Finite Sample Theory and Bias Correction of Maximum Likelihood Estimators in the EGARCH Model (Technical Appendix I)

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February 23, 2018

1 The Model

$$
\begin{aligned}
y_t &= z_t\sqrt{h_t}, \qquad t = 1, ...T, \qquad where \\
z_t &\sim iidD(0,1), \qquad \ln(h_t) = \alpha + \theta z_{t-1} + \gamma|z_{t-1}| + \beta\ln(h_{t-1}).
\end{aligned}
$$

with

$$
E\ln(h_t) = \frac{\alpha + \gamma E|z|}{1-\beta} = L.
$$

2 First Order Likelihood Derivatives:

Following, henceforth, the notation employed in Linton (1997), i.e. $h_{t;\circ} = \frac{\partial\ln(h_t)}{\partial\circ}$ and so on, the derivatives of the log-likelihood function with respect to all the parameters are: for $i, j, k \in \{\alpha, \theta, \gamma, \beta\}$ the derivatives are:

$$
\begin{aligned}
\ell(\mu, \alpha, \theta, \beta, \gamma|\, z_0, h_0) &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \sum_{t=1}^{T}\frac{(y_t-\mu)^2}{2h_t} = \\
&= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \frac{1}{2}\sum_{t=1}^{T}z_t^2.
\end{aligned}
$$

$$
\begin{aligned}
\ell_\circ &= -\frac{1}{2}\sum_{t=1}^{T}\frac{\partial\ln(h_t)}{\partial\circ} - \frac{1}{2}\sum_{t=1}^{T}2z_t\frac{\partial z_t}{\partial\circ} \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\circ}
\end{aligned}
$$

as

$$
\frac{\partial z_t}{\partial\circ} = \frac{\partial y_t e^{-1/2\ln h_t}}{\partial\circ} = y_t e^{-1/2\ln h_t}\frac{\partial(-1/2\ln h_t)}{\partial\circ} = -\frac{1}{2}z_t h_{t;\circ}.
$$

For $\circ \in \{\alpha, \theta, \gamma, \beta\}$ the derivatives are:

$$
\begin{aligned}
h_{t;\alpha} &= 1 - \frac{1}{2}\theta z_{t-1}h_{t-1;\alpha} - \frac{1}{2}\gamma|z_{t-1}|\,h_{t-1;\alpha} + \beta h_{t-1;\alpha} \\
&= 1 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)h_{t-1;\alpha},
\end{aligned}
$$

with

$$
E(h_{t;\alpha}) = \frac{1}{1-\left(\beta - \frac{1}{2}\gamma E|z|\right)} = E_{;\alpha}
$$

provided that $\left|\beta - \frac{1}{2}\gamma E|z|\right| < 1$ and

$$
h_{t;\alpha,a} = \frac{1}{4}(\theta z_{t-1} + \gamma|z_{t-1}|)\,h_{t-1;\alpha}^2 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)h_{t-1;\alpha,\alpha}.
$$

Now

$$\begin{aligned} h_{t;\theta} &= z_{t-1} + \theta \frac{\partial z_{t-1}}{\partial \theta} + \gamma \frac{\partial \left| z_{t-1} \right|}{\partial \theta} + \beta h_{t-1;\theta} \\ &= z_{t-1} + \left(\beta - \frac{1}{2} \theta z_{t-1} - \frac{1}{2} \gamma \left| z_{t-1} \right| \right) h_{t-1;\theta}. \end{aligned}$$

and

$$E h_{t;\theta} = E_{;\theta} = 0.$$

Also

$$h_{t;\gamma} = \left| z_{t-1} \right| + \left(\beta - \frac{1}{2} \theta z_{t-1} - \gamma \frac{1}{2} \left| z_{t-1} \right| \right) h_{t-1;\gamma}.$$

$$\begin{aligned} E h_{t;\gamma} &= E_{;\gamma} = \frac{E\left| z \right|}{1 - \left(\beta - \frac{1}{2} \gamma E \left| z \right| \right)} = \frac{E\left| z \right|}{1 - ST0}, \quad \textit{where} \\ ST0 &= \beta - \frac{1}{2} \gamma E \left| z \right|. \end{aligned}$$

Finally,

$$\begin{aligned} h_{t;\beta} &= \theta \frac{\partial z_{t-1}}{\partial \beta} + \gamma \frac{\partial \left| z_{t-1} \right|}{\partial \beta} + \ln\left(h_{t-1} \right) + \beta h_{t-1;\beta} \\ &= \ln\left(h_{t-1} \right) + \left(\beta - \frac{1}{2} \theta z_{t-1} - \frac{1}{2} \gamma \left| z_{t-1} \right| \right) h_{t-1;\beta}. \end{aligned}$$

and

$$E h_{t;\beta} = E_{;\beta} = \frac{L}{1 - \left(\beta - \frac{1}{2} \gamma E \left| z \right| \right)}.$$

3 Second Order Likelihood Derivatives:

For $\circ, * \in \{\alpha, \theta, \gamma, \beta\}$ we get that

$$\ell_{\circ,*} = \frac{1}{2} \sum_{t=1}^{T} \left(z_t^2 - 1 \right) h_{t;\circ,*} - \frac{1}{2} \sum_{t=1}^{T} z_t^2 h_{t;*} h_{t;\circ}$$

3.1 Alpha-Alpha

$$E\left(\ell_{\alpha,\alpha} \right) = E\left(\frac{1}{2} \sum_{t=1}^{T} \left(z_t^2 - 1 \right) h_{t;\alpha,\alpha} - \frac{1}{2} \sum_{t=1}^{T} z_t^2 \left(h_{t;\alpha} \right)^2 \right) = -\frac{T}{2} E\left(h_{t;\alpha} \right)^2$$

Now

$$h_{t;\alpha}^2 = 1 + 2 \left(\beta - \frac{1}{2} \theta z_{t-1} - \frac{1}{2} \gamma \left| z_{t-1} \right| \right) h_{t-1;\alpha} + \left(\beta - \frac{1}{2} \theta z_{t-1} - \frac{1}{2} \gamma \left| z_{t-1} \right| \right)^2 h_{t-1;\alpha}^2,$$

and as we have that

$$\begin{aligned} E\left(h_{t;\alpha}^2\right) &= 1+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\left(h_{t-1;\alpha}\right) \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\alpha}^2\right) \\ &= \frac{1+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{(;\alpha)^2}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

$$\begin{aligned} E\left(h_{t;\alpha}^2\right) &= \frac{1+2ST0E_{;\alpha}}{1-ST1}=EHA2, \quad where \\ ST1 &= \beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right). \end{aligned}$$

3.2 Alpha-Beta

Next

$$E\left(\ell_{\alpha,\beta}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\beta}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\beta}h_{t;\alpha}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha}\right).$$

Now to find the expctation we need he expecation of

$$\begin{aligned} h_{t;\alpha}\ln h_t &= \alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln h_{t-1}+\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha} \\ &\quad +\beta\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}\ln h_{t-1}, \end{aligned}$$

and it follows that

$$\begin{aligned} E\left[\ln\left(h_t\right)h_{t;\alpha}\right] &= \frac{\alpha+\gamma E\left|z\right|+\beta E\left[\ln\left(h_{t-1}\right)\right]+\begin{pmatrix} \alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2 \\ -\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right] \end{pmatrix}E_{;\alpha}}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)} \\ &= LE_{;\alpha} \end{aligned}$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1.$

Hence, as

$$\begin{aligned} h_{t;\alpha}h_{t;\beta} &= \ln\left(h_{t-1}\right)+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left[\ln\left(h_{t-1}\right)h_{t-1;\alpha}+h_{t-1;\beta}\right] \\ &\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\beta}, \end{aligned}$$

we have that

$$E\left[h_{t;\alpha}h_{t;\beta}\right]=\frac{L+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left[LE_{;\alpha}+E_{;\beta}\right]}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{;\alpha;\beta},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

3.3 Alpha-Gama

$$E\left(\ell_{\alpha,\gamma}\right) = E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\gamma} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\alpha}h_{t;\gamma}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma}\right).$$

Now, as

$$\begin{aligned} h_{t;\alpha}h_{t;\gamma} &= \left|z_{t-1}\right| + \left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left(h_{t-1;\gamma} + \left|z_{t-1}\right| h_{t-1;\alpha}\right) \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\gamma}, \end{aligned}$$

we get

$$E\left[h_{t;\alpha}h_{t;\gamma}\right] = \frac{E\left|z\right| + \left(\beta - \gamma\frac{1}{2}E\left|z\right|\right)E_{;\gamma} + \left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)} = E_{;\alpha;\gamma},$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

3.4 Alpha-Theta

$$E\left(\ell_{\alpha,\gamma}\right) = E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\theta} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\alpha}h_{t;\theta}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta}\right)$$

where

$$h_{t;\alpha,\theta} = -\frac{1}{2}z_{t-1}h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\theta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\theta}$$

Now, as

$$\begin{aligned} h_{t;\alpha}h_{t;\theta} &= z_{t-1} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta} + z_{t-1}\right)h_{t-1;\alpha} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\theta}h_{t-1;\alpha} \end{aligned}$$

and as $E_{;\theta} = 0$, we get

$$E\left[h_{t;\alpha}h_{t;\theta}\right] = \frac{-\frac{1}{2}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)E_{;\alpha}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)} = E_{;\alpha;\theta},$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

3.5 Beta-Beta

$$E\left(\ell_{\beta\beta}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\beta,\beta}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\beta}\right)^2\right)=-\frac{T}{2}E\left(h_{t;\beta}\right)^2.$$

First notice that

$$\begin{aligned}\ln^2\left(h_t\right) &= \alpha^2+\left(\theta^2+\gamma^2\right)z_{t-1}^2+2\alpha\theta z_{t-1}+2\alpha\gamma\left|z_{t-1}\right|+2\gamma\theta z_{t-1}\left|z_{t-1}\right| \\ &\quad +2\left(\alpha\beta+\beta\theta z_{t-1}+\gamma\beta\left|z_{t-1}\right|\right)\ln\left(h_{t-1}\right)+\beta^2\ln^2\left(h_{t-1}\right)\end{aligned}$$

and and as

$$E\ln\left(h_t\right)=\frac{\alpha+\gamma E\left|z\right|}{1-\beta}=L$$

we get

$$E\left(\ln\left(h_t\right)\right)^2=\frac{\alpha^2+\theta^2+\gamma^2+2\alpha\gamma E\left|z\right|+2\beta\left(\alpha+\gamma E\left|z\right|\right)L+2\gamma\theta E\left[z\left|z\right|\right]}{\left(1-\beta^2\right)}=L^2.$$

Second, as

$$\begin{aligned}h_{t;\beta}\ln h_t &= \left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\ln h_{t-1}+\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta} \\ &\quad +\beta\left(\ln h_{t-1}\right)^2+\beta\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}\ln h_{t-1},\end{aligned}$$

we have that

$$\begin{aligned}E\left[h_{t;\beta}\ln h_t\right] &= \frac{\left(\alpha+\gamma E\left|z\right|\right)L+\beta L^2+\left[\begin{array}{c}\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\theta^2 \\ -\frac{1}{2}\gamma^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\end{array}\right]E_{;\beta}}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)} \\ &= LE_{;\beta},\end{aligned}$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$.

Hence, as

$$h_{t;\beta}^2=\ln^2 h_{t-1}+2\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\beta}^2,$$

we get

$$E\left(h_{t;\beta}\right)^2=\frac{L^2+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{(;\beta)^2},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

3.6 Beta-Gama

$$E\left(\ell_{\beta\gamma}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\beta,\gamma}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\beta}h_{t;\gamma}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma}\right).$$

First,

$$\begin{aligned}E\left[\ln\left(h_t\right)h_{t;\gamma}\right] &= \frac{\gamma+\alpha E\left|z\right|+\theta E\left[z\left|z\right|\right]+\beta E\left|z\right|L+\begin{pmatrix}\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2\\ -\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\end{pmatrix}E_{;\gamma}}{1-\beta\left(\beta-\gamma\frac{1}{2}E\left|z\right|\right)}\\ &= LE_{;\gamma},\end{aligned}$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$.

It follows that

$$E\left(h_{t;\beta}h_{t;\gamma}\right)=\frac{E\left|z\right|L+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\gamma}+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{;\beta;\gamma},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

3.7 Beta-Theta

$$E\left(\ell_{\beta\theta}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\beta,\theta}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\beta}h_{t;\theta}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta}\right).$$

First, have to find $E\left[h_{t-1;\theta}\ln\left(h_{t-1}\right)\right]$. As now $E_{;\theta}=0$ we get

$$E\left[h_{t;\theta}\ln\left(h_t\right)\right]=\frac{\theta+\gamma E\left[z\left|z\right|\right]}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=LE_{;\theta},$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$.

Hence,

$$E\left(h_{t;\beta}h_{t;\theta}\right)=\frac{\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\theta}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{;\beta;\theta},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

3.8 Gama-Gama

$$E\left(\ell_{\gamma\gamma}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\gamma,\gamma}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\gamma}\right)^2\right)=-\frac{T}{2}E\left(h_{t;\gamma}\right)^2,$$

where

$$E\left(h_{t;\gamma}^2\right)=\frac{1+2\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{(;\gamma)^2},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

3.9 Gama-Theta

$$E\left(\ell_{\theta\gamma}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\theta,\gamma}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\theta}h_{t;\gamma}\right)=-\frac{T}{2}E\left(h_{t;\theta}h_{t;\gamma}\right)$$

Now as $E_{;\theta}=0$

$$E\left(h_{t;\gamma}h_{t;\theta}\right)=\frac{E\left[z\left|z\right|\right]-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{;\gamma;\theta}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$

3.10 Theta-Theta

$$E\left(\ell_{\theta\theta}\right)=E\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\theta,\theta}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\theta}\right)^2\right)=-\frac{T}{2}E\left(h_{t;\theta}^2\right)$$

$$Eh_{t;\theta}^2=\frac{1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{(;\theta)^2},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

4 Third Order Likelihood Derivatives

4.1 Alpha-Alpha-Alpha

$$E\left(\ell_{\alpha,\alpha,\alpha}\right)=-\frac{T}{2}\left[3E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right)-E\left(h_{t;\alpha}^3\right)\right].\ as\ before$$

Now, as

$$\begin{aligned}h_{t;\alpha}^3 &= 1+3\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}+3\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha}^2\\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^3h_{t-1;\alpha}^3,\end{aligned}$$

we get

$$E\left(h_{t;\alpha}^3\right)=\frac{1+3\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha}+3\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{(;\alpha)^2}}{1-\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)}=E_{(;\alpha)^3},$$

if $\left|\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\right|<1.$

Furthermore,

$$h_{t;\alpha,a}=\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha},$$

and

$$E\left(h_{t;\alpha,\alpha}\right)=E_{;\alpha,\alpha}=\frac{\frac{1}{4}\gamma E\left|z\right|E_{(;\alpha)^2}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)},$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$. Also

$$\begin{aligned}h_{t;\alpha}h_{t;\alpha,a} &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2\\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left[\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^3+h_{t-1;\alpha,\alpha}\right]\\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha,\alpha}h_{t-1;\alpha},\end{aligned}$$

and

$$\begin{aligned}E\left(h_{t;\alpha}h_{t;\alpha,a}\right) &= \frac{\frac{1}{4}\gamma E\left|z\right|E_{(;\alpha)^2}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\alpha)^3}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha,\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\ &= E_{;\alpha;\alpha,\alpha},\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.2 Alpha-Alpha-Beta

$$E\left(\ell_{\alpha\alpha\beta}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\alpha}-\left(h_{t;\alpha}\right)^2h_{t;\beta}+2h_{t;\alpha}h_{t;\alpha,\beta}\right)$$

First,

$$\begin{aligned}E\left(h_{t;\alpha}^2\ln h_t\right) &= \alpha+\gamma E\left|z\right|+\beta L+\left(\beta\alpha-\theta^2-\gamma^2+\beta\gamma E\left|z\right|-\alpha\gamma E\left|z\right|-2\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha}\\ &\quad+\beta\left(\beta-\gamma E\left|z\right|\right)LE_{;\alpha}+\left(\begin{array}{c}\alpha\beta^2+\left(\frac{1}{4}\alpha-\beta\right)\left(\gamma^2+\theta^2\right)\\ +\beta\gamma\left(\beta-\alpha\right)E\left|z\right|+\frac{1}{2}\gamma\theta\left(\alpha-4\beta\right)E\left[z\left|z\right|\right]\\ \frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3+\frac{1}{4}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)E_{(;\alpha)^2}\\ &\quad+\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\beta\gamma E\left|z\right|+\frac{1}{2}\gamma\theta E\left[z\left|z\right|\right]\right)E\left(h_{t-1;\alpha}^2\ln h_{t-1}\right),\end{aligned}$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$. It follows

$$\begin{aligned}E\left(h_{t;\alpha}^2h_{t;\beta}\right) &= E\ln h_{t-1}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left(E_{;\beta}+2LE_{;\alpha}\right)\\ &\quad+\left(\begin{array}{c}\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\\ -\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left[z\left|z\right|\right]\end{array}\right)\left(LE_{(;\alpha)^2}+2E_{;\alpha;\beta}\right)\\ &\quad+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)\\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left(\left|z\right|^3\right)\end{array}\right)E\left(h_{t-1;\beta}h_{t-1;\alpha}^2\right)\\ &= E_{(;\alpha)^2;\beta},\end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

Now

$$h_{t;\alpha,\beta} = h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\beta} h_{t-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\beta},$$

$$E\left(h_{t;\alpha,\beta}\right) = E_{;\alpha,\beta} = \frac{E_{;\alpha} + \frac{1}{4}\gamma E\left|z\right| E_{;\alpha;\beta}}{1 - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)},$$

if $\left|\beta - \frac{1}{2}\gamma E\left|z\right|\right| < 1$. Hence, as

$$\begin{aligned} h_{t;\alpha} h_{t;\alpha,\beta} &= h_{t-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^2 \\ &+ \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)\left[h_{t-1;\alpha} h_{t-1;\beta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}^2 h_{t-1;\beta}\right] \\ &+ \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha,\beta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha} h_{t-1;\alpha,\beta}, \end{aligned}$$

it follows that

$$\begin{aligned} E\left(h_{t;\alpha} h_{t;\alpha,\beta}\right) &= E_{;\alpha} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left(E_{(;\alpha)^2} + E_{;\alpha,\beta}\right) \\ &+ \frac{1}{4}\gamma E\left|z\right| E_{;\alpha;\beta} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\theta^2 - \gamma\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma^2\right) E_{(;\alpha)^2;\beta} \\ &+ \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) h_{t-1;\alpha} h_{t-1;\alpha,\beta} \\ &= E_{;\alpha;\alpha,\beta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

Finally, as

$$\begin{aligned} E\left(h_{t;\alpha,\alpha}\ln h_t\right) &= \frac{1}{4}\left(\theta^2 + 2\gamma\theta E\left[z\left|z\right|\right] + \alpha\gamma E\left|z\right| + \gamma^2\right) E_{(;\alpha)^2} \\ &+ \left[\alpha\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right] E_{;\alpha,\alpha} \\ &+ \frac{1}{4}\beta\gamma E\left|z\right| LE_{(;\alpha)^2} + \beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E\left(h_{t-1;\alpha,\alpha}\ln h_{t-1}\right) = LE_{;\alpha,\alpha} \end{aligned}$$

we get

$$\begin{aligned} Eh_{t;\beta} h_{t;\alpha,\alpha} &= \frac{1}{4}\gamma E\left|z\right| LE_{(;\alpha)^2} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) LE_{;\alpha,\alpha} \\ &+ \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E\left[z\left|z\right|\right]\right) h_{t-1;\beta} h_{t-1;\alpha}^2 \\ &+ \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) h_{t-1;\beta} h_{t-1;\alpha,\alpha} \\ &= E_{;\beta;\alpha,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

4.3 Alpha-Alpha-Gama

$$E\left(\ell_{\alpha\alpha\gamma}\right)=-\frac{T}{2}E\left(h_{t;\gamma}h_{t;\alpha,\alpha}-h_{t;\alpha}^{2}h_{t;\gamma}+2h_{t;\alpha}h_{t;\alpha,\gamma}\right)$$

First

$$\begin{aligned}
h_{t;\alpha}^{2}h_{t;\gamma} &= |z_{t-1}|+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)\left(h_{t-1;\gamma}+2|z_{t-1}|h_{t-1;\alpha}\right)\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)^{2}\left(2h_{t-1;\gamma}h_{t-1;\alpha}+|z_{t-1}|h_{t-1;\alpha}^{2}\right)\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma|z_{t-1}|\right)^{3}h_{t-1;\gamma}h_{t-1;\alpha}^{2},
\end{aligned}$$

with

$$\begin{aligned}
E\left(h_{t;\alpha}^{2}h_{t;\gamma}\right) &= E|z|+\left(\beta-\gamma\frac{1}{2}E|z|\right)E_{;\gamma}+2\left(\beta E|z|-\frac{1}{2}\theta E\left[z|z|\right]-\frac{1}{2}\gamma\right)E_{;\alpha}\\
&\quad+2\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right)E_{;\alpha;\gamma}\\
&\quad+\left(\begin{array}{c}\beta^{2}E|z|+\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)E|z|^{3}\\-\gamma\beta-\theta\beta E\left[z|z|\right]+\frac{1}{2}\gamma\theta E\left(z^{3}\right)\end{array}\right)E_{(;\alpha)^{2}}\\
&\quad+\left(\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)\\-\frac{3}{2}\beta^{2}\gamma E|z|+\frac{3}{2}\gamma\beta\theta E\left[z|z|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left(|z|^{3}\right)\end{array}\right)E\left(h_{t-1;\alpha}^{2}h_{t-1;\gamma}\right)\\
&= E_{(;\alpha)^{2};\gamma},
\end{aligned}$$

if $\left|\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)-\frac{3}{2}\beta^{2}\gamma E|z|\\+\frac{3}{2}\beta\gamma\theta E\left(z|z|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E|z|^{3}\end{array}\right|<1.$

Second

$$h_{t;\gamma,\alpha}=-\frac{1}{2}|z_{t-1}|h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma|z_{t-1}|\right)h_{t-1;\alpha}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)h_{t-1;\gamma,\alpha}$$

and

$$E\left(h_{t;\gamma,\alpha}\right)=E_{;\gamma,\alpha}=\frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\gamma}-\frac{1}{2}E|z|E_{;\alpha}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)},$$

if $\left|\beta-\frac{1}{2}\gamma E|z|\right|<1.$

Hence

$$\begin{aligned}
h_{t;\gamma,\alpha}h_{t;\alpha} &= +\frac{1}{4}\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma|z_{t-1}|\right)\left(\left(\theta z_{t-1}+\gamma|z_{t-1}|\right)h_{t-1;\alpha}^{2}h_{t-1;\gamma}-2|z_{t-1}|h_{t-1;\alpha}^{2}\right)\\
&\quad-\frac{1}{2}|z_{t-1}|h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma|z_{t-1}|\right)h_{t-1;\alpha}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)h_{t-1;\gamma,\alpha}\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma|z_{t-1}|\right)^{2}h_{t-1;\alpha}h_{t-1;\gamma,\alpha},
\end{aligned}$$

with

$$
\begin{aligned}
E\left(h_{t;\gamma,\alpha}h_{t;\alpha}\right) &= -\frac{1}{2}E\left|z\right|E_{;\alpha}-\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{(;\alpha)^2}+\frac{1}{4}\gamma E\left|z\right|E_{;\alpha;\gamma} \\
&\quad -\frac{1}{8}\left(\gamma^2+\theta^2-2\beta\gamma E\left|z\right|+2\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\alpha)^2;\gamma}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha,\gamma} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\alpha}h_{t-1;\gamma,\alpha}\right) \\
&= E_{;\alpha;\alpha,\gamma},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally,

$$
\begin{aligned}
h_{t;\gamma}h_{t;\alpha,\alpha} &= \left|z_{t-1}\right|\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2 \\
&\quad +\left(\frac{1}{4}\theta z_{t-1}+\frac{1}{4}\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)h_{t-1;\gamma}h_{t-1;\alpha}^2 \\
&\quad +\left|z_{t-1}\right|\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha,\alpha}h_{t-1;\gamma}
\end{aligned}
$$

with

$$
\begin{aligned}
E\left(h_{t;\gamma}h_{t;\alpha,\alpha}\right) &= \frac{1}{4}\left(\theta E\left[z\left|z\right|\right]+\gamma\right)E_{(;\alpha)^2}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\theta^2-\theta\gamma E\left[z\left|z\right|\right]-\frac{1}{2}\gamma^2\right)E_{(;\alpha)^2;\gamma} \\
&\quad +\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\alpha,\alpha} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\alpha,\alpha}h_{t-1;\gamma}\right) \\
&= E_{;\gamma;\alpha,\alpha},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.4 Alpha-Alpha-Theta

$$
E\left(\ell_{\alpha\alpha\theta}\right)=-\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\alpha}-h_{t;\alpha}^2h_{t;\theta}+2h_{t;\alpha}h_{t;\alpha,\theta}\right)
$$

First as $E_{;\theta}=0$

$$
\begin{aligned}
h_{t;\alpha}^2h_{t;\theta} &= z_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta}+2z_{t-1}h_{t-1;\alpha}\right) \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2\left(2h_{t-1;\theta}h_{t-1;\alpha}+z_{t-1}h_{t-1;\alpha}^2\right) \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^3h_{t-1;\theta}h_{t-1;\alpha}^2,
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
E\left(h_{t;\alpha}^2 h_{t;\theta}\right) &= -\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\alpha;\theta} \\
&\quad+\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\theta\beta \\ -\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\end{array}\right)E_{(;\alpha)^2} \\
&\quad+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left(\left|z\right|^3\right)\end{array}\right)E\left(h_{t-1;\theta}h_{t-1;\alpha}^2\right) \\
&= E_{(;\alpha)^2;\theta},
\end{aligned}
$$

if $\left|\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\right|<1$.

Next

$$
E\left(h_{t;\alpha,\theta}\right)=E_{;\alpha,\theta}=\frac{\frac{1}{4}\gamma E\left|z\right|E_{;\alpha;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)},
$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

$$
\begin{aligned}
h_{t;\alpha}h_{t;\alpha,\theta} &= -\frac{1}{2}z_{t-1}h_{t-1;\alpha}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\alpha,\theta}-\frac{1}{2}z_{t-1}\right)h_{t-1;\alpha}^2 \\
&\quad+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\alpha}h_{t-1;\theta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2h_{t-1;\theta}\right) \\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha}h_{t-1;\alpha,\theta},
\end{aligned}
$$

with

$$
\begin{aligned}
E\left(h_{t;\alpha}h_{t;\alpha,\theta}\right) &= \frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{(;\alpha)^2}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha,\theta} \\
&\quad+\frac{1}{4}\gamma E\left|z\right|E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\alpha)^2;\theta} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\alpha}h_{t-1;\alpha,\theta}\right) \\
&= E_{;\alpha;\alpha,\theta},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally,

$$
\begin{aligned}
h_{t;\theta}h_{t;\alpha,\alpha} &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left[z_{t-1}h_{t-1;\alpha}^2+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}^2h_{t-1;\theta}\right] \\
&\quad+z_{t-1}\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\theta}h_{t-1;\alpha,\alpha},
\end{aligned}
$$

with

$$
\begin{aligned}
E\left(h_{t;\theta}h_{t;\alpha,\alpha}\right) &= \frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)\left[E_{(;\alpha)^2}-2*h_{t-1;\alpha,\alpha}\right] \\
&\quad +\frac{1}{4}\left[\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E\left(z\left|z\right|\right)\right]*E_{(;\alpha)^2;\theta} \\
&\quad +\left(\beta^2+\frac{1}{4}\left(\theta^2+\gamma^2\right)-\beta\gamma E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\theta}h_{t-1;\alpha,\alpha}\right) \\
&= E_{;\theta;\alpha,\alpha},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.5 Alpha-Beta-Beta

$$
E\left(\ell_{\beta\beta\alpha}\right)=-\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\beta}+2h_{t;\beta}h_{t;\beta,\alpha}-h_{t;\alpha}h_{t;\beta}^2\right)
$$

First, from above we have

$$
E\left(h_{t;\alpha,\beta}\right)=E_{;\alpha,\beta}=\frac{E_{;\alpha}+\frac{1}{4}\gamma E\left|z\right|E_{;\alpha;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)},
$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

Now as

$$
\begin{aligned}
h_{t;\alpha}\ln^2 h_t &= \alpha^2+\left(\theta^2+\gamma^2\right)z_{t-1}^2+2\gamma\theta z_{t-1}\left|z_{t-1}\right|+2\alpha\theta z_{t-1}+2\alpha\gamma\left|z_{t-1}\right| \\
&\quad +2\beta\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\alpha\right)\ln h_{t-1}+\beta^2\ln^2 h_{t-1} \\
&\quad +\left[\begin{array}{c}\left(\theta^2+\gamma^2\right)z_{t-1}^2+2\alpha\theta z_{t-1}+\alpha^2 \\ +2\gamma\left(\alpha+\theta z_{t-1}\right)\left|z_{t-1}\right|\end{array}\right]\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha} \\
&\quad +2\left(\alpha\beta+\beta\theta z_{t-1}+\gamma\beta\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}\ln h_{t-1} \\
&\quad +\beta^2\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}\ln^2 h_{t-1},
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(h_{t;\alpha}\ln^2 h_t\right) &= \alpha^2+\theta^2+\gamma^2+2\alpha\gamma E\left|z\right|+2\gamma\theta E\left[z\left|z\right|\right]+2\beta\left(\gamma E\left|z\right|+\alpha\right)L+\beta^2L^2 \\
&\quad +\left[\begin{array}{c}\alpha^2\beta+\beta\gamma^2+\beta\theta^2-\alpha\gamma^2-\alpha\theta^2+\alpha\gamma\left[2\beta-\frac{1}{2}\alpha\right]E\left|z\right| \\ -\frac{1}{2}\gamma\left[\gamma^2+2\theta^2\right]E\left|z\right|^3+2\gamma\theta\left[\beta-\alpha\right]E\left[z\left|z\right|\right]-\frac{1}{2}\theta\left[2\gamma^2+\theta^2\right]Ez^3\end{array}\right]E_{;\alpha} \\
&\quad +\beta\left[2\alpha\beta-\theta^2-\gamma^2+\gamma\left(2\beta-\alpha\right)E\left|z\right|-2\gamma\theta E\left[z\left|z\right|\right]\right]LE_{;\alpha} \\
&\quad +\beta^2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\left(h_{t-1;\alpha}\ln^2 h_{t-1}\right) \\
&= L^2E_{;\alpha},
\end{aligned}
$$

if $\left|\beta^2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$.

Now given that,

$$
\begin{aligned}
h_{t;\alpha}h_{t;\beta} &= \ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}\ln h_{t-1} \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2}h_{t-1;\beta}h_{t-1;\alpha},
\end{aligned}
$$

we get

$$
\begin{aligned}
E\left(h_{t;\alpha}h_{t;\beta}\ln h_{t}\right) &= \left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)\left(LE_{;\alpha}+E_{;\beta}\right) \\
&\quad +\left(\alpha+\gamma E\left|z\right|\right)L+\begin{pmatrix}\alpha\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\\ -\beta\left(\theta^{2}+\gamma^{2}\right)+\beta^{2}\gamma E\left|z\right|+\frac{1}{4}\gamma\left(3\theta^{2}+\gamma^{2}\right)E\left|z\right|^{3}\\ +\frac{1}{4}\theta\left(\theta^{2}+3\gamma^{2}\right)Ez^{3}-2\beta\gamma\theta E\left[z\left|z\right|\right]\end{pmatrix}E_{;\alpha;\beta} \\
&\quad +\beta L^{2}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left(LE_{;\beta}+L^{2}E_{;\alpha}\right) \\
&\quad +\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\beta}h_{t-1;\alpha}\ln h_{t-1}\right) \\
&= LE_{;\alpha;\beta},
\end{aligned}
$$

if $\left|\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$.

Hence, it follows that

$$
\begin{aligned}
E\left(h_{t;\alpha}h_{t;\beta}^{2}\right) &= L^{2}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)L^{2}E_{;\alpha}+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\beta} \\
&\quad +\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\left(2LE_{;\alpha;\beta}+E_{(;\beta)^{2}}\right) \\
&\quad +\begin{pmatrix}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)-\frac{3}{2}\beta^{2}\gamma E\left|z\right|\\ +\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{pmatrix}E\left(h_{t-1;\beta}^{2}h_{t-1;\alpha}\right) \\
&= E_{;\alpha(;\beta)^{2}},
\end{aligned}
$$

if $\left|\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)+\frac{3}{2}\beta\gamma\left[\theta E\left(z\left|z\right|\right)-\beta E\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\right|<1$

Now for the second term of the $E\left(\ell_{\beta\beta\alpha}\right)$, we have

$$
\begin{aligned}
h_{t;\alpha,\beta}\ln h_{t} &= \left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\alpha} \\
&\quad +\left(\theta z_{t-1}+\alpha+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\beta}+\beta h_{t-1;\alpha}\ln h_{t-1} \\
&\quad +\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\beta h_{t-1;\beta}h_{t-1;\alpha}\ln h_{t-1} \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\beta h_{t-1;\alpha,\beta}\ln h_{t-1},
\end{aligned}
$$

and

$$\begin{aligned}
E\left(h_{t;\alpha,\beta}\ln h_t\right) &= \left(\alpha+\gamma E\left|z\right|\right)E_{;\alpha}+\frac{1}{4}\left[\alpha\gamma E\left|z\right|+\gamma^2+\theta^2+2\gamma\theta E\left[z\left|z\right|\right]\right]E_{;\alpha;\beta} \\
&\quad+\left[\alpha\beta-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\gamma\left(\beta-\frac{1}{2}\alpha\right)E\left|z\right|-E\left[z\left|z\right|\right]\right]E_{;\alpha,\beta} \\
&\quad+\beta LE_{;\alpha}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{;\alpha;\beta}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\left(h_{t-1;\alpha,\beta}\ln h_{t-1}\right) \\
&= LE_{;\beta,\alpha},
\end{aligned}$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$. Further,

$$\begin{aligned}
h_{t;\beta}h_{t;\alpha,\beta} &= h_{t-1;\alpha}\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\alpha} \\
&\quad+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\beta}\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}^2\right)h_{t-1;\alpha} \\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\beta}\ln h_{t-1} \\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\beta}h_{t-1;\alpha,\beta},
\end{aligned}$$

and hence,

$$\begin{aligned}
E\left(h_{t;\beta}h_{t;\alpha,\beta}\right) &= LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha;\beta}+\frac{1}{4}\gamma E\left|z\right|LE_{;\alpha;\beta} \\
&\quad+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha(;\beta)^2}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\alpha,\beta} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\alpha,\beta} \\
&= E_{;\beta;\alpha,\beta},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

For the first term of $E\left(\ell_{\beta\beta\alpha}\right)$, notice that

$$h_{t;\beta,\beta}=2h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}^2+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\beta}$$

and it follows that

$$E\left(h_{t;\beta,\beta}\right)=\frac{2E_{;\beta}+\frac{1}{4}\gamma E\left|z\right|E_{(;\beta)^2}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=E_{;\beta,\beta}$$

Hence,

$$\begin{aligned}
E\left(h_{t;\alpha}h_{t;\beta,\beta}\right) &= \frac{1}{4}\gamma E\left|z\right| E_{(;\beta)^2} + 2E_{;\beta} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E_{;\beta,\beta} + 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E_{;\alpha;\beta} \\
&\quad + \left(\frac{1}{4}\beta\gamma\left|z_{t-1}\right| - \frac{1}{8}\gamma^2 - \frac{1}{8}\theta^2 - \frac{1}{4}\theta\gamma E\left(z\left|z\right|\right)\right) E_{;\alpha(;\beta)^2} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) E\left(h_{t-1;\alpha}h_{t-1;\beta,\beta}\right) \\
&= E_{;\alpha;\beta,\beta}
\end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

4.6 Alpha-Beta-Gama

$$E\left(\ell_{\alpha\beta\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\beta} + h_{t;\alpha}h_{t;\beta,\gamma} - h_{t;\beta}h_{t;\alpha}h_{t;\gamma}\right)$$

First,

$$\begin{aligned}
h_{t;\alpha}h_{t;\gamma}\ln h_t &= \left[\left|z_{t-1}\right|\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) + \alpha\left|z_{t-1}\right|\right]\left[1 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha}\right] \\
&\quad + \left(\alpha + \theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)\left[h_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\gamma}\right] \\
&\quad + \beta\left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left(h_{t-1;\gamma}\ln h_{t-1} + \left|z_{t-1}\right| h_{t-1;\alpha}\ln h_{t-1}\right) \\
&\quad + \beta\left|z_{t-1}\right|\ln h_{t-1} + \beta\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\gamma}\ln h_{t-1},
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\alpha}h_{t;\gamma}\ln h_t &= \gamma + \alpha E\left|z\right| + \theta E\left[z\left|z\right|\right] + \left[\alpha\beta - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \gamma\left(\beta - \frac{1}{2}\alpha\right) E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right] E_{;\gamma} \\
&\quad + \beta E\left|z\right| L + \beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) LE_{;\gamma} + \beta\left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right) LE_{;\alpha} \\
&\quad + \left[\beta\gamma - \frac{1}{2}\alpha\gamma + \alpha\beta E\left|z\right| + \theta\left(\beta - \frac{1}{2}\alpha\right) E\left[z\left|z\right|\right] - \frac{1}{2}\left(\theta^2 + \gamma^2\right) E\left|z\right|^3 - \gamma\theta E z^3\right] E_{;\alpha} \\
&\quad + \left[\begin{array}{c} \alpha\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) \\ \theta\left(\begin{array}{c} \frac{1}{4}\left(\theta^2 + 3\gamma^2\right) E z^3 \\ -\theta\beta - \gamma\beta E\left[z\left|z\right|\right] \end{array}\right) + \gamma\left(\begin{array}{c} \beta^2 E\left|z\right| + \frac{1}{4}\left(3\theta^2 + \gamma^2\right) E\left|z\right|^3 \\ -\theta\beta E\left[z\left|z\right|\right] - \gamma\beta \end{array}\right) \end{array}\right] E_{;\alpha;\gamma} \\
&\quad + \beta\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) E\left(h_{t-1;\gamma}h_{t-1;\alpha}\ln h_{t-1}\right) \\
&= LE_{;\alpha;\gamma},
\end{aligned}$$

if $\left|\beta\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right| < 1$.

Then

$$
\begin{aligned}
h_{t;\alpha}h_{t;\beta}h_{t;\gamma} &= \left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left(h_{t-1;\gamma}\ln h_{t-1}+\left|z_{t-1}\right|h_{t-1;\beta}+\left|z_{t-1}\right|h_{t-1;\alpha}\ln h_{t-1}\right)\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)^2\left(h_{t-1;\beta}h_{t-1;\gamma}+h_{t-1;\alpha}h_{t-1;\gamma}\ln h_{t-1}+\left|z_{t-1}\right|h_{t-1;\alpha}h_{t-1;\beta}\right)\\
&\quad+\left|z_{t-1}\right|\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^3 h_{t-1;\alpha}h_{t-1;\beta}h_{t-1;\gamma},
\end{aligned}
$$

and the forth term in $E\left(\ell_{\alpha\beta\gamma}\right)$ is given,

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\beta}h_{t;\gamma} &= E\left|z\right|L+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\gamma}+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)\left(E_{;\beta}+LE_{;\alpha}\right)\\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\left(E_{;\beta}E_{;\gamma}+LE_{;\alpha}E_{;\gamma}\right)\\
&\quad+\left(\begin{array}{c}\beta^2E\left|z\right|-\gamma\beta+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3\\ -\theta\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta Ez^3\end{array}\right)E_{;\alpha;\beta}\\
&\quad+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)Ez^3-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left(\left|z\right|^3\right)\end{array}\right)Eh_{t-1;\alpha}h_{t-1;\beta}h_{t-1;\gamma}\\
&= E_{;\alpha;\beta;\gamma}
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left(\left|z\right|^3\right)\end{array}\right|<1.$

For the third term of $E\left(\ell_{\alpha\beta\gamma}\right)$, we have that

$$
h_{t;\beta,\gamma}=h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\beta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\gamma}
$$

with

$$
Eh_{t;\beta,\gamma}=\frac{E_{;\gamma}-\frac{1}{2}E\left|z\right|E_{;\beta}+\frac{1}{4}\gamma E\left|z\right|E_{;\beta;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=E_{;\beta,\gamma},
$$

and

$$
\begin{aligned}
h_{t;\alpha}h_{t;\beta,\gamma} &= h_{t-1;\gamma}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\gamma}\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right|\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\beta}\\
&\quad+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\beta}h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\beta}\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\beta,\gamma}
\end{aligned}
$$

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\beta,\gamma} &= E_{;\gamma} - \frac{1}{2}E\left|z\right|E_{;\beta} - \frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha;\beta} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha;\gamma} \\
&\quad + \frac{1}{4}\gamma E\left|z\right|E_{;\beta;\gamma} - \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\theta^2 - \frac{1}{2}\gamma^2 - \gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta;\gamma} \\
&\quad + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E_{;\beta,\gamma} + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\alpha}h_{t-1;\beta,\gamma} \\
&= E_{;\alpha;\beta,\gamma},
\end{aligned}
$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

For the second term of $E\left(\ell_{\alpha\beta\gamma}\right)$ we have that

$$
\begin{aligned}
h_{t;\gamma}h_{t;\alpha,\beta} &= \frac{1}{4}\left|z_{t-1}\right|\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\alpha} + \left|z_{t-1}\right|\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\beta} \\
&\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left[1 + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}\right]h_{t-1;\alpha}h_{t-1;\gamma} \\
&\quad + \left|z_{t-1}\right|h_{t-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\gamma}h_{t-1;\alpha,\beta},
\end{aligned}
$$

and it follows that

$$
\begin{aligned}
Eh_{t;\gamma}h_{t;\alpha,\beta} &= E\left|z\right|E_{;\alpha} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha;\gamma} + \left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha,\beta} \\
&\quad + \frac{1}{4}\left(\theta E\left[z\left|z\right|\right] + \gamma\right)E_{;\alpha;\beta} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\theta^2 - \frac{1}{2}\gamma\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma^2\right)E_{;\alpha;\beta;\gamma} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\gamma}h_{t-1;\alpha,\beta} \\
&= E_{;\gamma;\alpha,\beta},
\end{aligned}
$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

Finally, for the second term of $E\left(\ell_{\alpha\beta\gamma}\right)$ we have

$$
h_{t;\alpha,\gamma} = -\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\gamma},
$$

with

$$
\begin{aligned}
E\ln\left(h_t\right)h_{t;\alpha,\gamma} &= -\frac{1}{2}\left(\alpha E\left|z\right| + \theta E\left[z\left|z\right|\right] + \gamma\right)E_{;\alpha} + \frac{1}{4}\left[\alpha\gamma E\left|z\right| + \theta^2 + 2\gamma\theta E\left[z\left|z\right|\right] + \gamma^2\right]E_{;\alpha;\gamma} \\
&\quad + \left(\alpha\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha,\gamma} + \\
&\quad -\frac{1}{2}\beta E\left|z\right|LE_{;\alpha} + \frac{1}{4}\beta\gamma E\left|z\right|LE_{;\alpha;\gamma} + \beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E\ln\left(h_{t-1}\right)h_{t-1;\alpha,\gamma} \\
&= LE_{;\alpha,\gamma}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
h_{t;\beta}h_{t;\alpha,\gamma} &= -\frac{1}{2}\left|z_{t-1}\right|\ln\left(h_{t-1}\right)h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\ln\left(h_{t-1}\right)h_{t-1;\alpha}h_{t-1;\gamma} \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left[\begin{array}{c}\frac{1}{4}\left[\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}-2\left|z_{t-1}\right|\right]h_{t-1;\beta}h_{t-1;\alpha} \\ \ln\left(h_{t-1}\right)h_{t-1;\alpha,\gamma}\end{array}\right] \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\beta}h_{t-1;\alpha,\gamma},
\end{aligned}
$$

with

$$
\begin{aligned}
Eh_{t;\beta}h_{t;\alpha,\gamma} &= -\frac{1}{2}E\left|z\right|LE_{;\alpha}+\frac{1}{4}\gamma E\left|z\right|LE_{;\alpha;\gamma}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\alpha,\gamma} \\
&\quad -\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\alpha;\beta} \\
&\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta;\gamma} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\alpha,\gamma},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.7 Alpha-Beta-Theta

$$
E\left(\mathcal{L}_{\alpha\beta\theta}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\theta}+h_{t;\alpha}h_{t;\beta,\theta}+h_{t;\theta}h_{t;\alpha,\beta}-h_{t;\alpha}h_{t;\beta}h_{t;\theta}\right)
$$

For the last term of $E\left(\mathcal{L}_{\alpha\beta\theta}\right)$ we have that

$$
\begin{aligned}
h_{t;\alpha}h_{t;\theta} &= z_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta}+z_{t-1}\right)h_{t-1;\alpha} \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\theta},
\end{aligned}
$$

and as $E_{;\theta}=0$

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\theta}\ln h_t &= \theta+\gamma E\left[z\left|z\right|\right] \\
&\quad +\left[\begin{array}{c}\theta\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)-\frac{1}{2}\alpha\left(\theta+\gamma E\left[z\left|z\right|\right]\right) \\ +\gamma\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^3-\frac{1}{2}\gamma Ez^3\right)\end{array}\right]E_{;\alpha} \\
&\quad +\left[\begin{array}{c}\alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)+\beta\gamma\left[\left(\beta-\alpha\right)E\left|z\right|-2\theta E\left(z\left|z\right|\right)\right] \\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E\left|z\right|^3-\beta\left(\theta^2+\gamma^2\right)\end{array}\right]E_{;\alpha;\theta} \\
&\quad +\beta\left(-\frac{1}{2}\theta-\frac{1}{2}\gamma E\left[z\left|z\right|\right]\right)LE_{;\alpha}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\theta} \\
&\quad +\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\left(h_{t-1;\alpha}h_{t-1;\theta}\ln h_{t-1}\right) \\
&= LE_{;\alpha;\theta},
\end{aligned}
$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$, and

$$\begin{aligned} h_{t;\beta}h_{t;\theta} &= z_{t-1}\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta}\ln h_{t-1}+z_{t-1}h_{t-1;\beta}\right) \\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\beta}h_{t-1;\theta} \end{aligned}$$

with

$$Eh_{t;\beta}h_{t;\theta}=E_{;\beta;\theta}=\frac{\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\theta}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)},$$

and it follows that

$$\begin{aligned} Eh_{t;\alpha}h_{t;\beta}h_{t;\theta} &= -\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\theta}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta} \\ &\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left[z\left|z\right|\right]\right)\left[LE_{;\alpha;\theta}+E_{;\beta;\theta}\right] \\ &\quad+\left(\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\right)E_{;\alpha;\beta} \\ &\quad+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\left(\theta^2+\gamma^2\right)-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left(\left|z\right|^3\right)\end{array}\right)Eh_{t-1;\alpha}h_{t-1;\beta}h_{t-1;\theta} \\ &= E_{;\alpha;\beta;\theta}, \end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\left(\theta^2+\gamma^2\right)-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

For the third term in $E\left(\mathcal{L}_{\alpha\beta\theta}\right)$, notice

$$\begin{aligned} h_{t;\theta}h_{t;\alpha,\beta} &= z_{t-1}h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(z_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}\right)h_{t-1;\alpha}h_{t-1;\beta} \\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta}h_{t-1;\alpha}+z_{t-1}h_{t-1;\alpha,\beta}\right) \\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\theta}h_{t-1;\alpha,\beta} \end{aligned}$$

and it follows that

$$\begin{aligned} Eh_{t;\theta}h_{t;\alpha,\beta} &= \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha;\theta}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta} \\ &\quad+\frac{1}{4}\left[\gamma\left(\beta E\left|z\right|-\frac{1}{2}\gamma\right)-\frac{1}{2}\theta\left(\theta+2\gamma E\left[z\left|z\right|\right]\right)\right]E_{;\alpha;\beta;\theta}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha,\beta} \\ &\quad+\left(\beta^2-\beta\gamma E\left|z\right|+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\alpha,\beta} \\ &= E_{;\theta;\alpha,\beta} \end{aligned}$$

if $\left|\beta^2-\beta\gamma E\left|z\right|+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

For the second term in $E\left(\mathcal{L}_{\alpha\beta\theta}\right)$ we have that, as $E_{;\theta}=0$

$$Eh_{t;\beta,\theta}=\frac{1}{4}\frac{\gamma E\left|z\right|E_{;\beta;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=E_{t;\beta,\theta}$$

and

$$\begin{aligned}
Eh_{t;\alpha}h_{t;\beta,\theta} &= \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\alpha;\theta}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\beta,\theta}\\
&\quad+\frac{1}{4}\gamma E\left|z\right|E_{;\beta;\theta}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\theta^{2}-\gamma\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma^{2}\right)E_{;\alpha;\beta;\theta}\\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\alpha}h_{t-1;\beta,\theta}\\
&= E_{;\alpha;\beta,\theta},
\end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, for the first term $E\left(\mathcal{L}_{\alpha\beta\theta}\right)$ we have

$$\begin{aligned}
h_{t;\alpha,\theta} &= -\frac{1}{2}z_{t-1}h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\theta}\\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\theta},
\end{aligned}$$

and

$$\begin{aligned}
Eh_{t;\alpha,\theta}\ln h_{t} &= -\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{;\alpha;\theta}\\
&\quad+\frac{1}{4}\left[\alpha\gamma E\left|z\right|+\theta\left(\theta+\gamma E\left[z\left|z\right|\right]\right)+\gamma\left(\theta E\left[z\left|z\right|\right]+\gamma\right)\right]E_{;\alpha;\theta}\\
&\quad+\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\left(\gamma^{2}+\theta^{2}-2\beta\gamma E\left|z\right|+2\gamma\theta E\left[z\left|z\right|\right]\right)\right]E_{;\alpha,\theta}\\
&\quad+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\alpha,\theta}\ln h_{t-1}\\
&= LE_{;\alpha,\theta},
\end{aligned}$$

and also

$$\begin{aligned}
Eh_{t;\beta}h_{t;\alpha,\theta} &= \frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta}+\frac{1}{4}\gamma E\left|z\right|LE_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\alpha,\theta}\\
&\quad+\frac{1}{4}\left(-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\beta;\theta}\\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\alpha,\theta}\\
&= E_{;\beta;\alpha,\theta},
\end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.8 Alpha-Gama-Gama

$$E(\ell_{\alpha\gamma\gamma}) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\alpha,\gamma} - h_{t;\alpha}h_{t;\gamma}^2\right).$$

For the last term in $E(\ell_{\alpha\gamma\gamma})$, we have

$$Eh_{t;\gamma}^2 = \frac{1 + (2\beta E|z| - \theta E[z|z|] - \gamma)E_{;\gamma}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)} = E_{(;\gamma)^2},$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1$ and

$$\begin{aligned}
Eh_{t;\alpha}h_{t;\gamma}^2 &= 1 + \left(\beta - \frac{1}{2}\theta Ez^3 - \frac{1}{2}\gamma E|z|^3\right)E_{;\alpha} + (2\beta E|z| - \theta E[z|z|] - \gamma)E_{;\gamma} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\gamma)^2} \\
&\quad + 2\left(\beta^2 E|z| + \frac{1}{4}\left(\theta^2 + \gamma^2\right)E|z|^3 - \theta\beta E[z|z|] - \gamma\beta + \frac{1}{2}\gamma\theta z_{t-1}^3\right)E_{;\alpha;\gamma} \\
&\quad + \left(\begin{array}{c} \beta^3 + \frac{3}{4}\beta\left(\theta^2 + \gamma^2\right) - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\gamma\beta\theta E[z|z|] - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E|z|^3 \end{array}\right)Eh_{t-1;\alpha}h_{t-1;\gamma}^2 \\
&= E_{;\alpha(;\gamma)^2},
\end{aligned}$$

if $\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\left(\theta^2 + \gamma^2\right) - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\gamma\beta\theta E[z|z|] - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E|z|^3 \end{array}\right| < 1.$

For the second term in $E(\ell_{\alpha\gamma\gamma})$, we have that

$$h_{t;\alpha,\gamma} = -\frac{1}{2}|z_{t-1}|h_{t-1;\alpha} + \frac{1}{4}(\theta z_{t-1} + \gamma|z_{t-1}|)h_{t-1;\alpha}h_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)h_{t-1;\alpha,\gamma},$$

with

$$E(h_{t;\alpha,\gamma}) = E_{;\alpha,\gamma} = \frac{-\frac{1}{2}E|z|E_{;\alpha} + \frac{1}{4}\gamma E|z|E_{;\alpha;\gamma}}{1 - \left(\beta - \frac{1}{2}\gamma E|z|\right)}$$

if $\left|\beta - \frac{1}{2}\gamma E|z|\right| < 1$, and it follows that

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\alpha,\gamma} &= -\frac{1}{2}E_{;\alpha} + \left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right)\left(E_{;\alpha,\gamma} - \frac{1}{2}E_{;\alpha;\gamma}\right) \\
&\quad + \frac{1}{4}(\theta E[z|z|] + \gamma)E_{;\alpha;\gamma} + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\alpha(;\gamma)^2} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)Eh_{t-1;\gamma}h_{t-1;\alpha,\gamma} \\
&= E_{;\gamma;\alpha,\gamma},
\end{aligned}$$

if $\left|\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)\right| < 1.$

Finally, for the first term in $E(\ell_{\alpha\gamma\gamma})$, we have that, as

$$h_{t;\gamma,\gamma} = -|z_{t-1}|h_{t-1;\gamma} + \frac{1}{4}(\theta z_{t-1} + \gamma|z_{t-1}|)h_{t-1;\gamma}^2 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)h_{t-1;\gamma,\gamma}$$

we get

$$E\left(h_{t;\gamma,\gamma}\right)=\frac{-E\left|z\right|E_{;\gamma}+\frac{1}{4}\gamma E\left|z\right|E_{(;\gamma)^2}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=E_{;\gamma,\gamma},$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$, and it follows that

$$\begin{aligned}Eh_{t;\alpha}h_{t;\gamma,\gamma} &= -E\left|z\right|E_{;\gamma}-\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\alpha;\gamma}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\gamma,\gamma}\\ &\quad+\frac{1}{4}\gamma E\left|z\right|E_{(;\gamma)^2}+\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha(;\gamma)^2}\\ &\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\alpha}h_{t-1;\gamma,\gamma}\\ &= E_{;\alpha;\gamma,\gamma},\end{aligned}$$

if $\left|\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|E\left(z\left|z\right|\right)\right)\right|<1$.

4.9 Alpha-Gama-Theta

$$E\left(l_{\alpha\gamma\theta}\right)=-\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\gamma}+h_{t;\gamma}h_{t;\alpha,\theta}+h_{t;\alpha}h_{t;\gamma,\theta}-h_{t;\alpha}h_{t;\gamma}h_{t;\theta}\right)$$

First, for the last term in $E\left(l_{\alpha\gamma\theta}\right)$, as

$$\begin{aligned}h_{t;\gamma}h_{t;\theta} &= z_{t-1}\left|z_{t-1}\right|+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left(z_{t-1}h_{t-1;\gamma}+\left|z_{t-1}\right|h_{t-1;\theta}\right)\\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)^2h_{t-1;\gamma}h_{t-1;\theta}\end{aligned}$$

and as $E_{;\theta}=0$

$$Eh_{t;\gamma}h_{t;\theta}=\frac{E\left[z\left|z\right|\right]-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{;\gamma;\theta},$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$, and we get

$$\begin{aligned}Eh_{t;\alpha}h_{t;\gamma}h_{t;\theta} &= E\left[z\left|z\right|\right]+\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^3-\frac{1}{2}\gamma Ez^3\right)E_{;\alpha}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma}\\ &\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\gamma;\theta}\\ &\quad+\left(\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\right)E_{;\alpha;\gamma}\\ &\quad+\left(\beta^2E\left|z\right|+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3-\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta z_{t-1}^3\right)E_{;\alpha;\theta}\\ &\quad+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\left(\theta^2+\gamma^2\right)-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)Ez^3\\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]\end{array}\right)h_{t-1;\alpha}h_{t-1;\gamma}h_{t-1;\theta}\\ &= E_{;\alpha;\gamma;\theta}\end{aligned}$$

if $\left| \begin{array}{c} \beta^3 + \frac{3}{4}\beta\left(\theta^2+\gamma^2\right) - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E\left|z\right| + \frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3 \end{array} \right| < 1.$

Second, the first term in $E\left(l_{\alpha\gamma\theta}\right)$ is given by

$$\begin{aligned} Eh_{t;\theta}h_{t;\alpha,\gamma} &= -\frac{1}{2}E\left[z\left|z\right|\right]E_{;\alpha} - \frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha;\theta} \\ &+\frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\gamma;\theta} \\ &+\frac{1}{4}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)\left(E_{;\alpha;\gamma} - 2E_{;\alpha,\gamma}\right) \\ &+\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\theta}h_{t-1;\alpha,\gamma} \\ &= E_{;\theta;\alpha,\gamma} \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Third, the second term in $E\left(l_{\alpha\gamma\theta}\right)$ is given by

$$\begin{aligned} Eh_{t;\gamma}h_{t;\alpha,\theta} &= -\frac{1}{2}E\left[z\left|z\right|\right]E_{;\alpha} + \frac{1}{4}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\gamma} + \left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha,\theta} \\ &+\frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\gamma;\theta} + \frac{1}{4}\left(\theta E\left[z\left|z\right|\right] + \gamma\right)E_{;\alpha;\theta} \\ &+\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\gamma}h_{t-1;\alpha,\theta} \\ &= E_{;\gamma;\alpha,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, for the third term in $E\left(l_{\alpha\gamma\theta}\right)$ we have that

$$\begin{aligned} h_{t;\gamma,\theta} &= -\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\theta} - \frac{1}{2}z_{t-1}h_{t-1;\gamma} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma}h_{t-1;\theta} \\ &+\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\gamma,\theta} \end{aligned}$$

and as $E_{;\theta} = 0$

$$Eh_{t;\gamma,\theta} = \frac{1}{4}\frac{\gamma E\left|z\right|E_{;\gamma;\theta}}{1-\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)} = E_{;\gamma,\theta}$$

and it follows that

$$\begin{aligned} Eh_{t;\alpha}h_{t;\gamma,\theta} &= \frac{1}{4}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\gamma} - \frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)E_{;\alpha;\theta} + \frac{1}{4}\gamma E\left|z\right|E_{;\gamma;\theta} \\ &+\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E_{;\gamma,\theta} + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha;\gamma;\theta} \\ &+\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\alpha}h_{t-1;\gamma,\theta} \\ &= E_{;\alpha;\gamma,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

4.10 Alpha-Theta-Theta

$$E\left(l_{\alpha\theta\theta}\right)=-\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\theta}+2h_{t;\theta}h_{t;\alpha,\theta}-h_{t;\alpha}h_{t;\theta}^{2}\right)$$

First, for the third term in $E\left(l_{\alpha\theta\theta}\right)$, we have that

$$\begin{aligned} h_{t;\theta}^{2} &= z_{t-1}^{2}+2z_{t-1}\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta} \\ &\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2}h_{t-1;\theta}^{2} \end{aligned}$$

and, as $E_{;\theta}=0$,

$$Eh_{t;\theta}^{2}=\frac{1}{1-\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}=E_{(;\theta)^{2}}$$

and it follows that

$$\begin{aligned} Eh_{t;\alpha}h_{t;\theta}^{2} &= \left(\beta-\frac{1}{2}\theta Ez^{3}-\frac{1}{2}\gamma E\left|z\right|^{3}\right)E_{;\alpha}+\left(\begin{array}{c}\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}\\ -\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\end{array}\right)E_{(;\theta)^{2}} \\ &\quad+1+2\left(\begin{array}{c}\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)Ez^{3}-\theta\beta\\ -\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^{3}\end{array}\right)E_{;\alpha;\theta} \\ &\quad+\left(\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\left(\theta^{2}+\gamma^{2}\right)-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)\\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right)h_{t-1;\alpha}h_{t-1;\theta}^{2} \\ &= E_{;\alpha(;\theta)^{2}} \end{aligned}$$

if $\left|\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)\\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right|<1.$

Second, for the first term in $E\left(l_{\alpha\theta\theta}\right)$, we have that

$$h_{t;\theta,\theta}=-z_{t-1}h_{t-1;\theta}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}^{2}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta,\theta}$$

with

$$Eh_{t;\theta,\theta}=\frac{\frac{1}{4}\gamma E\left|z\right|E_{(;\theta)^{2}}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}=E_{;\theta,\theta}.$$

It follows that

$$\begin{aligned} Eh_{t;\alpha}h_{t;\theta,\theta} &= \frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\alpha;\theta}+\frac{1}{4}\gamma E\left|z\right|E_{(;\theta)^{2}}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\theta,\theta} \\ &\quad+\frac{1}{4}\left(-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\alpha(;\theta)^{2}} \\ &\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\alpha}h_{t-1;\theta,\theta} \\ &= E_{;\alpha;\theta,\theta}, \end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

Finally, the second term in $E(l_{\alpha\theta\theta})$ is given by

$$\begin{aligned} Eh_{t;\theta}h_{t;\alpha,\theta} &= -\frac{1}{2}E_{;\alpha} + \frac{1}{2}(\theta + \gamma E[z|z|])E_{;\alpha;\theta} - \frac{1}{2}(\theta + \gamma E[z|z|])E_{;\alpha,\theta} \\ &\quad + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\alpha(;\theta)^2} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)h_{t-1;\theta}h_{t-1;\alpha,\theta} \\ &= E_{;\theta;\alpha,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1$.

4.11 Beta-Beta-Beta

$$E(\ell_{\beta\beta\beta}) = -\frac{T}{2}\left(3E(h_{t;\beta}h_{t;\beta,\beta}) - E\left(h_{t;\beta}^3\right)\right).$$

First, for the last term of $E(\ell_{\beta\beta\beta})$, notice that

$$\begin{aligned} E\ln^2(h_t) &= \alpha^2 + \theta^2 + \gamma^2 + 2\alpha\gamma E|z| + 2\gamma\theta E[z|z|] \\ &\quad + 2\beta[\alpha + \gamma E|z|]L + \beta^2 E\ln^2(h_{t-1}) = L^2 \end{aligned}$$

$$\begin{aligned} E\ln^3(h_t) &= \alpha^3 + 3\alpha\gamma^2 + 3\alpha\theta^2 + 3\alpha^2\gamma E|z| + 6\alpha\gamma\theta E[z|z|] + \theta\left(3\gamma^2 + \theta^2\right)Ez^3 \\ &\quad + \gamma\left(3\theta^2 + \gamma^2\right)E|z|^3 + 3\beta\left[\alpha^2 + \gamma^2 + \theta^2 + 2\alpha\gamma E|z| + 2\gamma\theta E[z|z|]\right]L \\ &\quad + 3\beta^2[\alpha + \gamma E|z|]L^2 + \beta^3\ln^3(h_{t-1}) = L^3 \end{aligned}$$

and

$$\begin{aligned} Eh_{t;\beta}\ln^2 h_t &= \left\{\alpha^2 + \gamma^2 + \theta^2 + 2\gamma\theta E[z|z|] + 2\alpha\gamma E|z|\right\}L + 2\beta(\alpha + \gamma E|z|)L^2 + \beta^2 L^3 \\ &\quad + \left\{\begin{array}{c} \alpha^2\left(\beta - \frac{1}{2}\gamma E|z|\right) + (\theta + \gamma^2)\left(\beta - \frac{1}{2}\theta E z^3 - \frac{1}{2}\gamma E|z|^3\right) \\ -\alpha\theta(\theta + \gamma E[z|z|]) + 2\alpha\gamma\left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right) \\ +2\gamma\theta\left(\beta E[z|z|] - \frac{1}{2}\theta E|z|^3 - \frac{1}{2}\gamma E z^3\right) \end{array}\right\}E_{;\beta} \\ &\quad + 2\beta\left[\alpha\left(\beta - \frac{1}{2}\gamma E|z|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right]LE_{;\beta} \\ &\quad + \beta^2\left(\beta - \frac{1}{2}\gamma E|z|\right)Eh_{t-1;\beta}\ln^2 h_{t-1} \\ &= L^2 E_{;\beta}. \end{aligned}$$

Also

$$\begin{aligned} h_{t;\beta}^2 &= \ln^2(h_{t-1}) + 2\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)\ln(h_{t-1})h_{t-1;\beta} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma|z_{t-1}|\right)^2 h_{t-1;\beta}^2, \end{aligned}$$

and it follows

$$\begin{aligned}
Eh_{t;\beta}^{2}\ln h_{t} &= \left[\alpha+\gamma E\left|z\right|\right]L^{2}+\beta L^{3}+2\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)L^{2}E_{;\beta} \\
&+2\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right]LE_{;\beta} \\
&+\left(\begin{array}{c} \alpha\beta^{2}+\left(\frac{1}{4}\alpha-\beta\right)\left(\theta^{2}+\gamma^{2}\right)+\beta\gamma\left(\beta-\alpha\right)E\left|z\right| \\ +\gamma\theta\left(\frac{1}{2}\alpha-2\beta\right)E\left[z\left|z\right|\right]+\frac{1}{4}\gamma\left(3\theta^{2}+\gamma^{2}\right)E\left|z\right|^{3} \\ +\frac{1}{4}\theta\left(\theta^{2}+3\gamma^{2}\right)Ez^{3} \end{array}\right)E_{;(\beta)^{2}} \\
&+\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}^{2}\ln h_{t-1} \\
&= LE_{(;\beta)^{2}},
\end{aligned}$$

if $\left|\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$.

Hence

$$\begin{aligned}
Eh_{t;\beta}^{3} &= 3\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)L^{2}E_{;\beta}+3\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{(;\beta)^{2}} \\
&+L^{3}+\left(\begin{array}{c} \beta^{3}+\frac{3}{4}\beta\left(\theta^{2}+\gamma^{2}\right)-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right) \\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3} \end{array}\right)h_{t-1;\beta}^{3} \\
&= E_{(;\beta)^{3}},
\end{aligned}$$

if $\left|\begin{array}{c} \beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right) \\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3} \end{array}\right|<1$.

Finally, for the first term in $E\left(\ell_{\beta\beta\beta}\right)$ we have that

$$\begin{aligned}
Eh_{t;\beta,\beta}\ln h_{t} &= 2\left(\alpha+\gamma E\left|z\right|\right)E_{;\beta}+2\beta LE_{;\beta}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{(;\beta)^{2}} \\
&+\left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta,\beta} \\
&+\frac{1}{4}\left[\alpha\gamma E\left|z\right|+\theta^{2}+2\gamma\theta E\left[z\left|z\right|\right]+\gamma^{2}\right]E_{(;\beta)^{2}} \\
&+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\beta,\beta}\ln h_{t-1} \\
&= LE_{;\beta,\beta}
\end{aligned}$$

and it follows

$$\begin{aligned}
Eh_{t;\beta}h_{t;\beta,\beta} &= 2LE_{;\beta}+\frac{1}{4}\gamma E\left|z\right|LE_{(;\beta)^{2}}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\beta,\beta} \\
&+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{(;\beta)^{2}}+\frac{1}{4}\left(-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\beta)^{3}} \\
&+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\beta,\beta} \\
&= E_{;\beta;\beta,\beta},
\end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.12 Beta-Beta-Gama

$$E\left(\ell_{\beta\beta\gamma}\right)=-\frac{T}{2}E\left(2h_{t;\beta}h_{t;\gamma,\beta}+h_{t;\gamma}h_{t;\beta,\beta}-h_{t;\beta}^{2}h_{t;\gamma}\right)$$

First, for the last term in $E\left(\ell_{\beta\beta\gamma}\right)$, we have that

$$\begin{aligned}
Eh_{t;\gamma}\ln^2 h_t &= 2\alpha\gamma+\alpha^2E|z|+\left(\theta^2+\gamma^2\right)E|z|^3+2\alpha\theta E\left[z|z|\right]+2\gamma\theta Ez^3\\
&\quad +2\beta\left[\alpha E|z|+\theta E\left[z|z|\right]+2\gamma\right]L+\beta^2E|z|L^2\\
&\quad +\left\{\begin{array}{c}\alpha^2\left(\beta-\frac{1}{2}\gamma E|z|\right)+\left[\theta^2+\gamma^2\right]\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E|z|^3\right)\\ -\alpha\theta\left(\theta+\gamma E\left[z|z|\right]\right)+2\alpha\gamma\left(\beta E|z|-\frac{1}{2}\theta E\left[z|z|\right]-\frac{1}{2}\gamma\right)\\ +2\gamma\theta\left(\beta E\left[z|z|\right]-\frac{1}{2}\theta E|z|^3-\frac{1}{2}\gamma Ez^3\right)\end{array}\right\}E_{;\gamma}\\
&\quad +\beta\left[2\alpha\left(\beta-\gamma\frac{1}{2}E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta E\left[z|z|\right]\right]LE_{;\gamma}\\
&\quad +\beta^2\left(\beta-\gamma\frac{1}{2}E|z|\right)Eh_{t-1;\gamma}\ln^2 h_{t-1}\\
&= L^2E_{;\gamma},
\end{aligned}$$

if $\left|\beta^2\left(\beta-\frac{1}{2}\gamma E|z|\right)\right|<1$.

Also, as

$$\begin{aligned}
h_{t;\beta}h_{t;\gamma} &= |z_{t-1}|\ln\left(h_{t-1}\right)+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)\left(h_{t-1;\gamma}\ln h_{t-1}+|z_{t-1}|h_{t-1;\beta}\right)\\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}|z_{t-1}|\right)^2h_{t-1;\beta}h_{t-1;\gamma},
\end{aligned}$$

we get

$$\begin{aligned}
Eh_{t;\beta}h_{t;\gamma}\ln h_t &= \left[\alpha E|z|+\theta E\left[z|z|\right]+\gamma\right]L+\beta E|z|L^2+\left[\begin{array}{c}\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2\\ -\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta E\left[z|z|\right]\end{array}\right]LE_{;\gamma}\\
&\quad +\left[\begin{array}{c}\alpha\left[\beta E|z|-\frac{1}{2}\theta E\left[z|z|\right]-\frac{1}{2}\gamma\right]+\gamma\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E|z|^3\right)\\ +\theta\left[\beta E\left[z|z|\right]-\frac{1}{2}\theta E|z|^3-\frac{1}{2}\gamma Ez^3\right]\end{array}\right]E_{;\beta}\\
&\quad +\beta\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2E_{;\gamma}+\beta\left(\beta E|z|-\frac{1}{2}\theta E\left[z|z|\right]-\frac{1}{2}\gamma\right)LE_{;\beta}\\
&\quad +\left[\begin{array}{c}\alpha\left[\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right]\\ +\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E|z|^3-\beta\left(\theta^2+\gamma^2\right)\\ +\beta^2\gamma E|z|+\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E\left[z|z|\right]\end{array}\right]E_{;\beta;\gamma}\\
&\quad +\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right)Eh_{t-1;\beta}h_{t-1;\gamma}\ln h_{t-1}\\
&= LE_{;\beta;\gamma},
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\beta}^2 h_{t;\gamma} &= E|z|L^2 + \left(\beta - \frac{1}{2}\gamma E|z|\right) L^2 E_{;\gamma} + 2\left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right) LE_{;\beta} \\
&\quad + 2\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right) LE_{;\beta;\gamma} \\
&\quad + \left(\begin{array}{c} \beta^2 E|z| + \frac{1}{4}\left(\theta^2+\gamma^2\right) E|z|^3 \\ -\theta\beta E[z|z|] - \gamma\beta + \frac{1}{2}\gamma\theta E z^3 \end{array}\right) E_{(;\beta)^2} \\
&\quad + \left(\begin{array}{c} \beta^3 + \frac{3}{4}\beta\left(\theta^2+\gamma^2\right) - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right) E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\gamma\beta\theta E[z|z|] - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right) E|z|^3 \end{array}\right) h_{t-1;\beta}^2 h_{t-1;\gamma} \\
&= E_{(;\beta)^2;\gamma},
\end{aligned}$$

if $\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right) E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\beta\gamma\theta E(z|z|) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right) E|z|^3 \end{array}\right| < 1.$

Also for the first term in $E\left(\ell_{\beta\beta\gamma}\right)$, we have

$$\begin{aligned}
Eh_{t;\beta,\gamma}\ln h_t &= (\alpha + \gamma E|z|) E_{;\gamma} + \beta LE_{;\gamma} - \frac{1}{2}\left((\beta+\alpha) E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right) E_{;\beta} \\
&\quad + \frac{1}{4}\left[\left(\theta^2 + 2\gamma\theta E[z|z|] + \gamma^2\right) + \alpha\gamma E|z|\right] E_{;\beta;\gamma} + \beta LE_{;\gamma} - \frac{1}{2}\beta E|z| LE_{;\beta} \\
&\quad + \left[\alpha\left(\beta - \frac{1}{2}\gamma E|z|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right] E_{;\beta,\gamma} \\
&\quad + \frac{1}{4}\beta\gamma E|z| LE_{;\beta;\gamma} + \beta\left(\beta - \frac{1}{2}\gamma E|z|\right) Eh_{t-1;\beta,\gamma}\ln h_{t-1} \\
&= LE_{;\beta,\gamma}
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\beta,\gamma} &= LE_{;\gamma} - \frac{1}{2}E|z| LE_{;\beta} + \frac{1}{4}\gamma E|z| LE_{;\beta;\gamma} - \frac{1}{2}\left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right) E_{(;\beta)^2} \\
&\quad + \left(\beta - \frac{1}{2}\gamma E|z|\right)\left(LE_{;\beta,\gamma} + E_{;\beta;\gamma}\right) + \frac{1}{4}\left(\beta\gamma E|z| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E[z|z|]\right) E_{(;\beta)^2;\gamma} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right) h_{t-1;\beta}h_{t-1;\beta,\gamma} \\
&= E_{;\beta;\beta,\gamma},
\end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1.$

Finally, the second term of $E\left(\ell_{\beta\beta\gamma}\right)$ is given by

$$\begin{aligned}
h_{t;\gamma}h_{t;\beta,\beta} &= 2E|z| E_{;\beta} + \frac{1}{4}(\theta E z|z| + \gamma) E_{(;\beta)^2} + \left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right) E_{;\beta,\beta} \\
&\quad + 2\left(\beta - \frac{1}{2}\gamma E|z|\right) E_{;\beta;\gamma} + \frac{1}{4}\left(\beta\gamma E|z| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E z|z|\right) E_{(;\beta)^2;\gamma} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right) Eh_{t-1;\gamma}h_{t-1;\beta,\beta} \\
&= E_{;\gamma;\beta,\beta}
\end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1.$

4.13 Beta-Beta-Theta

$$E\left(\ell_{\beta\beta\theta}\right)=-\frac{T}{2}E\left(2h_{t;\beta}h_{t;\theta,\beta}+h_{t;\theta}h_{t;\beta,\beta}-h_{t;\beta}^{2}h_{t;\theta}\right).$$

First, for the last term of $E\left(\ell_{\beta\beta\theta}\right)$ and as $E_{;\theta}=0$

$$\begin{aligned}
Eh_{t;\theta}\ln^{2}h_{t} &= 2\alpha\theta+\left(\theta^{2}+\gamma^{2}\right)Ez^{3}+2\alpha\gamma E\left[z\left|z\right|\right]+2\gamma\theta E\left|z\right|^{3}+2\beta\left[\theta+\gamma E\left[z\left|z\right|\right]\right]L \\
&\quad +2\beta\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right]LE_{;\theta} \\
&\quad +\beta^{2}\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\theta}\ln^{2}h_{t-1} \\
&= L^{2}E_{;\theta}
\end{aligned}$$

Also

$$\begin{aligned}
h_{t;\beta}h_{t;\theta} &= z_{t-1}\ln h_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(h_{t-1;\theta}\ln h_{t-1}+z_{t-1}h_{t-1;\beta}\right) \\
&\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2}h_{t-1;\beta}h_{t-1;\theta}
\end{aligned}$$

with

$$\begin{aligned}
Eh_{t;\beta}h_{t;\theta}\ln h_{t} &= \left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right]LE_{;\theta} \\
&\quad +\left(\theta+\gamma E\left[z\left|z\right|\right]\right)L+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)L^{2}E_{;\theta}-\frac{1}{2}\beta\left(\theta+\gamma E\left[z\left|z\right|\right]\right)LE_{;\beta} \\
&\quad +\left[\begin{array}{c}\gamma\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^{3}-\frac{1}{2}\gamma Ez^{3}\right)-\frac{1}{2}\alpha\left(\theta+\gamma E\left[z\left|z\right|\right]\right) \\ +\theta\left(\beta-\frac{1}{2}\theta Ez^{3}-\frac{1}{2}\gamma E\left|z\right|^{3}\right)\end{array}\right]E_{;\beta} \\
&\quad +\left[\begin{array}{c}\theta FORM6+\gamma FORM4 \\ +\alpha FORM2\end{array}\right]E_{;\beta;\theta} \\
&\quad +\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\theta}\ln h_{t-1} \\
&= LE_{;\beta;\theta}
\end{aligned}$$

if $\left|\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$, and it follows that

$$\begin{aligned}
Eh_{t;\beta}^{2}h_{t;\theta} &= \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)L^{2}E_{;\theta}-\left(\theta+\gamma E\left[z\left|z\right|\right]\right)LE_{;\beta} \\
&\quad +2\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{;\beta;\theta} \\
&\quad +\left(\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)Ez^{3}-\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^{3}\right)E_{(;\beta)^{2}} \\
&\quad +\left(\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\left(\theta^{2}+\gamma^{2}\right)-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right) \\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right)h_{t-1;\beta}^{2}h_{t-1;\theta} \\
&= E_{(;\beta)^{2};\theta}
\end{aligned}$$

if $\left| \begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\beta\gamma\theta E\left(z|z|\right) - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E|z|^3 \end{array} \right| < 1.$

For the first term of $E\left(\ell_{\beta\beta\theta}\right)$ we have that

$$\begin{aligned} h_{t;\beta,\theta} &= h_{t-1;\theta} - \frac{1}{2}z_{t-1}h_{t-1;\beta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\beta}h_{t-1;\theta} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\theta} \end{aligned}$$

and as $E_{;\theta} = 0$

$$\begin{aligned} Eh_{t;\beta,\theta}\ln h_t &= -\frac{1}{2}\left[\theta + \gamma E\left[z\left|z\right|\right]\right]E_{;\beta} + \frac{1}{4}\left(\alpha\gamma E|z| + \theta^2 + 2\gamma\theta E\left[z\left|z\right|\right] + \gamma^2\right)E_{;\beta;\theta} \\ &\quad + \left[\alpha\left(\beta - \frac{1}{2}\gamma E|z|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E\left[z\left|z\right|\right]\right]E_{;\beta,\theta} \\ &\quad + \beta L E_{;\theta} + \frac{1}{4}\beta\gamma E|z|\,LE_{;\beta;\theta} + \beta\left(\beta - \frac{1}{2}\gamma E|z|\right)Eh_{t-1;\beta,\theta}\ln h_{t-1} \\ &= LE_{;\beta,\theta}. \end{aligned}$$

It follows that

$$\begin{aligned} Eh_{t;\beta}h_{t;\beta,\theta} &= LE_{;\theta} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\beta;\theta} + \frac{1}{4}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)E_{(;\beta)^2} \\ &\quad + \frac{1}{4}\gamma E|z|\,LE_{;\beta;\theta} + \left(\beta - \frac{1}{2}\gamma E|z|\right)LE_{;\beta,\theta} \\ &\quad + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\beta)^2;\theta} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\beta,\theta} \\ &= E_{;\beta;\beta,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, the second term of $E\left(\ell_{\beta\beta\theta}\right)$ is given,

$$\begin{aligned} Eh_{t;\theta}h_{t;\beta,\beta} &= 2\left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\beta;\theta} + \frac{1}{4}\left(\theta + \gamma E\left[z\left|z\right|\right]\right)\left(E_{(;\beta)^2} - 2E_{;\beta,\beta}\right) \\ &\quad + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\beta)^2;\theta} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\beta,\beta} \\ &= E_{;\theta;\beta,\beta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

4.14 Beta-Gama-Gama

$$E\left(\ell_{\beta\gamma\gamma}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\gamma}+2h_{t;\gamma}h_{t;\beta,\gamma}-h_{t;\beta}h_{t;\gamma}^{2}\right)$$

First, for the last term in $E\left(\ell_{\beta\gamma\gamma}\right)$ we have

$$h_{t;\gamma}^{2}=z_{t-1}^{2}+2\left|z_{t-1}\right|\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)h_{t-1;\gamma}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)^{2}h_{t-1;\gamma}^{2}$$

and

$$\begin{aligned}
Eh_{t;\gamma}^{2}\ln h_{t} &= \alpha+\theta Ez^{3}+\gamma E\left|z\right|^{3}+\beta L+2\beta\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)LE_{;\gamma}\\
&\quad+2\left[\begin{array}{c}\alpha\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)+\theta\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^{3}\right)\\-\gamma\theta Ez^{3}+\gamma\left(\beta-\frac{1}{2}\gamma E\left|z\right|^{3}\right)\end{array}\right]E_{;\gamma}\\
&\quad+\left(\begin{array}{c}\alpha\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\\-\beta\left(\theta^{2}+\gamma^{2}\right)+\beta^{2}\gamma E\left|z\right|+\frac{1}{4}\gamma\left(3\theta^{2}+\gamma^{2}\right)E\left|z\right|^{3}\\+\frac{1}{4}\theta\left(\theta^{2}+3\gamma^{2}\right)Ez^{3}-2\beta\gamma\theta E\left[z\left|z\right|\right]\end{array}\right)E_{(;\gamma)^{2}}\\
&\quad+\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}^{2}\ln h_{t-1}\\
&= LE_{(;\gamma)^{2}},
\end{aligned}$$

if $\left|\beta\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$. It follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\gamma}^{2} &= L+\left(\beta-\frac{1}{2}\theta Ez^{3}-\frac{1}{2}\gamma E\left|z\right|^{3}\right)E_{;\beta}+2\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)LE_{;\gamma}\\
&\quad+2\left(\begin{array}{c}\beta^{2}E\left|z\right|+\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)E\left|z\right|^{3}\\-\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^{3}\end{array}\right)E_{;\beta;\gamma}\\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{(;\gamma)^{2}}\\
&\quad+\left(\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\left(\theta^{2}+\gamma^{2}\right)-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)\\-\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right)h_{t-1;\beta}h_{t-1;\gamma}^{2}\\
&= E_{;\beta(;\gamma)^{2}},
\end{aligned}$$

if $\left|\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)\\-\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right|<1.$

First, for the first term in $E\left(\ell_{\beta\gamma\gamma}\right)$ we have,

$$\begin{aligned}
E\ln\left(h_{t}\right)h_{t;\gamma,\gamma} &= -\left[\alpha E\left|z\right|+\theta E\left[z\left|z\right|\right]+\gamma\right]E_{;\gamma}+\frac{1}{4}\left[\alpha\gamma E\left|z\right|+\theta^{2}+2\gamma\theta E\left[z\left|z\right|\right]+\gamma^{2}\right]E_{(;\gamma)^{2}}\\
&\quad+\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right]E_{;\gamma,\gamma}\\
&\quad-\beta E\left|z\right|LE_{;\gamma}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{(;\gamma)^{2}}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\ln\left(h_{t-1}\right)h_{t-1;\gamma,\gamma}\\
&= LE_{;\gamma,\gamma},
\end{aligned}$$

and it follows

$$\begin{aligned} Eh_{t;\beta}h_{t;\gamma,\gamma} &= -E\left|z\right|LE_{;\gamma}-\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\beta;\gamma}+\frac{1}{4}\gamma E\left|z\right|LE_{(;\gamma)^2} \\ &\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta(;\gamma)^2}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\gamma,\gamma} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\gamma,\gamma} \\ &= E_{;\beta;\gamma,\gamma}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, the second term of $E\left(\ell_{\beta\gamma\gamma}\right)$ is given

$$\begin{aligned} Eh_{t;\gamma}h_{t;\beta,\gamma} &= E\left|z\right|E_{;\gamma}-\frac{1}{2}E_{;\beta}+\frac{1}{4}\left(\theta Ez\left|z\right|+\gamma\right)E_{;\beta;\gamma}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{(;\gamma)^2} \\ &\quad +\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)\left\{E_{;\beta,\gamma}-\frac{1}{2}E_{;\beta;\gamma}\right\} \\ &\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta(;\gamma)^2} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\beta,\gamma} \\ &= E_{;\gamma;\beta,\gamma} \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.15 Beta-Gama-Theta

$$E\left(\ell_{\beta\gamma\theta}\right)=-\frac{T}{2}E\left(h_{t;\theta}h_{t;\beta,\gamma}+h_{t;\gamma}h_{t;\beta,\theta}+h_{t;\beta}h_{t;\gamma,\theta}-h_{t;\beta}h_{t;\gamma}h_{t;\theta}\right).$$

For the last term in $E\left(\ell_{\beta\gamma\theta}\right)$,

$$\begin{aligned} h_{t;\gamma}h_{t;\theta} &= z_{t-1}\left|z_{t-1}\right|+\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left(z_{t-1}h_{t-1;\gamma}+\left|z_{t-1}\right|h_{t-1;\theta}\right) \\ &\quad +\left(\beta-\frac{1}{2}\theta z_{t-1}-\gamma\frac{1}{2}\left|z_{t-1}\right|\right)^2 h_{t-1;\gamma}h_{t-1;\theta}, \end{aligned}$$

and as $E_{;\theta}=0$

$$Eh_{t;\gamma}h_{t;\theta}=\frac{E\left[z\left|z\right|\right]-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}.$$

Further,

$$
\begin{aligned}
Eh_{t;\gamma}h_{t;\theta}\ln h_t &= \alpha E\left[z\left|z\right|\right]+\theta E\left|z\right|^3+\gamma Ez^3+\beta E\left[z\left|z\right|\right]L \\
&+\begin{bmatrix} -\frac{1}{2}\alpha\left(\theta+\gamma E\left[z\left|z\right|\right]\right) \\ +\theta\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right) \\ +\gamma\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^3-\frac{1}{2}\gamma Ez^3\right)\end{bmatrix}E_{;\gamma} \\
&+\begin{pmatrix} \alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) \\ -\beta\left(\theta^2+\gamma^2\right)+\beta^2\gamma E\left|z\right|+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E\left|z\right|^3 \\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E\left[z\left|z\right|\right]\end{pmatrix}E_{;\gamma;\theta} \\
&-\frac{1}{2}\beta\left(\theta+\gamma E\left[z\left|z\right|\right]\right)LE_{;\gamma}+\beta\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)LE_{;\theta} \\
&+\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\theta}\ln h_{t-1} \\
&= LE_{;\gamma;\theta},
\end{aligned}
$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$.

It follows that

$$
\begin{aligned}
Eh_{t;\beta}h_{t;\gamma}h_{t;\theta} &= E\left[z\left|z\right|\right]L+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{;\gamma;\theta} \\
&+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)LE_{;\theta}+\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^3-\frac{1}{2}\gamma Ez^3\right)E_{;\beta} \\
&+\begin{pmatrix} \frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 \\ -\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\end{pmatrix}E_{;\beta;\gamma} \\
&-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)LE_{;\gamma}+\begin{pmatrix} \beta^2E\left|z\right|+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3 \\ -\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^3\end{pmatrix}E_{;\beta;\theta} \\
&+\begin{pmatrix} \beta^3+\frac{3}{4}\beta\left(\theta^2+\gamma^2\right)-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{pmatrix}Eh_{t-1;\beta}h_{t-1;\gamma}h_{t-1;\theta} \\
&= E_{;\beta;\gamma;\theta},
\end{aligned}
$$

if $\left|\begin{pmatrix} \beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{pmatrix}\right|<1$.

Second,for the third term in $E\left(\ell_{\beta\gamma\theta}\right)$, as $E_{;\theta}=0$

$$
\begin{aligned}
E\ln\left(h_t\right)h_{t;\gamma,\theta} &= -\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma}-\frac{1}{2}\beta E\left|z\right|LE_{;\theta}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{;\gamma;\theta} \\
&+\frac{1}{4}\left(\theta^2+2\gamma\theta E\left[z\left|z\right|\right]+\gamma^2+\alpha E\left|z\right|\right)E_{;\gamma;\theta} \\
&+\left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\gamma,\theta} \\
&+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\ln\left(h_{t-1}\right)h_{t-1;\gamma,\theta} \\
&= LE_{;\gamma,\theta},
\end{aligned}
$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$, and

$$\begin{aligned} Eh_{t;\beta}h_{t;\gamma,\theta} &= \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\gamma,\theta}-\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\beta;\theta} \\ &\quad -\frac{1}{2}E\left|z\right|LE_{;\theta}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma}+\frac{1}{4}\gamma E\left|z\right|LE_{;\gamma;\theta} \\ &\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma;\theta} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\beta}h_{t-1;\gamma,\theta} \\ &= E_{;\beta;\gamma,\theta}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

As $E_{;\theta}=0$, the second term in $E\left(\ell_{\beta\gamma\theta}\right)$ becomes

$$\begin{aligned} Eh_{t;\gamma}h_{t;\beta,\theta} &= \frac{1}{4}\left(\theta E\left[z\left|z\right|\right]+\gamma\right)E_{;\beta;\theta}+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\beta,\theta} \\ &\quad -\frac{1}{2}\theta E\left|z\right|E_{;\beta}+\left(\beta-\gamma\frac{1}{2}E\left|z\right|\right)E_{;\gamma;\theta}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma} \\ &\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma;\theta} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\gamma}h_{t-1;\beta,\theta} \\ &= E_{;\gamma;\beta,\theta}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, the first term in $E\left(\ell_{\beta\gamma\theta}\right)$ equals

$$\begin{aligned} Eh_{t;\theta}h_{t;\beta,\gamma} &= -\frac{1}{2}E\left[z\left|z\right|\right]E_{;\beta}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\beta,\gamma} \\ &\quad +\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\gamma;\theta}-\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\beta;\theta} \\ &\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{;\beta;\gamma;\theta} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\theta}h_{t-1;\beta,\gamma} \\ &= E_{;\theta;\beta,\gamma}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.16 Beta-Theta-Theta

$$E\left(l_{\beta\theta\theta}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\theta}+2h_{t;\theta}h_{t;\beta,\theta}-h_{t;\beta}h_{t;\theta}^2\right)$$

First, for the last term in $E(l_{\beta\theta\theta})$

$$
\begin{aligned}
Eh_{t;\theta}^2 \ln h_t &= \alpha + \theta E z^3 + \gamma E|z|^3 + \beta L - \beta(\theta + \gamma E[z|z|]) LE_{;\theta} \\
&+ \begin{pmatrix} \alpha\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2\right) - \beta\left(\theta^2+\gamma^2\right) + \gamma\beta(\beta-\alpha)E|z| \\ +\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E|z|^3 + \frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3 \\ +\gamma\theta\left(\frac{1}{2}\alpha - 2\beta\right)E[z|z|] \end{pmatrix} E_{(;\theta)^2} \\
&+\beta\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right) Eh_{t-1;\theta}^2 \ln h_{t-1} \\
&= LE_{(;\theta)^2},
\end{aligned}
$$

if $\left|\beta\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)\right| < 1$, and

$$
\begin{aligned}
Eh_{t;\beta}h_{t;\theta}^2 &= -(\theta+\gamma E[z|z|])LE_{;\theta} + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)LE_{(;\theta)^2} \\
&+L + \left(\beta - \frac{1}{2}\theta E z^3 - \frac{1}{2}\gamma E|z|^3\right)E_{;\beta} + 2\begin{pmatrix} \frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 - \theta\beta \\ -\gamma\beta E[z|z|] + \frac{1}{2}\gamma\theta E|z|^3 \end{pmatrix} E_{;\beta;\theta} \\
&+ \begin{pmatrix} \beta^3 + \frac{3}{4}\beta\left(\theta^2+\gamma^2\right) - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\gamma\beta\theta E[z|z|] - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3 \end{pmatrix} h_{t-1;\beta}h_{t-1;\theta}^2 \\
&= E_{;\beta(;\theta)^2},
\end{aligned}
$$

if $\left| \begin{matrix} \beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 E z^3 - \frac{3}{2}\gamma\left(\beta^2 E|z| - \beta\theta E(z|z|) + \frac{1}{4}\theta^2 E|z|^3\right) \\ +\frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta E z^3\right) - \frac{1}{8}\gamma^3 E|z|^3 \end{matrix} \right| < 1.$

Second, the second term in $E(l_{\beta\theta\theta})$ equals

$$
\begin{aligned}
Eh_{t;\theta}h_{t;\beta,\theta} &= \frac{1}{2}(\theta+\gamma E[z|z|])E_{;\beta;\theta} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{(;\theta)^2} - \frac{1}{2}(\theta+\gamma E[z|z|])E_{;\beta,\theta} \\
&-\frac{1}{2}E_{;\beta} + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\beta(;\theta)^2} \\
&+\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)Eh_{t-1;\theta}h_{t-1;\beta,\theta} \\
&= E_{;\theta;\beta,\theta},
\end{aligned}
$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1$.

Finally, for the first term in $E(l_{\beta\theta\theta})$, as $E_{;\theta} = 0$,

$$
\begin{aligned}
Eh_{t;\theta,\theta}\ln h_t &= \frac{1}{4}\left[\theta^2 + 2\gamma\theta E[z|z|] + \gamma^2 + \alpha\gamma E|z|\right]E_{(;\theta)^2} \\
&+\left(\alpha\left(\beta - \frac{1}{2}\gamma E|z|\right) - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\theta,\theta} \\
&+\frac{1}{4}\beta(\theta+\gamma E[z|z|])LE_{(;\theta)^2} + \beta\left(\beta - \frac{1}{2}\gamma E|z|\right)Eh_{t-1;\theta,\theta}\ln h_{t-1} \\
&= LE_{;\theta,\theta},
\end{aligned}
$$

and it follows that

$$\begin{aligned} Eh_{t;\beta}h_{t;\theta,\theta} &= \frac{1}{4}\gamma E\left|z\right| LE_{(;\theta)^2} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) LE_{;\theta,\theta} + \frac{1}{2}\left(\theta + \gamma E\left[z\left|z\right|\right]\right) E_{;\beta;\theta} \\ &+ \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right) E_{;\beta(;\theta)^2} \\ &+ \left(\beta^2 + \frac{1}{4}\left(\theta^2 + \gamma^2\right) - \gamma\beta E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right) Eh_{t-1;\beta}h_{t-1;\theta,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$.

4.17 Gama-Gama-Gama

$$E\left(\ell_{\gamma\gamma\gamma}\right) = -\frac{T}{2}\left(3E\left(h_{t;\gamma}h_{t;\gamma,\gamma}\right) - E\left(h_{t;\gamma}^3\right)\right).$$

First, for the last term in $E\left(\ell_{\gamma\gamma\gamma}\right)$

$$\begin{aligned} h_{t;\gamma}^3 &= 3\left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)\left[z_{t-1}^2 h_{t-1;\gamma} + \left|z_{t-1}\right|\left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right) h_{t-1;\gamma}^2\right] \\ &+ \left|z_{t-1}\right|^3 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \gamma\frac{1}{2}\left|z_{t-1}\right|\right)^3 h_{t-1;\gamma}^3, \end{aligned}$$

and

$$\begin{aligned} Eh_{t;\gamma}^3 &= \frac{E\left|z\right|^3 + 3\begin{pmatrix} \beta - \frac{1}{2}\theta Ez^3 \\ -\frac{1}{2}\gamma E\left|z\right|^3 \end{pmatrix} E_{;\gamma} + 3\begin{pmatrix} \beta^2 E\left|z\right| + \frac{1}{4}\left(\theta^2 + \gamma^2\right) E\left|z\right|^3 \\ -\theta\beta E\left[z\left|z\right|\right] - \gamma\beta + \frac{1}{2}\gamma\theta Ez^3 \end{pmatrix} E_{(;\gamma)^2}}{1 - \begin{pmatrix} \beta^3 + \frac{3}{4}\beta\left(\theta^2 + \gamma^2\right) - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right) E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E\left|z\right| + \frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right] - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right) E\left|z\right|^3 \end{pmatrix}} \\ &= E_{(;\gamma)^3}, \end{aligned}$$

if $\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E\left|z\right| - \beta\theta E\left(z\left|z\right|\right) + \frac{1}{4}\theta^2 E\left|z\right|^3\right) \\ +\frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E\left|z\right|^3 \end{array}\right| < 1$.

Finally, the first term in $E\left(\ell_{\gamma\gamma\gamma}\right)$ equals

$$\begin{aligned} Eh_{t;\gamma}h_{t;\gamma,\gamma} &= -E_{;\gamma} + \left(\frac{1}{4}\left(\theta E\left[z\left|z\right|\right] + \gamma\right) - \left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right)\right) E_{(;\gamma)^2} \\ &+ \left(\beta E\left|z\right| - \frac{1}{2}\theta E\left[z\left|z\right|\right] - \frac{1}{2}\gamma\right) E_{;\gamma,\gamma} \\ &+ \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E\left|z\right| - \gamma\theta E\left[z\left|z\right|\right]\right) E_{(;\gamma)^3} \\ &+ \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) h_{t-1;\gamma}h_{t-1;\gamma,\gamma} \\ &= E_{;\gamma;\gamma,\gamma}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right|\right| < 1$.

4.18 Gama-Gama-Theta

$$E\left(\ell_{\gamma\gamma\theta}\right)=-\frac{T}{2}E\left(h_{t;\theta}h_{t;\gamma,\gamma}+2h_{t;\gamma}h_{t;\gamma,\theta}-h_{t;\gamma}^{2}h_{t;\theta}\right)$$

First, the lest term in $E\left(\ell_{\gamma\gamma\theta}\right)$, since $E_{;\theta}=0$, equals

$$\begin{aligned}
Eh_{t;\gamma}^{2}h_{t;\theta} &= Ez^{3}+2\left(\beta E\left[z\left|z\right|\right]-\frac{1}{2}\theta E\left|z\right|^{3}-\frac{1}{2}\gamma Ez^{3}\right)E_{;\gamma}+\left(\beta-\frac{1}{2}\theta Ez^{3}-\frac{1}{2}\gamma E\left|z\right|^{3}\right)E_{;\theta} \\
&\quad+\begin{pmatrix}\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)Ez^{3}-\theta\beta \\ -\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^{3}\end{pmatrix}E_{(;\gamma)^{2}}+2\begin{pmatrix}\beta^{2}E\left|z\right|+\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)E\left|z\right|^{3} \\ -\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^{3}\end{pmatrix}E_{;\gamma;\theta} \\
&\quad+\begin{pmatrix}\beta^{3}+\frac{3}{4}\beta\left(\theta^{2}+\gamma^{2}\right)-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right) \\ -\frac{3}{2}\beta^{2}\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{pmatrix}h_{t-1;\theta}h_{t-1;\gamma}^{2} \\
&= E_{;\theta(;\gamma)^{2}}
\end{aligned}$$

if $\left|\begin{matrix}\beta^{3}+\frac{3}{4}\beta\theta^{2}-\frac{1}{8}\theta^{3}Ez^{3}-\frac{3}{2}\gamma\left(\beta^{2}E\left|z\right|-\beta\theta E\left(z\left|z\right|\right)+\frac{1}{4}\theta^{2}E\left|z\right|^{3}\right) \\ +\frac{3}{4}\gamma^{2}\left(\beta-\frac{1}{2}\theta Ez^{3}\right)-\frac{1}{8}\gamma^{3}E\left|z\right|^{3}\end{matrix}\right|<1.$

Second, the second term in $E\left(\ell_{\gamma\gamma\theta}\right)$, since $E_{;\theta}=0$, equals

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\gamma,\theta} &= -\frac{1}{2}E\left[z\left|z\right|\right]E_{;\gamma}+\frac{1}{2}\left[\frac{1}{2}\left(\theta E\left[z\left|z\right|\right]+\gamma\right)-\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)\right]E_{;\gamma;\theta} \\
&\quad+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{(;\gamma)^{2}}+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\gamma,\theta} \\
&\quad+\frac{1}{4}\left(-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\gamma)^{2};\theta} \\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\gamma}h_{t-1;\gamma,\theta} \\
&= E_{;\gamma;\gamma,\theta}
\end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

Finally, the first term in $E\left(\ell_{\gamma\gamma\theta}\right)$,

$$\begin{aligned}
Eh_{t;\theta}h_{t;\gamma,\gamma} &= -E\left[z\left|z\right|\right]E_{;\gamma}+\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{(;\gamma)^{2}}-\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)E_{;\gamma;\theta} \\
&\quad-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\gamma,\gamma}+\frac{1}{4}\left(-\frac{1}{2}\gamma^{2}-\frac{1}{2}\theta^{2}+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\gamma)^{2};\theta} \\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)h_{t-1;\theta}h_{t-1;\gamma,\gamma} \\
&= E_{;\theta;\gamma,\gamma}
\end{aligned}$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

4.19 Gama-Theta-Theta

$$E\left(l_{\gamma\theta\theta}\right)=-\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\theta}+2h_{t;\theta}h_{t;\gamma,\theta}-h_{t;\gamma}h_{t;\theta}^{2}\right).$$

First, the last term of $E(l_{\gamma\theta\theta})$ equals

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\theta}^2 &= E|z|^3 + \left(\begin{array}{c} \beta^2 E|z| + \frac{1}{4}\left(\theta^2+\gamma^2\right)E|z|^3 \\ -\theta\beta E[z|z|] - \gamma\beta + \frac{1}{2}\gamma\theta E z^3 \end{array}\right) E_{(;\theta)^2} \\
&\quad +2\left(\begin{array}{c} \frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 - \theta\beta \\ -\gamma\beta E[z|z|] + \frac{1}{2}\gamma\theta E|z|^3 \end{array}\right) E_{;\gamma;\theta} + \left(\beta - \frac{1}{2}\theta E z^3 - \frac{1}{2}\gamma E|z|^3\right) E_{;\gamma} \\
&\quad +\left(\begin{array}{c} \beta^3 + \frac{3}{4}\beta\left(\theta^2+\gamma^2\right) - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) \\ -\frac{3}{2}\beta^2\gamma E|z| + \frac{3}{2}\gamma\beta\theta E[z|z|] - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3 \end{array}\right) Eh_{t-1;\gamma}h_{t-1;\theta}^2 \\
&= E_{;\gamma(;\theta)^2},
\end{aligned}$$

if $\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 E z^3 - \frac{3}{2}\gamma\left(\beta^2 E|z| - \beta\theta E\left(z|z|\right) + \frac{1}{4}\theta^2 E|z|^3\right) \\ +\frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta E z^3\right) - \frac{1}{8}\gamma^3 E|z|^3 \end{array}\right| < 1.$

Second, the second term of $E(l_{\gamma\theta\theta})$ equals

$$\begin{aligned}
Eh_{t;\theta}h_{t;\gamma,\theta} &= -\frac{1}{2}E_{;\gamma} + \frac{1}{2}\left(\theta + \gamma E[z|z|]\right)E_{;\gamma;\theta} - \frac{1}{2}\left(\theta + \gamma E[z|z|]\right)E_{;\gamma,\theta} \\
&\quad -\frac{1}{2}\left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right)E_{(;\theta)^2} \\
&\quad +\frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\gamma(;\theta)^2} \\
&\quad +\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z|z|\right)\right)h_{t-1;\theta}h_{t-1;\gamma,\theta} \\
&= E_{;\theta;\gamma,\theta},
\end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z|z|\right)\right| < 1.$

Finally, the first term of $E(l_{\gamma\theta\theta})$ equals

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\theta,\theta} &= \frac{1}{2}\left(\theta + \gamma E[z|z|]\right)E_{;\gamma;\theta} + \left(\beta E|z| - \frac{1}{2}\theta E[z|z|] - \frac{1}{2}\gamma\right)E_{;\theta,\theta} \\
&\quad +\frac{1}{4}\left(\theta E[z|z|] + \gamma\right)E_{(;\theta)^2} + \frac{1}{4}\left(-\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]\right)E_{;\gamma(;\theta)^2} \\
&\quad +\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z|z|\right)\right)h_{t-1;\gamma}h_{t-1;\theta,\theta} \\
&= E_{;\gamma;\theta,\theta},
\end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z|z|\right)\right| < 1.$

4.20 Theta-Theta-Theta

$$E\left(\ell_{\theta\theta\theta}\right) = -\frac{T}{2}\left(3E\left(h_{t;\theta}h_{t;\theta,\theta}\right) - E\left(h_{t;\theta}^3\right)\right).$$

First, the last term of $E\left(\ell_{\theta\theta\theta}\right)$, and as $E_{;\theta}=0$, equals

$$\begin{aligned} Eh_{t;\theta}^3 &= \frac{Ez^3+3\begin{pmatrix} \frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 \\ -\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3 \end{pmatrix} E_{(;\theta)^2}}{1-\begin{pmatrix} \beta^3+\frac{3}{4}\beta\left(\theta^2+\gamma^2\right)-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)Ez^3 \\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\gamma\beta\theta E\left[z\left|z\right|\right]-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3 \end{pmatrix}} \\ &= E_{(;\theta)^3}, \end{aligned}$$

if $\left|\begin{matrix} \beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)Ez^3 \\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3 \end{matrix}\right|<1.$

Finally, as $E_{;\theta}=0$, the first term of $E\left(\ell_{\theta\theta\theta}\right)$ is given by

$$\begin{aligned} Eh_{t;\theta}h_{t;\theta,\theta} &= \frac{3}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{(;\theta)^2}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)E_{;\theta,\theta} \\ &+\frac{1}{4}\left(-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right)E_{(;\theta)^3} \\ &+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\theta,\theta} \\ &= E_{;\theta;\theta,\theta}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

5 Expectations of Cross-Products

For this section let @ be any of the parameter $\alpha,\beta,\gamma,$ or θ, i.e. $@\in\{\alpha,\beta,\gamma,\theta\}$. Further, from the expected values only the terms of order T are kept. This is due to the fact that all expectations are devided by T. In this section we have repeatedly employed the following formulae

$$\begin{aligned} \sum_{t=1}^{T-1}\sum_{k=1}^{T-t}S^{k-1} &= \sum_{t=1}^{T-1}\frac{S^{T-t}-1}{S-1}=\frac{T}{1-S}+O\left(1\right) \\ \sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\beta^{k-1} &= \frac{T}{1-\beta}+O\left(1\right) \\ \sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\beta^{k-1}S^{k-1} &= \frac{T}{1-\beta S}+O\left(1\right) \end{aligned}$$

provided that $|S|<1$ and $|\beta|<1$.

5.1 @*Alpha-Alpha

We have $E\sum\sum_{s<t}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\alpha}^2h_{t;@}$ where:

$$h_{t+k;\alpha}=1+\left(\beta-\frac{1}{2}\theta z_{t+k-1}-\frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)h_{t+k-1;\alpha}$$

and

$$h^2_{t+k;\alpha} = 1 + 2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right)^2 h^2_{t+k-1;\alpha}.$$

Hence

$$E\left(z_t^2-1\right) h^2_{t+k;\alpha} h_{t;@} = 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) * E\left(z_t^2-1\right) h_{t+k-1;\alpha} h_{t;@} + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) E\left(z_t^2-1\right) h^2_{t+k-1;\alpha} h_{t;@}$$

But

$$E\left(z_t^2-1\right) h_{t+k;\alpha} h_{t;@} = \frac{1}{2}\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1}\left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right) * E h_{t;\alpha} h_{t;@}$$

Hence

$$\begin{aligned} E\left(z_t^2-1\right) h^2_{t+k;\alpha} h_{t;@} &= \frac{\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)^{k-1} - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^k}{\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)} \\ &\times \left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right) * E h_{t;\alpha} h_{t;@} \\ &+ \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)^{k-1} \\ &\times \left[\begin{array}{c} \left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right) E h_{t;\alpha} h_{t;@} \\ + \frac{1}{4}\left(\theta^2+\gamma^2\right)\left(E z^4 - 1\right) - \beta\theta E \\ z^3 + \beta\gamma\left(E\left|z\right| - E\left|z\right|^3\right) + \frac{1}{2}\gamma\theta\left(E z^3\left|z\right| - E z\left|z\right|\right) E h^2_{t-1;\alpha} h_{t;@} \end{array}\right] \end{aligned}$$

and it follows that $E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h^2_{t+k;\alpha} h_{t;@}$ equals

$$\begin{aligned} & T\frac{1}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)} \frac{\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3}{1 - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)} E_{;\alpha;@} \\ & + T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(E z^4 - 1\right) - \beta\theta E z^3 + \beta\gamma\left(E\left|z\right| - E\left|z\right|^3\right) + \frac{1}{2}\gamma\theta\left(E z^3\left|z\right| - E z\left|z\right|\right)}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)} E_{(;\alpha)^2;@} \\ & + O(1). \end{aligned}$$

5.2 @*Alpha-Beta

We have $E\sum_{s<t}\sum\left(z_s^2-1\right) z_t^2 h_{t;\alpha} h_{t;\beta} h_{s;@} = E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h_{t+k;\alpha} h_{t+k;\beta} h_{t;@}$ where:

$$h_{t+k;\alpha} = 1 + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha}$$

and

$$h_{t+k;\beta} = \ln\left(h_{t+k-1}\right) + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma\left|z_{t+k-1}\right|\right) h_{t+k-1;\beta}.$$

First as

$$E\left(z_t^2-1\right)\ln h_{t+k} h_{t;@} = \beta^{k-1}\left(\theta E z_t^3 + \gamma E\left|z_t\right|^3 - \gamma E\left|z_t\right|\right) E h_{t;@}$$

and

$$\begin{aligned} E\left(z_t^2-1\right) h_{t+k;\beta} h_{t;@} &= -2\left(\theta E z_t^3 + \gamma E\left|z_t\right|^3 - \gamma E\left|z_t\right|\right) \frac{\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1} - \beta^{k-1}}{\gamma E\left|z\right|} E h_{t;@} \\ &\quad + \frac{1}{2}\left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right)\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1} E h_{t;\beta} h_{t;@} \end{aligned}$$

we get

$$\begin{aligned} E\left(z_t^2-1\right) h_{t+k;\alpha}\ln\left(h_{t+k}\right) h_{t;@} &= \left[\frac{\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1} - \beta^{k-1}}{\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) - 1} + \left(\beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1}\right] \\ &\quad \times\left(\theta E z^3 + \gamma E\left|z\right|^3 - \gamma E\left|z\right|\right) E h_{t;@} \\ &\quad + \left[\begin{array}{c} \frac{\alpha\left(\beta - \frac{1}{2}\gamma E|z|\right) + -\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 + \beta\gamma E|z| - \gamma\theta E[z|z|]}{2} \\ \times\left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right) \\ \times\frac{\left(\beta\left(\beta - \frac{1}{2}\gamma E|z|\right)\right)^{k-1} - \left(\beta - \frac{1}{2}\gamma E|z|\right)^{k-1}}{(\beta-1)\left(\beta - \frac{1}{2}\gamma E|z|\right)} \\ + \left(\begin{array}{c} \alpha\frac{\gamma E|z| - \theta E z^3 - \gamma E|z|^3}{2} + \beta\theta E z^3 - \frac{\theta^2+\gamma^2}{2} E z^4 \\ + \frac{1}{2}\gamma^2 + \gamma\theta\left(E z\left|z\right| - E z\left|z\right|^3\right) \\ + \beta\gamma\left(E\left|z\right|^3 - E\left|z\right|\right) + \frac{1}{2}\theta^2 \end{array}\right) \end{array}\right] E_{;\alpha;@} \\ &\quad + \frac{\left(\gamma E\left|z\right| - \theta E z^3 - \gamma E\left|z\right|^3\right)\beta}{2} \\ &\quad \times\left(\beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1} E\ln\left(h_t\right) h_{t;\alpha} h_{t;@} \end{aligned}$$

Hence $E\left(z_t^2-1\right)\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}h_{t+k;\alpha}h_{t+k;\beta}h_{t;@}$ equals

$$
\begin{aligned}
&T\frac{\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{1}{1-\beta}\\
&\times\left(\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}+\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\right)E_{;@}\\
&+\frac{T}{2}\frac{\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}Eh_{t;\beta}h_{t;@}\\
&+T\frac{1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\\
&\times\left[\begin{array}{c}\frac{1}{2}\frac{\alpha-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta E[z|z|]}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\\+\beta\theta Ez^3-\frac{1}{2}\left(\theta^2+\gamma^2\right)Ez^4+\gamma\theta\left(Ez\left|z\right|-Ez\left|z\right|^3\right)\\+\beta\gamma\left(E\left|z\right|^3-E\left|z\right|\right)+\frac{1}{2}\gamma^2+\frac{1}{2}\theta^2\end{array}\right]E_{;\alpha;@}\\
&+\frac{T}{2}\frac{\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{1}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}LE_{;\alpha;@}\\
&+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{t;\alpha;\beta;@}\\
&+O\left(1\right),
\end{aligned}
$$

provided $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

5.3 @*Alpha-Gamma

We have $E\sum_{s<t}\sum\left(z_s^2-1\right)z_t^2h_{t;\alpha}h_{t;\gamma}h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\alpha}h_{t+k;\gamma}h_{t;@}$ where:

$$
\begin{aligned}
E\left(z_t^2-1\right)h_{t+k;\alpha}h_{t+k;\gamma}h_{t;@} &= \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)*E\left(z_t^2-1\right)h_{t+k-1;\gamma}h_{t;@}\\
&+\left(\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma\right)\frac{\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3}{2}\\
&\times\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-2}Eh_{t;\alpha}h_{t;@}\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\\
&\times E\left(z_t^2-1\right)h_{t+k-1;\alpha}h_{t+k-1;\gamma}h_{t;@}
\end{aligned}
$$

$$
E\left(z_t^2-1\right)h_{t+k-1;\alpha}h_{t;@}=\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-2}\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)*E\left(h_{t;\alpha}h_{t;@}\right).
$$

Also

$$
E\left(z_t^2-1\right)h_{t+k;\gamma}h_{t;@}=\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}\left[\begin{array}{c}\left(E\left|z\right|^3-\left|z\right|\right)Eh_{t;@}\\+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)Eh_{t;\gamma}h_{t;@}\end{array}\right] \qquad (1)
$$

It follows that $E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\alpha}h_{t+k;\gamma}h_{t;@}$ equals

$$\begin{aligned}
&\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\
&\times\frac{\left(E\left|z\right|^3-\left|z\right|\right)*Eh_{t;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)E_{;\gamma;@}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\\
&+\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\
&\times\left\{\begin{array}{c}\frac{\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\frac{\gamma E|z|-\theta Ez^3-\gamma E|z|^3}{2}+\beta\left(E\left|z\right|^3-E\left|z\right|\right)\\-\frac{1}{2}\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)-\frac{1}{2}\gamma\left(E\left|z\right|^4-1\right)\end{array}\right\}Eh_{t;\alpha}h_{t;@}\\
&+\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\
&\times\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)\\-\beta\theta Ez^3+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\end{array}\right)Eh_{t;\gamma}h_{t;\alpha}h_{t;@}+O\left(1\right)
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

5.4 @*Alpha-Theta

We have $E\sum\sum_{s<t}\left(z_s^2-1\right)z_t^2h_{t;\alpha}h_{t;\theta}h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\alpha}h_{t+k;\theta}h_{t;@}$ where:

$$\begin{aligned}
E\left(z_t^2-1\right)h_{t+k;\alpha}h_{t+k;\theta}h_{t;@} &= \frac{\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)^{k-1}-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}}{\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta Ez\left|z\right|-\beta+\frac{1}{2}\gamma E\left|z\right|}\\
&\times\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\left[\begin{array}{c}Ez^3E_{;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)E_{;\theta;@}\\-\frac{1}{4}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)E_{;\alpha;@}\end{array}\right]\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)^{k-1}\\
&\times\left[Ez^3Eh_{t;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)h_{t;\theta}h_{t;@}\right]\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)^{k-1}\\
&\times\left[\begin{array}{c}\left(\beta Ez^3-\frac{1}{2}\theta\left(Ez^4-1\right)-\frac{1}{2}\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\right)*Eh_{t;\alpha}h_{t;@}\\+\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3\\+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\end{array}\right)\\\times Eh_{t;\alpha}h_{t;\theta}h_{t;@}\end{array}\right],
\end{aligned}$$

as

$$E\left(z_t^2-1\right)h_{t+k;\theta}h_{t;@}=\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}*\left[Ez^3Eh_{t;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)h_{t;\theta}h_{t;@}\right] \tag{2}$$

It follows that $E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}(z_t^2-1)\,h_{t+k;\alpha}h_{t+k;\theta}h_{t;@}$ is equal to

$$\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$$
$$\times\left(Ez^3E_{;@}+\frac{1}{2}\left(\gamma E|z|-\theta Ez^3-\gamma E|z|^3\right)*E_{;\theta;@}\right)$$
$$-\frac{T}{2}\frac{1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$
$$\times\left\{\begin{array}{c}\frac{\theta+\gamma E[z|z|]}{2}\frac{\left(\gamma E|z|-\theta Ez^3-\gamma E|z|^3\right)+\beta Ez^3-\frac{1}{2}\theta\left(Ez^4-1\right)}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\\+\gamma\left(Ez^3|z|-Ez|z|\right)\end{array}\right\}E_{;\alpha;@}$$
$$+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)(Ez^4-1)-\beta\theta Ez^3+\beta\gamma\left(E|z|-E|z|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3|z|-Ez|z|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}E_{;\alpha;\theta;@}$$

if $\left|\beta-\frac{1}{2}\gamma E|z|\right|<1$ and $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right|<1$.

5.5 @*Beta-Beta

We have $E\sum\sum_{s<t}(z_s^2-1)\,z_t^2h_{t;\beta}^2h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}(z_t^2-1)\,h_{t+k;\beta}^2h_{t;@}$ where first,

$$\begin{aligned}E\left(z_t^2-1\right)h_{t;@}\ln^2h_{t+k} &= 2\left(\alpha+\gamma E|z|\right)\frac{\beta^{k-1}-\beta^{2(k-1)}}{1-\beta}\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)E_{;@}\\ &\quad+\beta^{2(k-1)}\left[\left(\begin{array}{c}2\alpha\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\\+\left(\theta^2+\gamma^2\right)(Ez^4-1)+2\gamma\theta\left(Ez^3|z|-Ez|z|\right)\end{array}\right)E_{;@}\\+2\beta\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)LE_{;@}\right],\end{aligned}$$

as

$$E\left(z_t^2-1\right)h_{t;@}\ln h_{t+k}=\beta^{k-1}\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)E_{;@}$$

Second, as

$$\begin{aligned}E\left(z_t^2-1\right)h_{t+k;\beta}h_{t;@} &= \frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}-\beta^{k-1}}{\left(\beta-\frac{1}{2}\gamma E|z|\right)-\beta}\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)Eh_{t;@}\\ &\quad+\frac{\gamma E|z|-\theta Ez^3-\gamma E|z|^3}{2}\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}Eh_{t;\beta}h_{t;@},\end{aligned}\tag{3}$$

we get that $E\left(z_t^2-1\right)h_{t+k;\beta}h_{t;@}\ln h_{t+k}$ equals

$$
\begin{aligned}
&(\alpha+\gamma E|z|)\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}*\beta^{k-2}-\beta^{k-2}}{\beta-\frac{1}{2}\gamma E|z|-1}E_{;@}\\
&-2\left(\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta Ez|z|\right)\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\\
&\times\frac{\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}*\beta^{k-1}-\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}}{\beta-1}-\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-1}*\beta^{k-2}-\beta^{k-2}}{\beta-\frac{1}{2}\gamma E|z|-1}}{\gamma E|z|}E_{;@}\\
&-2\frac{\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)*\beta\right)^{k-1}-\beta^{2(k-1)}}{\gamma E|z|}\left[\begin{array}{c}2\alpha\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\\+\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)+2\gamma\theta\left(Ez^3|z|-Ez|z|\right)\end{array}\right]E_{;@}\\
&+2(\alpha+\gamma E|z|)\frac{\frac{\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)*\beta\right)^{k-1}-\beta^{k-1}}{\beta-\frac{1}{2}\gamma E|z|-1}+2\frac{\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)*\beta\right)^{k-1}-\beta^{2(k-1)}}{\gamma E|z|}}{1-\beta}\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)E_{;@}\\
&+\frac{1}{2}\left(\gamma E|z|-\theta Ez^3-\gamma E|z|^3\right)\left[\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta Ez|z|\right]\\
&\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}\beta^{k-1}-\left(\beta-\frac{1}{2}\gamma E|z|\right)^{k-2}}{\beta-1}E_{;\beta;@}\\
&+\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)\beta\right)^{k-1}\left[\begin{array}{c}\frac{\alpha}{2}\left(\gamma E|z|-\theta Ez^3-\gamma E|z|^3\right)+\beta\theta Ez^3-\frac{1}{2}\left(\theta^2+\gamma^2\right)Ez^4\\+\gamma\theta\left(Ez|z|-Ez|z|^3\right)+\beta\gamma\left(E|z|^3-E|z|\right)+\frac{1}{2}\gamma^2+\frac{1}{2}\theta^2\end{array}\right]E_{;\beta;@}\\
&-\beta\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\frac{\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)\beta\right)^{k-1}-\beta^{2(k-1)}}{\gamma E|z|}LE_{;@}\\
&+\left(\theta Ez^3+\gamma E|z|^3-\gamma E|z|\right)\left(\left(\beta-\frac{1}{2}\gamma E|z|\right)*\beta\right)^{k-1}\left(LE_{;@}-\frac{\beta}{2}*LE_{;\beta;@}\right)
\end{aligned}
$$

It follows that $E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\beta}^2 h_{t;@}$ equals

$$\frac{T}{1-\beta^2}\frac{1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{1+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}$$
$$\times\left(2\alpha*\left(\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|\right)+\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)+2\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\right)E_{;@}$$
$$+\frac{2T}{1-\beta}\frac{\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{1}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}$$
$$\times\left[\begin{array}{c}\left(\alpha+\gamma E\left|z\right|\right)\frac{2\beta-\frac{1}{2}\gamma E|z|}{1-\beta^2}\\+\left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta Ez\left|z\right|\right)\frac{\beta-\frac{1}{2}\gamma E|z|}{1-\beta+\frac{1}{2}\gamma E|z|}\end{array}\right]E_{;@}$$
$$+\frac{2T}{1-\beta^2}\frac{\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{2\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}LE_{;@}$$
$$+\frac{2T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}$$
$$\times\left[\begin{array}{c}\frac{1}{2}\left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta Ez\left|z\right|\right)\frac{\gamma E|z|-\theta Ez^3-\gamma E|z|^3}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\\+\frac{\alpha}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)+\beta\theta Ez^3-\frac{1}{2}\left(\theta^2+\gamma^2\right)Ez^4\\+\gamma\theta\left(Ez\left|z\right|-Ez\left|z\right|^3\right)+\beta\gamma\left(E\left|z\right|^3-E\left|z\right|\right)+\frac{1}{2}\gamma^2+\frac{1}{2}\theta^2\end{array}\right]E_{;\beta;@}$$
$$+\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\beta}LE_{;\beta;@}$$
$$+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{(;\beta)^2;@}+O\left(1\right)$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$ and $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$

5.6 @*Beta-Gamma

We have $E\sum_{s<t}\sum\left(z_s^2-1\right)z_t^2 h_{t;\gamma}h_{t;\beta}h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\gamma}h_{t+k;\beta}h_{t;@}$ where first, employing equation (3) we get that $E\left(z_t^2-1\right)h_{t+k;\gamma}\ln\left(h_{t+k}\right)h_{t;@}$ equals

$$E\left|z\right|\beta^{k-1}\frac{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}}{1-\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\left(\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|\right)E_{;@}$$
$$+\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta E\left[z\left|z\right|\right]\right]\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-2}\frac{1-\beta^{k-1}}{1-\beta}$$
$$\times\left[\left(E\left|z\right|^3-\left|z\right|\right)Eh_{t;@}+\left(\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\right)Eh_{t;\gamma}h_{t;@}\right]$$
$$+\left(\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1}\left\{\begin{array}{c}\left[\alpha\left(E\left|z\right|^3-E\left|z\right|\right)+\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)+\gamma\left(Ez^4-1\right)\right]E_{;@}\\+\beta\left(E\left|z\right|^3-E\left|z\right|\right)LE_{;@}\end{array}\right\}$$
$$+\left(\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1}\left\{\begin{array}{c}\left[\begin{array}{c}\frac{\alpha}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)+\beta\theta Ez^3-\frac{1}{2}\left(\theta^2+\gamma^2\right)Ez^4\\+\gamma\theta\left(Ez\left|z\right|-Ez\left|z\right|^3\right)+\beta\gamma\left(E\left|z\right|^3-E\left|z\right|\right)+\frac{1}{2}\gamma^2+\frac{1}{2}\theta^2\end{array}\right]E_{;\gamma;@}\\+\beta\left(\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\right)Eh_{t;\gamma}\ln\left(h_t\right)h_{t;@}\end{array}\right\},$$

as

$$E\left(z_t^2-1\right)h_{t+k;\gamma}h_{t;@}=\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-1}\left[\begin{array}{c}\left(E\left|z\right|^3-\left|z\right|\right)E_{;@}\\+\left(\left(\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\right)\right)E_{;\gamma;@}\end{array}\right].$$

Hence, it follows that $E\left(z_t^2-1\right)\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}h_{t+k;\gamma}h_{t+k;\beta}h_{t;@}$ equals

$$\begin{aligned}
&\frac{T}{1-\beta}\frac{\theta Ez^3+\gamma E\left|z\right|^3-\gamma E\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\
&\times\left[\frac{E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}+\frac{\beta E\left|z\right|-\frac{1}{2}\theta E\left[z\left|z\right|\right]-\frac{1}{2}\gamma}{1-\beta+\frac{1}{2}\gamma E\left|z\right|}\right]E_{;@}\\
&+T\frac{\begin{array}{c}\alpha\left(E\left|z\right|^3-E\left|z\right|\right)+\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)+\gamma\left(Ez^4-1\right)\\+\frac{\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta Ez|z|}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(E\left|z\right|^3-\left|z\right|\right)\end{array}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\\
&\times\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}E_{;@}\\
&+T\frac{\begin{array}{c}\frac{\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\right)\\+\beta\left(E\left|z\right|^3-E\left|z\right|\right)-\frac{1}{2}\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)-\frac{1}{2}\gamma\left(E\left|z\right|^4-1\right)\end{array}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\beta;@}\\
&+\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\\
&\times\left[\begin{array}{c}\frac{\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta Ez|z|}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)\right)\\+\beta\theta Ez^3-\frac{1}{2}\left(\theta^2+\gamma^2\right)Ez^4\\+\gamma\theta\left(Ez\left|z\right|-Ez\left|z\right|^3\right)+\beta\gamma\left(E\left|z\right|^3-E\left|z\right|\right)+\frac{1}{2}\gamma^2+\frac{1}{2}\theta^2\end{array}\right]E_{;\gamma;@}\\
&+\frac{T}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{E\left|z\right|^3-E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}LE_{;@}\\
&+T\frac{1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}LE_{;\gamma;@}\\
&+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\beta;\gamma;@}+O\left(1\right)
\end{aligned}$$

if $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$ and $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

5.7 @*Beta-Theta

We have $E\sum_{s<t}\sum \left(z_s^2-1\right) z_t^2 h_{t;\beta} h_{t;\theta} h_{s;@} = E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h_{t+k;\beta} h_{t+k;\theta} h_{t;@}$ where employing equation (2) we get that $E\left(z_t^2-1\right)\ln\left(h_{t+k}\right) h_{t+k;\theta} h_{t;@}$ equals

$$\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta Ez\left|z\right|\right]$$
$$\times\left[\begin{array}{c}\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)^{k-2}\frac{1-\beta^{k-1}}{1-\beta}\left(Ez^3Eh_{t;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)E_{;\theta;@}\right)\\ +\left(\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1}E_{;\theta;@}\end{array}\right]$$
$$+\left(\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right)^{k-1}\left(\begin{array}{c}\left[\alpha Ez^3+\theta Ez^4+\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\right]E_{;@}\\ +\beta\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)LE_{;\theta;@}+\beta Ez^3LE_{;@}\end{array}\right)$$

Employing equation (3) we get that $E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h_{t+k;\beta} h_{t+k;\theta} h_{t;@}$ equals

$$T\frac{\frac{\beta-\frac{1}{2}\gamma E|z|}{1-\beta\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left[\begin{array}{c}\frac{\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E|z|-\gamma\theta Ez|z|}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}Ez^3\\ +\alpha Ez^3+\theta Ez^4+\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\end{array}\right]-\frac{1}{2}\frac{\theta+\gamma E[z|z|]}{1-\beta+\frac{1}{2}\gamma E|z|}\frac{\theta Ez^3+\gamma E|z|^3-\gamma E|z|}{1-\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;@}$$
$$+T\frac{\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2+\beta\gamma E\left|z\right|-\gamma\theta Ez\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}\frac{\beta-\frac{1}{2}\gamma E\left|z\right|}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}$$
$$\times\left(\frac{1}{2}\frac{\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3}{1-\beta+\frac{1}{2}\gamma E\left|z\right|}+1\right)E_{;\theta;@}$$
$$+T\frac{\beta Ez^3-\frac{1}{2}\theta\left(Ez^4-1\right)-\frac{1}{2}\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right)-\frac{1}{4}\frac{\left(\theta+\gamma E[z|z|]\right)\left(\gamma E|z|-\theta Ez^3-\gamma E|z|^3\right)}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\beta;@}$$
$$+\frac{T}{1-\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)}\frac{Ez^3*LE_{;@}+\frac{1}{2}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)*LE_{;\theta;@}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}$$
$$+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\beta;\theta;@}+O\left(1\right)$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

5.8 @*Gamma-Gamma

We have $E\sum_{s<t}\sum \left(z_s^2-1\right) z_t^2 h_{t;\gamma}^2 h_{s;@} = E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h_{t+k;\gamma}^2 h_{t;@}$, which is equal, employing equation (1),

$$T\frac{2\frac{\left(\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma\right)\left(E|z|^3-|z|\right)}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+Ez^4-1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;@}$$
$$+T\frac{\frac{\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)+2\beta\left(E\left|z\right|^3-E\left|z\right|\right) -\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)-\gamma\left(E\left|z\right|^4-1\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\gamma;@}$$
$$+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right) +\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{(;\gamma)^2;@}+O\left(1\right)$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

5.9 @*Gamma-Theta

We have $E\sum_{s<t}\sum \left(z_s^2-1\right) z_t^2 h_{t;\gamma} h_{t;\theta} h_{s;@} = E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right) h_{t+k;\gamma} h_{t+k;\theta} h_{t;@}$, which is equal, employing equations (1) and (2),

$$T\frac{Ez^3\frac{\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)\frac{E|z|^3-E|z|}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}+Ez^3\left|z\right|-Ez\left|z\right|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;@}$$
$$+T\frac{\beta\left(E\left|z\right|^3-E\left|z\right|\right)-\frac{1}{2}\theta\left(Ez\left|z\right|^3-Ez\left|z\right|\right)-\frac{1}{2}\gamma\left(E\left|z\right|^4-1\right) +\frac{1}{2}\frac{\beta E|z|-\frac{1}{2}\theta E[z|z|]-\frac{1}{2}\gamma}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}\left(\gamma E\left|z\right|-\theta Ez^3-\gamma E\left|z\right|^3\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\theta;@}$$
$$+T\frac{\beta Ez^3-\frac{1}{2}\theta\left(Ez^4-1\right)-\frac{1}{2}\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right) -\frac{1}{4}\left(\theta+\gamma Ez\left|z\right|\right)\frac{\gamma E|z|-\theta Ez^3-\gamma E|z|^3}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\gamma;@}$$
$$+T\frac{\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right) +\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\gamma;\theta;@}+O\left(1\right)$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

5.10 @*Theta-Theta

We have $E\sum_{s<t}\sum\left(z_s^2-1\right)z_t^2h_{t;\theta}^2h_{s;@}=E\sum_{t=1}^{T-1}\sum_{k=1}^{T-t}\left(z_t^2-1\right)h_{t+k;\theta}^2h_{t;@}$, due to equation (2), equals

$$T\frac{Ez^4-\frac{\theta+\gamma E[z|z|]}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}Ez^3-1}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;@}$$
$$+T\frac{2\beta Ez^3-\theta\left(Ez^4-1\right)-\gamma\left(Ez^3\left|z\right|-Ez\left|z\right|\right)-\frac{1}{2}\left(\theta+\gamma E\left[z\left|z\right|\right]\right)\frac{\gamma E|z|-\theta Ez^3-\gamma E|z|^3}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{;\theta;@}$$
$$+T\frac{\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(Ez^4-1\right)-\beta\theta Ez^3+\beta\gamma\left(E\left|z\right|-E\left|z\right|^3\right)\\+\frac{1}{2}\gamma\theta\left(Ez^3\left|z\right|-Ez\left|z\right|\right)\end{array}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)}E_{(;\theta)^2;@}h_t+O\left(1\right)$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

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