0. \tag{16}
$$

where H_d is a highly nonlinear function of the gradient DV defined by

$$H_d(DV) := \max_{(c,q,z,\phi)} \min_{\upsilon} \left[\sum_{i=1}^{N} \omega_i \ell_i U(\frac{c_i}{\ell_i}) + \sum_{i=1}^{N} \frac{\theta_i}{2} \left(\sum_{j=1}^{J} \sigma_{ij} \upsilon_j \right)^2 + \right.$$
$$\left. \mathcal{L}^{(T)}V + \mathcal{L}^{(M)}V + \mathcal{L}^{(K)}V + \mathcal{L}^{(\Omega)}V \right].$$

Remark 1 *In the model presented here, for simplicity, we have only included risk and uncertainty in the determination of the damage factors for each site due to climate change. As a result of that, the second order term of the resulting HJBI equation only depends on the second order derivatives with respect to the state variable Ω. One can easily envisage situations where the fluctuations and the uncertainty will also affect the other state variables, e.g. temperature, etc. The effects of that would be two-fold. First, it would introduce the second order derivatives with respect to the other state variables into the HJBI equation, thus leading to a modification of the highest order term to $\frac{1}{2}Tr\left(\mathsf{C}\mathsf{C}^{(tr)}\mathsf{D}^2V\right)$, where now D^2V is the full Hessian matrix (i.e. with respect to all variables (T, M, K, Ω)), thus alleviating possible degeneracy of the nonlinear elliptic HJBI equation. However, at the same time it will modify the $H_d(DV)$ term as now the information drift υ, will also affect the operators $\mathcal{L}^{(T)}, \mathcal{L}^{(M)}, \mathcal{L}^{(K)}$ as*

$$\mathcal{L}^{(T)}V = (\mathsf{A}_d T - \mathsf{A}_r T - \mathsf{B}_z(\mathbf{1}_N z) + \mathsf{B}_e(M) + F + \mathsf{C}^{(T)}\upsilon) \cdot \mathsf{D}_T V$$
$$\mathcal{L}^{(M)}V = (\beta \mathbf{1}_N q - \delta_m M + \mathsf{C}^{(M)}\upsilon)\mathsf{D}_M V$$
$$\mathcal{L}^{(K)}V = \Big(\Omega(T, \phi, \mathbf{1}_N z)\, F(K, \mathbf{1}_N q) - \delta K - [\mathbf{1}_N c + \mathbf{1}_N z + \mathbf{1}_N \phi] + \mathsf{C}^{(K)}\upsilon\Big)\, \mathsf{D}_K V.$$

5.4 Solvability of the HJBI equation, Viscosity Solutions and Hot Spot Formation

The solvability of the robust control problem depends on the solvability of the related HJBI equation (16). The solution of (16) will be used to obtain the optimizers (c, ϕ, q, z), υ which are all defined in terms of $\mathsf{D}V$, and obtain therefore the robust feedback control policy. By inserting the feedback rules into the state equation, we obtain the optimum path for the controlled system.

Since our problem does not have a linear-quadratic structure, the solution of (16) is not an easy task. To address the issue we use the concept of viscosity solutions (e.g., Bardi and Capuzzo-Dolcetta 2008), which are continuous but not necessarily differentiable functions that solve the HJBI equation in a weak sense. This approach can prove very useful in addressing robust control problems without linear-quadratic structure, which are exactly the problems associated with climate change. Let $x = (T, M, K)$.

Definition (Viscosity solutions of HJBI equation)

1. $v \in C(\mathbb{R}^{N+2})$ is a viscosity subsolution of (16) if for any test function $\varphi \in C^2(\mathbb{R}^{N+2})$ such that x is a local maximum of $v - \varphi$,

   $\mathsf{F}(x, v(x), \mathsf{D}\varphi(x), \mathsf{D}^2\varphi(x)) \leq 0.$

2. $v \in C(\mathbb{R}^{N+2})$ is a viscosity supersolution of (16) if for any test function $\varphi \in C^2(\mathbb{R}^{N+2})$ such that x is a local minimum of $v - \varphi$,

   $\mathsf{F}(v(x), \mathsf{D}\varphi(x), \mathsf{D}^2\varphi(x)) \geq 0.$

3. $v \in C(\mathbb{R}^{N+2})$ is a viscosity solution of (16) if it is both a viscosity subsolution and a viscosity supersolution.

Regarding the solvability of the problem, we consider the finite horizon version, i.e., $t \in [0, T]$ for large T and treat a parabolic version of the HJBI equation of the form

$$\frac{\partial V}{\partial t} + \mathsf{F}_1(\mathsf{D}V) + \mathsf{F}_2(\mathsf{D}^2V) = 0.$$

It can be shown that if the controls (c, ϕ, q, z) are allowed to take values in a compact subset of $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, then under certain regularity assumptions there exists a $T^* \in [0, T]$, such that equation (16) admits a unique viscosity solution, such that $|V(t, x)| \leq C(1 + |x|^2)$. Furthermore it can be proved that the value of the game is the viscosity solution of the relevant HJBI equation. The derivatives of the viscosity solution v can be used to construct satisfactory approximate feedback controls. The optimal state of the system can then be calculated using a forward integration of the state equation.[10]

Following Da Lio and Ley (2006), the condition for existence of a supersolution would be of the general form $r - C_0 - \frac{C_0}{\theta} e^{rT^*} > 0$ $(r < \rho)$, where C_0

[10] There is a well established literature on numerical methods for the calculation of viscosity solutions of fully nonlinear elliptic and parabolic equations of the general type of the HJBI equation obtained here (e.g., Souganidis 1985, Nikolopoulos and Yannacopoulos 2010).

is a constant. When $\theta = (\theta_1, ..., \theta_N, \theta_{N+1}, \theta_{N+2}) \to 0$, the last term which is negative dominates, and this condition cannot hold at all. This means that the robustness parameter θ plays an important role in the loss of solutions for the system. The next theorem whose proof follows along the lines of Felmer, Quaas, and Sirakov (2013) states that the HJBI equation (16) does not have a solution (even in the viscosity sense) in the limit as $\theta \to 0$.

Theorem 2 (The $\theta \to 0$ limit) *Equation (16) does not have a solution in the limit as $\theta \to 0$ in the classical or in the viscosity sense.*

This breakdown of solutions at the deep uncertainty limit, as $\theta \to 0$, induces type I hot spots. In fact, the breakdown can occur even when just one of the θ_i tends to zero, as is indicated by exact results in the linear-quadratic case (see Brock Xepapadeas and Yannacopoulos 2012, 2013). This can be shown for general problems, by a proper modification of arguments along the lines of Felmer, Quaas, and Sirakov (2013) which essentially boil down to the nonexistence of a positive solution for ODEs of the form $\rho u + \frac{1}{\theta_i} |u'|^2 - u'' = 0$ for $\theta_i \to 0$. We call this breakdown of solutions in the deep uncertainty limit a type I hot spot. This means breakdown of the solution for the whole system because there is "too much" uncertainty for just one site which propagates to the other sites through spatial interactions.

As shown in Athanassoglou and Xepapadeas (2012) for a linear-quadratic problem, solving the HJBI equation for a given robustness parameter θ is equivalent to finding a robust policy for all probability models having relative entropy less than the worst-case model, and allows us to estimate the deviation between the benchmark and the worst case. This implies that if the actual deviation between the worst case and the benchmark case can be inferred from existing knowledge, then by repetitive solving of the model for different values of θ, a value θ^0 that corresponds to the actual deviation can be calculated. This will be the 'correct value' of the robustness parameter.

Therefore by combining solutions of (16) for vectors of robustness parameters θ and existing knowledge about possible deviations between the benchmark and the worst case, the robustness parameters can be calibrated. In this context two types of *hot spots* can be defined:

Type I hot spot. Assume that the realistic deviations between the benchmark and the worst-case model imply low values $\theta_i^0 \in \theta$ such the HJBI equation (16) does not have a solution. This is a type I hot spot which means that misspecification concerns for a location cause regulation to break down for the whole spatial domain. Thus local ambiguity breaks down regulation globally.

Type II hot spot. Assume that the HJBI equation (16) has a solution, either classical or viscosity, for realistic deviations between the benchmark and the worst case. This can be associated with a weak PP and robust control regulation is feasible. The value function in this case will be a function of the states of the system and the robustness parameters θ. At the risk limit $(\theta_1, ..., \theta_N) \to \infty$, there is complete trust in the benchmark model, with no entropic constraints. Let

$$\left(c(t)^U, q(t)^U, z(t)^U, \phi(t)^U; T(t)^U, K(t)^U, M(t)^U\right), \tag{17}$$

$$\left(c(t)^R, q(t)^R, z(t)^R, \phi(t)^R; T(t)^R, K(t)^R, M(t)^R\right) \tag{18}$$

denote, for all i, the time paths for the control and the state variables that correspond to the solution of (13) and the risk limit case respectively.

Define by

$$W^U\left(c^U, \theta^U\right) = \mathbb{E}_{\mathbb{Q}_{v^*}}\left[\int_0^\infty e^{-\rho t}\left\{\sum_{n=1}^N \omega_{nt}^U\left[\ell_i U\left(\frac{c^U(t)}{\ell_i}\right)\right]\right\} dt\right] \tag{19}$$

the welfare measure for the planner where $c_i^U(t), n = 1, ...N$ is the Nash equilibrium consumption path for (13), and by

$$W^R\left(c^R\right) = \mathbb{E}_{\mathbb{P}}\left[\int_0^\infty \mathrm{e}^{-\rho t}\left\{\sum_{n=1}^N \omega_{nt}^R\left[\ell_i U\left(\frac{c_n^R(t)}{\ell_i}\right)\right]\right\} dt\right] \tag{20}$$

the welfare measure for the regulator in the risk limit case. Then if $\Delta W = W^R - W^U > 0$, this difference can be interpreted as the cost of following robust control rules or the cost of been precautious. If $\frac{\partial \Delta W}{\partial \theta_i}$ is high for some locations, these locations can be characterized as type II hot spots. On the other hand if $\Delta W < 0$, precaution is desirable and one may even discuss the optimal level of precaution in the sense that robustness parameters $(\theta_1^*, ..., \theta_N^*)$ may exist such that they maximize the difference $W^U - W^R$, under the constraint that these parameters correspond to realistic deviations between the benchmark and the worst case.

5.5 Extension of the model under recursive utility

The literature on recursive utility under uncertainty indicates that welfare weights are not constant, except for special time additive model where the "aggregator" function f takes the form $f(c, V) = u(c/\ell) - \rho V$, with V been the stochastic utility process (Duffie and Epstein 1992). The multiplier robust problem (13) is observational equivalent with a corresponding risk-sensitive problem since the have the same value functions (Hansen et al. 2006). Furthermore risk-sensitive preferences (e.g. Anderson 2005)) are a special case of recursive preferences, which means that in problem (13) without time additive utilities welfare weights are not constant over time.[11]

In this section we extend our model to include the effects of recursive utility, which according to some authors is a better description of the agents' attitude towards discount factors and intertemporal allocation of resources in the presence of uncertainty and over long time horizons as compared to the standard exponential discount model, used in the model of equation (9)).

Let us assume that the agents $i = 1, \cdots N$ evaluate at time t the consumption stream under the probability model (probability measure Q) using recursive utility functionals $V_i(t; c)$ characterized by felicity function $F_i(c, r)$ where r refers to the instantaneous discount factor. Following Dumas et al. (2000) the recursive utility functional for agent i is defined by

$$V_i(t; c_i) = \mathbb{E}_Q \left[\int_t^T \exp(-\int_t^s r_i(\tau) d\tau) F_i(c_i(s), r_i(s)) \mid \mathcal{F}_t \right], \quad (21)$$

where

$$r_i(t) := \arg\min_{\nu \in \mathbb{R}} \{ F_i(c_i(t), \nu) - \nu V_i(t; c_i).$$

Note that (21) is a nonlinear stochastic integral equation, equivalent to a backward stochastic differential equation, the solution of which will provide the utility functional that can be used for the evaluation of the consumption stream $c_i = \{c_i(t) : t \in [0, \infty]\}$. If the agent wishes to evaluate the consumption stream at $t = 0$ then he must calculate the value of the solution at $t = 0$ and use the resulting functional c_i in order to calculate the utility derived from receiving the consumption stream c_i. As Dumas et al (2000) has

[11] For the modeling of the time evolution of welfare weigts in models with heterogeneous agents and recursive utility see Dumas, et al. (2000), while of the time evolution of welfare weigts in risk-sensitive models with heterogenous agents see Anderson (2005)

shown the determination of the utility functional $V_i(t; c_i)$ can be expressed in terms of a dynamic programming problem of the form

$$\lambda_i(t)V_i(t;c) = \inf_{r_i \in D} \mathbb{E}_Q \left[\int_t^T \lambda_i(s) F(c_i(s), r_i(s)) \mid \mathcal{F}_t \right] \tag{22}$$
$$\text{s.t } \; d\lambda_i(t) = -r_i(t)\lambda_i(t)dt, \;\; \lambda_i(0) = 1.$$

We now consider the effect of model uncertainty in this formulation. The risk in the model is introduced by the N dimensional Wiener process $w = (w_1, \cdots, w_N)$ which is used in order to model stochastic fluctuations of the damage factors Ω_i at each site in equation (10. Model uncertainty is introduced as information drift $\upsilon = (\upsilon_1, \cdots, \upsilon_N)$ for each of the stochastic factors $w = (w_1, \cdots, w_N)$, (see equation (12) which is in fact via Girsanov's theorem equivalent to viewing the evolution of the stochastic fluctuations under the equivalent measure Q_υ with Radon-Nikodym derivative with respect to the reference measure Q

$$\left.\frac{dQ_\upsilon}{dQ}\right|_{\mathcal{F}_t} = \mathcal{M}_\upsilon(t) := \exp\left(\int_0^t \upsilon(s)dw(s) - \frac{1}{2}\int_0^t |\upsilon|(s)^2 ds \right)$$
$$= \exp\left(\int_0^t \sum_{i=1}^N \upsilon_i(s)dw_i(s) - \frac{1}{2}\int_0^t \sum_{i=1}^N \int_0^t \upsilon_i(s)^2 ds \right).$$

Note that the exponential density process is a martingale with respect to the reference measure (under which w is a Wiener process).

Assume that the agents $i = 1, \cdots, N$, adopt the distorted probability measure Q_υ, for some information drift υ chosen by nature. Then, effectively, they calculate their recursive utility functionals using the recursion formula (21) however, they condition events under the distorted probability measure Q_υ rather than the reference probability measure Q, i.e.,

$$V_i(t; c_i, \upsilon) = \mathbb{E}_{Q_\upsilon} \left[\int_t^T \exp(-\int_t^s r_i(\tau)d\tau) F_i(c_i(s), r_i(s)) \mid \mathcal{F}_t \right], \tag{23}$$

where we denote the utility functional as $V_i(t; c_i, \upsilon)$ to emphasize the fact that it is estimated under the distorted probability measure. We may use the exponential density and its martingale property to express the conditional expectation in (23) as an expectation under the reference measure Q, in terms of

$$V_i(t; c_i, \upsilon) = \mathbb{E}_Q[\exp(\int_t^T \upsilon(s)dw(s) - \frac{1}{2}\int_t^T |\upsilon|(s)^2 ds)$$
$$\int_t^T \exp(-\int_t^s r_i(\tau)d\tau)F_i(c_i(s), r_i(s)) \mid \mathcal{F}_t],$$

which is in fact equivalent to expressing its calculation in terms of the modified dynamic programming problem of the form

$$\lambda_i(t)V_i(t; c, \upsilon) = \inf_{r_i \in D} \mathbb{E}_Q\left[\int_t^T \lambda_i(s)F(c_i(s), r_i(s)) \mid \mathcal{F}_t\right] \tag{24}$$
$$\text{s.t } \; d\lambda_i(t) = -(r_i(t) + \frac{1}{2}|\upsilon(t)|^2)\lambda_i(t)dt + \lambda_i \upsilon dw(t), \;\; \lambda_i(0) = 1.$$

where by $\upsilon dw = \sum_{j=1}^N \upsilon_j(t)dw_j(t)$.

Remark 3 *Our approach towards combining model uncertainty with recursive utilities is inspired by the approach of Borovicka (2016). Note that here, in contrast to Borovicka (2016), where each agent uses its own information drift υ_i, in order to define the distorted probability measure for each agent, we assume that all agents share the same distorted probability measure Q_υ, so that υ couples the dynamics for λ_i which model the evolution of the local discount factors. The reason for doing that is because here, unlike Borovicka (2016) we do not wish to model belief dispersion in the agents, and furthermore, it is not the agents themselves who chose the information drifts υ_i but rather an independent agent, Nature. The choice of the model by Nature, affects all agents and their corresponding attitudes towards discounting, and this is shown in the dynamics in (24).*

Since Nature chooses the probability measure which governs the fluctuations, and Nature is considered as a malevolent agent, the agents should consider the worst case scenario under an entropic penalization of all scenarios, which leads to a utility functional provided by the problem

$$\lambda_i(t)V_i(t; c, \upsilon) = \inf_{r_i, \upsilon \in D \times \mathcal{P}} \mathbb{E}_Q\left[\int_t^T \lambda_i(s)F(c_i(s), r_i(s)) \mid \mathcal{F}_t\right], \tag{25}$$
$$\text{s.t } \; d\lambda_i(t) = -(r_i(t) + \frac{1}{2}|\upsilon(t)|^2)\lambda_i(t)dt + \lambda_i \upsilon dw(t), \;\; \lambda_i(0) = 1.$$

where by $\upsilon dw = \sum_{j=1}^N \upsilon_j(t)dw_j(t)$, and $\mathcal{P}$ is the set of possible allowed models, as parameterized by the stochastic processes υ. The allowed set $\mathcal{P}$ can be determined e.g. by an allowed entropy ball or otherwise.

The problem encountered now by the social planner is to maximize the global recursive utility functional

$$\sum_{i=1}^{N} \mathfrak{a}_i V_i(0; c_i), \tag{26}$$

subject to (3),(4),(8), and initial and boundary conditions where $\mathfrak{a} = (\mathfrak{a}_1, \cdots, \mathfrak{a}_N)$ are initial welfare weights associated with the utility of consumption of each location. In contrast with problem (9) where the temporal evolution of the welfare weights was introduced in an ad hoc fashion by the social planner, and assumed to be of the form $e^{-\rho t}$ with the same constant exponential decay factor ρ for each geographic site, here, the evolution of the welfare welfare weights is **endogenous** and determined by the discount factor dynamics provided by the recursive utility functions, and are spatially dependent. In order to take into account the effect of model uncertainty we have to modify problem (27) to

$$\max_{c} \inf_{\upsilon \in \mathcal{P}} \sum_{i=1}^{N} \mathfrak{a}_i V_i(0; c_i, \upsilon), \tag{27}$$

subject to the same dynamic state constraints.

Following Dumas et al. (2000) and assuming that we start at time t (i.e. that the social planner starts the maximization procedure at time t rather than at time 0, and the current state of the system is given by $\mathfrak{S}_0 = (T, K, M, \Omega, \lambda)$, we may treat the more general problem

$$J(t, \mathfrak{S}) := \sup_{c} \inf_{\upsilon \in \mathcal{P}} \sum_{i=1}^{N} \mathfrak{a}_i V_i(t; c_i, \upsilon),$$

subject to the dynamic constraints which are considered as starting at time t at state $\mathfrak{S}$. Once this problem is solved, substitute $t = 0$ and $\mathfrak{S} = \mathfrak{S}_0$ to the general solution in order to obtain the solution to the original problem required.

By the discussion in the previous paragraphs, this problem can be represented in the equivalent form

$$J(t, \mathfrak{S}_E) := \sup_c \inf_{r_i \in D, \upsilon \in \mathcal{P}} \sum_{i=1}^{N} \mathbb{E}[\int_t^T \lambda_i(s) F_i(c_i(s), r_i(s)) ds \mid \mathcal{F}_t],$$

subject to the dynamic constraints plus the extra dynamic constraint for the new state variables $\lambda = (\lambda_1, \cdots, \lambda_N)$, which are given in the second equation of (25). Here we assume again that the regulator starts at t and that the current (at time t) state of the system which is now extended to include λ is $\mathfrak{S}_E = (\mathfrak{S}, \lambda)$, where λ is the new starting point for the newly introduced state variable. If this solution is obtained, then setting $t = 0$ and $\lambda = \mathfrak{a}$ in it, we obtain the solution to the original problem. The constraint $\upsilon \in \mathcal{P}$ can be further treated by using a quadratic penalty function of the form $\mathcal{R}(\upsilon, \theta, t)$ where $\theta = (\theta_1, \cdots, \theta_N)$ is a vector valued parameter modelling the uncertainty aversion at different sites. Such a quadratic penalty function will be compatible with an entropic constraint, but in a non homogeneous fashion. If geometric intuition is of any help to the reader, consider the entropy ball being treated as an entropy ellipsoid instead, with the principal axes corresponding to different uncertainty aversion in the various directions in state space. This would lead to the equivalent problem:

$$J(t, \mathfrak{S}_E) := \sup_c \inf_{r = r_i \in D^N, \upsilon = (\upsilon_i) \in D^N} \sum_{i=1}^{N} \mathbb{E}[\int_t^T \lambda_i(s) F_i(c_i(s), r_i(s)) ds \mid \mathcal{F}_t + \mathcal{R}(\upsilon; \theta, s)], \tag{28}$$

subject to the dynamic constraints .

Remark 4 *An example of such a penalty function could be*

$$\mathcal{R}(\upsilon; \theta, s) = \sum_{i=1}^{N} \theta_i \upsilon_i(s)^2 ds,$$

or the more general form

$$\mathcal{R}(\upsilon; \theta, s) = \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_i \sigma_{ij} \upsilon_i(s)^2,$$

see e.g. (13).

The solution of the optimization problem 28 can be obtained in terms of an appropriate HJBI equation. In particular, the value function V will satisfy the nonlinear PDE

$$0 = \sup_{c} \inf_{r,\upsilon} \left(\sum_{i=1}^{N} \lambda_i F_i(c,r) + \mathcal{R}(\upsilon; \theta s) + \mathcal{L}_E V \right)$$

where $\mathcal{L}_E = \mathcal{L} + \mathcal{L}^{\lambda} + \mathcal{L}_0$ where $\mathcal{L} = \mathcal{L}^{(T)} + \mathcal{L}^{(M)} + \mathcal{L}^{(K)} + \mathcal{L}^{(\Omega)}$ is the generator operator defined in the previous section (see (14) whereas

$$\mathcal{L}^{(\lambda)} V = - \sum_{i=1}^{N} (r_i + \frac{1}{2} |\upsilon|^2) \lambda_i \frac{\partial V}{\partial \lambda_i},$$
$$\mathcal{L}_0 V = \frac{1}{2} Tr(S S^T \mathsf{D}^2_{\lambda} V),$$

and $S = (S_{ij})_{i,j=1,\cdots,N}$, with $S_{ij} = \lambda_i \upsilon_j$. A quick calculation shows that

$$S S^T = |\upsilon|^2 \Lambda,$$

where $\Lambda = \lambda \lambda^T = (\Lambda_{ij})_{i,j=1,\cdots,N}$ with $\Lambda_{ij} = \lambda_i \lambda_j$, so that

$$\mathcal{L}_0 V = \frac{1}{2} |\upsilon|^2 Tr(\lambda \lambda^T \mathsf{D}^2_{\lambda} V).$$

Further algebraic manipulation leads to a simplification of the resulting HJBI as

$$0 = \sup_{c} \inf_{r,\upsilon} \left(\sum_{i=1}^{N} \lambda_i (F_i(c,r) - (r_i + \frac{1}{2} |\upsilon|^2) \frac{\partial V}{\partial \lambda_i}) + \mathcal{R}(\upsilon; \theta s) + \mathcal{L} V + \mathcal{L}_0 V \right)$$

This equation has a striking difference compared to the equation we encountered in the previous section where additive utilities were assumed, as now the operator $\mathcal{L}_0$ contains the control parameter υ, and as a result of the optimization over υ we will end up with an equation that will be fully nonlinear in $\mathsf{D}^2_{\lambda} V$. For example assuming for simplicity the quadratic penalty

$$\mathcal{R}(\upsilon; \theta, s) = \frac{1}{2} \sum_{i=1}^{N} \theta_i \upsilon_i(s)^2,$$

the minimization over υ would involve the minimization of the term

$$-|\upsilon|^2\frac{1}{2}\sum_{i=1}^{N}\lambda_i V_{\lambda_i}+\frac{1}{2}\sum_{i=1}^{N}\theta_i\upsilon_i^2+\frac{1}{2}|\upsilon|^2 Tr(\lambda\lambda^T \mathsf{D}_\lambda^2 V)+\sum_{i=1}^{N}(\zeta_i+\sum_{j=1}^{N}\sigma_{ij}\upsilon_j)\Omega_i V_{\Omega_i},$$

leading to a minimizer of the form

$$\upsilon_j=\frac{\theta_j Tr(\lambda\lambda^T \mathsf{D}_\lambda^2 V)-\sum_{i=1}^{N}\sigma_{ij}\Omega_i V_{\Omega_i}}{\sum_{i=1}^{N}\lambda_i V_{\lambda_i}-\theta_j},\ \ j=1,\cdots,N.$$

Substituting this into the Hamiltonian, we derive a HJB which will be quadratic in D_λ^2. This is obtained easily since

$$|\upsilon|^2=J_0(\mathsf{D}_\lambda^2 V, D_\Omega V):=\sum_{j=1}^{N}\left(\frac{\theta_j Tr(\lambda\lambda^T \mathsf{D}_\lambda^2 V)-\sum_{k=1}^{N}\sigma_{kj}\Omega_k V_{\Omega_k}}{\sum_{k=1}^{N}\lambda_k V_{\lambda_k}-\theta_j}\right)^2,$$

and

$$\mathcal{R}(\upsilon;\theta,s)=J_1(\mathsf{D}_\lambda^2 V, D_\Omega V):=\frac{1}{2}\sum_{j=1}^{N}\theta_j\left(\frac{\theta_j Tr(\lambda\lambda^T \mathsf{D}_\lambda^2 V)-\sum_{k=1}^{N}\sigma_{kj}\Omega_k V_{\Omega_k}}{\sum_{k=1}^{N}\lambda_k V_{\lambda_k}-\theta_j}\right)^2$$

while

$$\begin{aligned}\sum_{i=1}^{N}\sum_{j=1}^{N}\sigma_{ij}\upsilon_j\Omega_i D_{\Omega_i}V &= J_2(\mathsf{D}_\lambda^2 V, D_\Omega V)\\ :=\sum_{i}^{N}\sum_{j=1}^{N}\sigma_{ij}&\left(\frac{\theta_j Tr(\lambda\lambda^T \mathsf{D}_\lambda^2 V)-\sum_{k=1}^{N}\sigma_{kj}\Omega_k V_{\Omega_k}}{\sum_{k=1}^{N}\lambda_k V_{\lambda_k}-\theta_j}\right)\Omega_i D_{\Omega_i}V,\end{aligned}$$

with J_0, J_1 being quadratic in $D_\lambda^2 V$ and all of the terms J_0, J_1 and J_2 containing products of $D_\lambda^2 V$ with $D_\Omega V$.

The minimization over $r=(r_i)$ and the maximization over $c=(c_i)$ is more involved, as the felicity function F_i couples c_i and r_i. This step involves the following terms of the Hamiltonian

$$\sum_{i=1}\lambda_i(F_i(c_i,r_i)-r_i V_{\lambda_i})-(\sum_{i=1}^{N}c_i)\mathsf{D}_K V.$$

The first order conditions are the system of nonlinear equations

$$\frac{\partial}{\partial r_i} F_i(c_i, r_i) - V_{\lambda_i} = 0,$$
$$\lambda_i \frac{\partial}{\partial c_i} F_i(c_i, r_i) - \mathsf{D}_K V = 0,$$

for $i = 1, \cdots, N$. This has to be solved as a system, but in order to compare with the non recursive case, we adopt the following parameterization of the solution to the first order conditions. The solution to the first equation, given c_i, can be expressed as

$$r_i = \Phi_i(V_{\lambda_i}; c_i),$$

where Φ_i is the inverse function, with respect to r of the function $\frac{\partial F_i}{\partial r}(c, r)$. For example, for the choice

$$F_i(c, r) = \beta_i \frac{c^{\gamma_i}}{\gamma_i} \left[-\frac{\rho_i - \gamma_i}{\gamma_i - \frac{\rho_i r}{\beta_i}} \right]^{\frac{\gamma_i}{\rho_i} - 1}, \tag{29}$$

for the felicity function (see e.g. Dumas et al. 2000) after some rather painful algebra we obtain

$$\Phi_i(x; c) = \frac{\beta_i}{\rho_i} \left(\gamma_i - \gamma_i^{-\frac{\rho_i}{\gamma_i}} (\gamma_i - \rho_i) c^{\rho_i} x^{-\frac{\rho_i}{\gamma_i}} \right)$$

Then the determination of c_i is obtained by the solution of the nonlinear algebraic equation

$$\lambda_i \frac{\partial}{\partial c_i} F_i(c_i, \Phi_i(V_{\lambda_i}; c_i)) - \mathsf{D}_K V = 0,$$

with respect to c_i. Unfortunately, this algebraic equation cannot be solved analytically with respect to c_i, in the general case, so one has to resort to numerical evaluation.

For the case of the explicit felicity function of (29), one may compute the first order conditions as

$$
\begin{aligned}
A_{1i} c_i^{\gamma_i} \left(\gamma_i - \frac{\rho_i}{\beta_i} r_i \right)^{-\gamma_i/\beta_i} &= D_{\lambda_i} V, \\
A_{2i} c_i^{\gamma_i - 1i} \left(\gamma_i - \frac{\rho_i}{\beta_i} r_i \right)^{1-\gamma_i/\beta_i} &= \frac{1}{\lambda_i} D_K V
\end{aligned}
$$

where

$$
\begin{aligned}
A_{1i} = \frac{\rho_i}{\gamma_i} \left(1 - \frac{\gamma_i}{\beta_i} \right) (\gamma_i - \rho_i)^{\gamma_i/\rho_i - 1}, \\
A_{2i} = \beta_i (\gamma_i - \rho_i)^{\gamma_i/\rho_i - 1}.
\end{aligned}
$$

Taking logarithms, we end up with the linear system

$$
\begin{aligned}
\gamma_i \ln c_i - \frac{\gamma_i}{\beta_i} \ln \left(1 - \frac{\rho_i}{\beta_i} r_i \right) &= \ln \left(\frac{D_{\lambda_i} V}{A_{1i}} \right), \\
(\gamma_i - 1) \ln c_i + (1 - \gamma_i/\beta_i) \ln \left(1 - \frac{\rho_i}{\beta_i} r_i \right) &= \ln \left(\frac{1}{A_{2i} \lambda_i} D_K V \right),
\end{aligned}
$$

which can be solved to obtain

$$
\begin{aligned}
1 - \frac{\rho_i}{\beta_i} r_i = B_i \frac{(D_K V)^{\beta_i/(\beta_i - 1)}}{\lambda_i^{\beta_i/(\beta_i-1)} (D_{\lambda_i} V)^{\frac{\gamma_i - 1}{\beta_i - 1} \frac{\beta_i}{\gamma_i}}}, \\
c_i = C_i \lambda_i^{-\frac{1}{\beta_i - 1}} (D_{\lambda_i} V)^{\frac{1}{\gamma_i} \frac{\beta_i - \gamma_i}{\beta_i - 1}} (D_K V)^{\frac{1}{\beta_i - 1}}
\end{aligned}
\tag{30}
$$

with

$$
\begin{aligned}
B_i = \frac{A_{1i}^{\frac{\gamma_i - 1}{\beta_i - 1} \frac{\beta_i}{\gamma_i}}}{A_{2i}^{\frac{\beta_i}{\beta_i - 1}}}, \\
C_i = \frac{A_{1i}^{-\frac{1}{\gamma_i} \frac{\gamma_i - 1}{\beta_i - 1}}}{A_{2i}^{\frac{1i}{\beta_i - 1}}}
\end{aligned}
$$

We may then define

$$
J_3(D_\lambda V, D_K V) := \sum_{i=1}^{N} \lambda_i (F_i(c_i, r_i) - D_{\lambda_i} V),
$$

where r_i, c_i are as defined in (30).

We are now in position to write down the resulting HJBI. The remaining maximizations over (q, z, ϕ) will essentially remain unaffected, so that the HJBI equation is the same as before with but with the extra term

$$\Delta J_{rec} = \frac{1}{2} J_0(D^2_\lambda V, D_\Omega V) Tr(\lambda\lambda^T D^2_\lambda V) - \frac{1}{2} J_0 \lambda^T D_\lambda V + J_1(D^2_\lambda V, D_\Omega V) + J_2(D^2_\lambda V, D_\Omega V) + J_3(D_\lambda V, D_K V),$$

which is nonlinear in the second order derivatives with respect to λ.

The HJBI for the recursive model will now be of the form

$$J_{rec} = J_{nonrec} + \Delta J_{rec} = 0.$$

The nonlinear terms in the higher order derivatives introduce serious difficulties as well as possible new features to the HJBI equation for the recursive utility case as compared to the HJBI equation for the additive utility, which is linear in the second order derivatives. The mathematical treatment of this equation is under active consideration, with our prime interest being in the qualitative features of this equation and the possibility of occurence of hot spot formation behaviour.

The above analysis suggest that the solution of the problem with recursive utilities is a formidable task. In this section we provided a set up of the problem which we think it constitutes an interesting area for further research.

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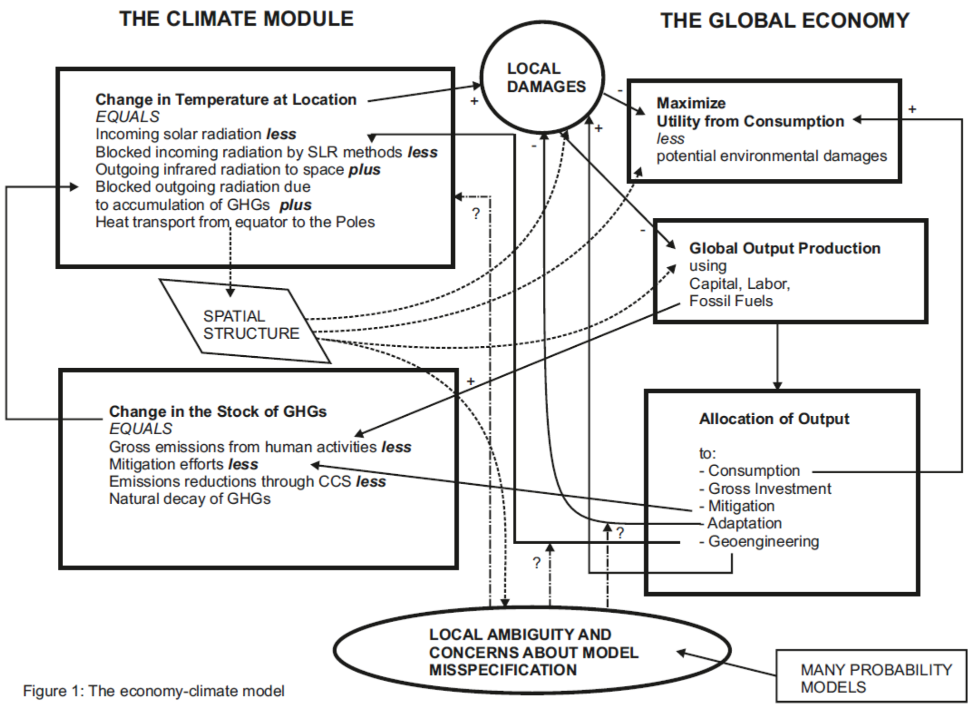

Figure 1: The economy-climate model

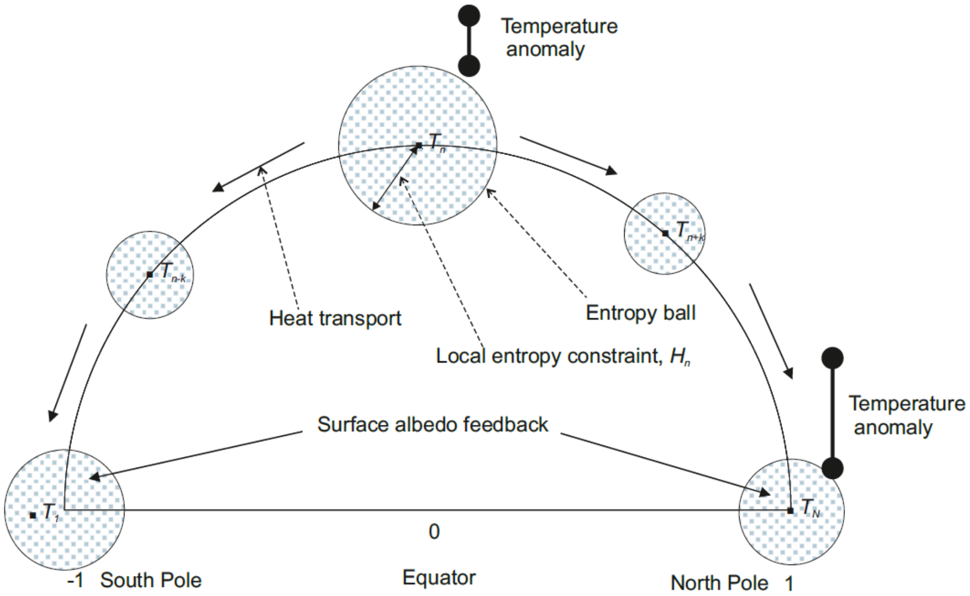

Figure 2: Spatially structured damage ambiguity for the climate system

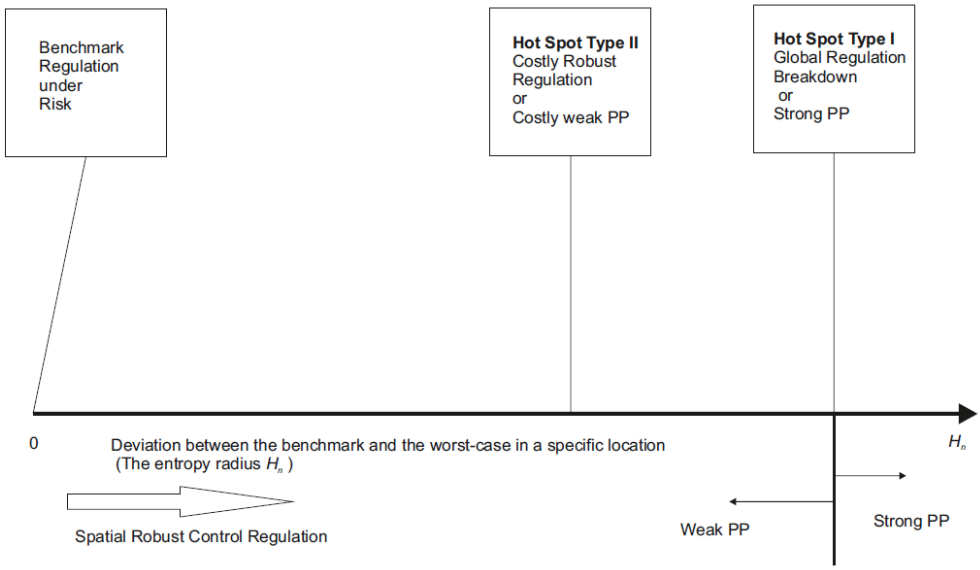

Figure 3: Emergence of Hot Spots, Weak and Strong PP