Finite Sample Theory and Bias Correction of Maximum Likelihood Estimators in the EGARCH Model (Technical Appendix II)

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February 23, 2018

1 The Model

$$\begin{aligned} y_t &= \mu + z_t\sqrt{h_t}, \qquad t = 1, ...T, \qquad where \\ z_t &\sim iidD(0,1), \qquad \ln(h_t) = \alpha + \theta z_{t-1} + \gamma |z_{t-1}| + \beta \ln(h_{t-1}). \end{aligned}$$

with

$$E \ln(h_t) = \frac{\alpha + \gamma E|z|}{1-\beta} = L.$$

2 First Order Likelihood Derivatives:

Following, henceforth, the notation employed in Linton (1997), i.e. $h_{t;\circ} = \frac{\partial \ln(h_t)}{\partial \circ}$ and so on, the 1^{st} derivatives of the log-likelihood function with respect to μ is:

$$\ell_\mu = -\frac{1}{2}\sum_{t=1}^{T}\frac{\partial \ln(h_t)}{\partial \mu} - \frac{1}{2}\sum_{t=1}^{T} 2z_t \frac{\partial z_t}{\partial \mu} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right) h_{t;\mu} + \sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}},$$

as

$$\frac{\partial z_t}{\partial \mu} = -\frac{1}{\sqrt{h_t}} - \frac{1}{2} z_t h_{t;\mu}.$$

Now

$$h_{t;\mu} \equiv \frac{\partial \ln(h_t)}{\partial \mu} = -\left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma |z_{t-1}|\right) h_{t-1;\mu},$$

as

$$E\left[I(z \geq 0) - I(z < 0)\right] z = E|z|,$$

and

$$\begin{aligned} Eq\exp(\ln h_t) &= \exp(q\alpha^*) E\exp\left[\sum_{i=0}^{\infty} q\left(\theta\beta^i z_{t-1-i} + \gamma\beta^i |z_{t-1-i}|\right)\right] \\ &= \exp(q\alpha^*) \prod_{i=0}^{\infty} E\exp\left[q\beta^i\left(\theta z_{t-1-i} + \gamma|z_{t-1-i}|\right)\right] \\ &= \exp(q\alpha^*) \prod_{i=0}^{\infty} E\exp\left[q\beta^i\left(\theta z + \gamma|z|\right)\right] \\ &= \exp\left(q\frac{\alpha}{1-\beta}\right)\prod_{i=0}^{\infty} A_{q\beta^i} = E_q \end{aligned}$$

where $A_{q\beta^i} = E\exp\left[q\beta^i\left(\theta z + \gamma|z|\right)\right]$ and it can be evaluated as in the Appendix below, it follows that

$$\begin{aligned} E(h_{t;\mu}) &= -\left(\theta + \gamma E_I\right) E\left(\frac{1}{\sqrt{h}}\right) + \left[\beta - \frac{1}{2}\gamma E(|z|)\right] E(h_{t-1;\mu}) \\ &= -\frac{\theta + \gamma E_I}{1 - \left(\beta - \frac{1}{2}\gamma E|z|\right)} E_{-\frac{1}{2}} = E_{;\mu} \end{aligned}$$

if $\left|\beta - \frac{1}{2}\gamma E\left|z\right|\right| < 1$, and as

$$E\left[I\left(z \geq 0\right) - I\left(z < 0\right)\right] = \int_{0}^{\infty} f\left(z\right) dz - \int_{-\infty}^{0} f\left(z\right) dz = E_I$$

not necessarily equal to 0, unless z has a symmetric distribution.

3 Second Order Likelihood Derivatives:

In this section we present the second order derivatives, involving μ, and their expectation. Notice that, for $\circ \in \{\alpha, \theta, \gamma, \beta\}$ we have that

$$\ell_{\mu\circ} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right) h_{t;\mu,\circ} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\mu} h_{t;\circ} - \sum_{t=1}^{T} z_t \frac{1}{\sqrt{h_t}} h_{t;\circ}$$

and it follows that

$$E\left(\ell_{\mu\circ}\right) = -\frac{1}{2}\sum_{t=1}^{T} E\left(h_{t;\mu} h_{t;\circ}\right).$$

However, the second derivative of the Likelihood function with respect to μ is given in the following subsection.

3.1 Miu-Miu

$$\ell_{\mu\mu} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right) h_{t;\mu,\mu} - \sum_{t=1}^{T}\left(\frac{1}{h_t} + 2\frac{1}{\sqrt{h_t}} z_t h_{t;\mu} + \frac{1}{2} z_t^2 h_{t;\mu}^2\right),$$

where

$$\begin{aligned} h_{t;\mu,\mu} &= \left\{\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right\} \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\mu} \\ &\quad + \frac{1}{4}\left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right) z_{t-1} h_{t-1;\mu}^2 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\mu,\mu}, \end{aligned}$$

with

$$Eh_{t;\mu,\mu} = \frac{\left(\theta + \gamma E_I\right) E_{-\frac{1}{2}} E_{;\mu} + \frac{1}{4}\gamma E\left|z\right| E_{(;\mu)^2}}{1 - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)} = E_{;\mu,\mu},$$

if $\left|\beta - \frac{1}{2}\gamma E\left|z\right|\right| < 1$ and where $E_{-\frac{1}{2}}E_{;\mu} = E\left(\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\mu}\right)$ is given by

$$\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\mu}\right) &= -B_q E_{\left(q\beta-\frac{1}{2}\right)} \\
&\quad + E\left[e^{q(\alpha+\theta z_{t-1}+\gamma|z_{t-1}|)}\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] E\left[e^{q\beta\ln(h_{t-1})}h_{t-1;\mu}\right] \\
&= -B_q E_{\left(q\beta-\frac{1}{2}\right)} - C_q B_{q\beta} E_{\left(q\beta^2-\frac{1}{2}\right)} - C_q C_{q\beta} B_{q\beta^2} E_{\left(q\beta^3-\frac{1}{2}\right)} \\
&\quad + C_q C_{q\beta} C_{q\beta^2} E\left[\left(\beta - \frac{1}{2}\theta z_{t-3} - \frac{1}{2}\gamma\left|z_{t-3}\right|\right) h_{t-3;\mu}\right] \\
&= -\sum_{i=0}^{\infty} B_{q\beta^i} E_{\left(q\beta^{i+1}-\frac{1}{2}\right)} \prod_{j=0}^{i} C_{q^j} = E_q E_{;\mu}.
\end{aligned}$$

It follows that

$$E\left(\ell_{\mu\mu}\right) = -TE\left(\frac{1}{h_t}\right) - \frac{T}{2}E\left(h_{t;\mu}^2\right) = -TE_{-1} - \frac{T}{2}E\left(h_{t;\mu}^2\right).$$

Now

$$\begin{aligned}
h_{t;\mu}^2 &= \left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)^2 \frac{1}{h_{t-1}} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\mu}^2 \\
&\quad -2\left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}}\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\mu},
\end{aligned}$$

and it follows that

$$Eh_{t;\mu}^2 = \frac{\left(\theta^2 + \gamma^2 + 2\gamma\theta E_I\right) E_{\frac{1}{h_{t-1}}} - 2\left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right) E_{-\frac{1}{2}}E_{;\mu}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma Ez\left|z\right|\right)} = E_{(;\mu)^2},$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma Ez\left|z\right|\right| < 1$ and as

$$E\left[\left(\theta + \gamma\left[I\left(z \geq 0\right) - I\left(z < 0\right)\right]\right)\left(\beta - \frac{1}{2}\theta z - \frac{1}{2}\gamma\left|z\right|\right)\right] = \beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right| \quad (1)$$

due to

$$E\left[I\left(z \geq 0\right) - I\left(z < 0\right)\right]\left|z\right| = \int_0^{\infty} zf\left(z\right)dz + \int_{-\infty}^{0} zf\left(z\right)dz = Ez = 0.$$

3.2 Miu-Alpha

Now

$$\ell_{\mu\alpha} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right)h_{t;\mu,\alpha} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\mu}h_{t;\alpha} - \sum_{t=1}^{T} z_t \frac{1}{\sqrt{h_t}}h_{t;\alpha},$$

where

$$h_{t;\mu,\alpha} = \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\alpha},$$

with

$$h_{t;\mu,\alpha} = \frac{1}{2}\left(\theta + \gamma E_I\right)E_{-\frac{1}{2}}E_{;\alpha} + \frac{1}{4}\gamma E\left|z\right|E_{;\alpha;\mu} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)h_{t-1;\mu,\alpha}$$

where $E_{-\frac{1}{2}}E_{;\alpha} = E\left(\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\alpha}\right)$ is given by

$$\begin{aligned} E\left(e^{q\ln h_t}h_{t;\alpha}\right) &= E_q + C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\alpha}\right) \\ &= E_q + C_q E_{q\beta} + C_q C_{q\beta} E\left(e^{q\beta^2\ln(h_{t-1})}h_{t-2;\alpha}\right) \\ &= \sum_{i=0}^{\infty}\left(E_{q\beta^i}\prod_{j=0}^{i-1}C_{q\beta^j}\right) = E_q E_{;\alpha} \end{aligned}$$

with the convention that $\prod_{j=0}^{-1}C_{q\beta^j} = 1$.

It follows that

$$E\left(\ell_{\mu\alpha}\right) = -\frac{T}{2}E\left(h_{t;\mu}h_{t;\alpha}\right)$$

Notice that

$$\begin{aligned} Eh_{t;\mu}h_{t;\alpha} &= E_{;\mu} - \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma Ez\left|z\right|\right)Eh_{t-1;\alpha}h_{t-1;\mu} \\ &= E_{;\alpha;\mu}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma Ez\left|z\right|\right| < 1$.

3.3 Miu-Beta

Now

$$\ell_{\mu\beta} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right)h_{t;\beta,\mu} - \frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;\beta}h_{t;\mu} - \sum_{t=1}^{T}z_t\frac{1}{\sqrt{h_t}}h_{t;\beta}$$

where

$$h_{t;\beta,\mu} = \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right)h_{t-1;\beta} + h_{t-1;\mu} + \frac{1}{4}\left(\theta + \gamma\right)z_{t-1}^2 h_{t-1;\beta}h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\beta,\mu}$$

It follows that

$$E\left(\ell_{\mu\beta}\right)=-\frac{T}{2}E\left(h_{t;\beta}h_{t;\mu}\right).$$

Now, due to equation (1) and as

$$\begin{aligned}
E\left[e^{q\ln h_t}\ln\left(h_t\right)\right] &= E\left(e^{q[\alpha+\theta z_{t-1}+\gamma|z_{t-1}|+\beta\ln(h_{t-1})]}\left[\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln\left(h_{t-1}\right)\right]\right)\\
&= D_qE_{q\beta}+\beta e^{q\alpha}A_qE\left(e^{q\beta\ln(h_{t-1})}\ln\left(h_{t-1}\right)\right)\\
&= D_qE_{q\beta}+\beta e^{q\alpha}A_qD_{q\beta}E_{q\beta^2}+\beta^2e^{q\alpha}A_qe^{q\beta\alpha}A_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}\ln\left(h_{t-2}\right)\right)\\
&= D_qE_{q\beta}+\beta D_{q\beta}E_{q\beta^2}e^{q\alpha}A_q+\beta^2D_{q\beta^2}E_{q\beta^3}e^{q\alpha(1+\beta)}A_qA_{q\beta}\\
&\quad +\beta^3e^{q\alpha}A_qe^{q\beta\alpha}A_{q\beta}e^{q\beta^2\alpha}A_{q\beta^2}E\left(e^{q\beta^3\ln(h_{t-3})}\ln\left(h_{t-3}\right)\right)\\
&= \sum_{i=0}^{\infty}\beta^iD_{q\beta^i}E_{q\beta^{i+1}}\prod_{j=0}^{i-1}A_{q\beta^j}e^{q\alpha\left(\sum_{k=0}^{j}\beta^k\right)}=E_qL,
\end{aligned}$$

with the convention that $\prod_{j=0}^{-1}A_{q\beta^j}e^{q\alpha\left(\sum_{k=0}^{j}\beta^k\right)}=1$, and as

$$\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\beta}\right) &= e^{q\alpha}A_qE_{q\beta}+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta}\right)\\
& e^{q\alpha}A_qLE_{q\beta}+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta}\right)\\
&= e^{q\alpha}A_qLE_{q\beta}+C_qe^{q\beta\alpha}A_{q\beta}LE_{q\beta^2}+C_qC_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}h_{t21;\beta}\right)\\
&= \sum_{i=0}^{\infty}\left(e^{q\alpha\beta^i}LE_{q\beta^{i+1}}\prod_{j=0}^{i-1}C_{q\beta^j}\right)=E_qE_{;\beta}
\end{aligned}$$

with the convention that $\prod\limits_{j=0}^{-1}C_{q\beta^j}=1$, we get

$$\begin{aligned}
Eh_{t;\mu}h_{t;\beta} &= -\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}L+\left(\beta-\frac{1}{2}E\gamma\left|z\right|\right)LE_{;\mu}\\
& \left(-\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta}\\
& +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\mu}\\
&= E_{;\beta;\mu},
\end{aligned}$$

where

$$\begin{aligned}
LE_{;\mu} &= E\ln\left(h_t\right)h_{t;\mu}=-\left(\alpha\left(\theta+E_I\right)+2\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}-\beta\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}L\\
& +\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\theta\left(\theta+\gamma Ez\left|z\right|\right)+\gamma\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)\right]E_{;\mu}\\
& +\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\mu}\ln\left(h_{t-1}\right)
\end{aligned}$$

if $\left|\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\right|<1$, and as

$$E\left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)\left[\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right]=\alpha\left(\theta+E_I\right)+2\gamma\theta E\left|z\right|$$

and

$$E\left[\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\left(\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\right] = \alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)-\frac{1}{2}\theta\left(\theta+\gamma Ez\left|z\right|\right) +\gamma\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right).$$

3.4 Miu-Gamma

Now

$$\ell_{\mu\gamma}=\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\gamma,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\beta}h_{t;\gamma}-\sum_{t=1}^{T}z_t\frac{1}{\sqrt{h_t}}h_{t;\gamma}$$

where

$$\begin{aligned} h_{t;\mu,\gamma} &= \frac{1}{2}\left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\gamma}-\frac{1}{2}\left|z_{t-1}\right|h_{t-1;\mu} \\ &\quad +\frac{1}{4}\left(\theta+\gamma\right)z_{t-1}^2h_{t-1;\gamma}h_{t-1;\mu}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\gamma}, \end{aligned}$$

with

$$\begin{aligned} Eh_{t;\mu,\gamma} &= \frac{1}{2}\left(\theta+\gamma E_I\right)E\left(\frac{1}{\sqrt{h_{t-1}}}h_{t;\gamma}\right)-\frac{1}{2}E\left|z\right|E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)E_{;\gamma;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\mu,\gamma} \\ &= E_{;\mu,\gamma}. \end{aligned}$$

It follows that

$$E\left(\ell_{\mu\gamma}\right)=-\frac{T}{2}E\left(h_{t;\gamma}h_{t;\mu}\right).$$

Now, due to equation (1) and as

$$\begin{aligned} E\left(e^{q\ln h_t}h_{t;\gamma}\right) &= e^{q\alpha}G_qE_{q\beta}+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}\right) \\ &= e^{q\alpha}G_qE_{q\beta}+e^{q\beta\alpha}C_qG_{q\beta}E_{q\beta^2}+C_qC_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}h_{t-2;\gamma}\right) \\ &= \sum_{i=0}^{\infty}\left(e^{q\alpha\beta^i}G_{q\beta^i}E_{q\beta^{i+1}}\prod_{j=0}^{i-1}C_{q\beta^j}\right)=E_qE_{;\gamma} \end{aligned}$$

with the convention that $\prod_{j=0}^{-1}C_{q\beta^j}=1$, again, we get

$$\begin{aligned} E\left(h_{t;\gamma}h_{t;\mu}\right) &= -\theta E\left|z\right|E_{-\frac{1}{2}}+\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\mu} \\ &\quad -\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\mu} \\ &= E_{;\gamma;\mu}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

3.5 Miu-Theta

Now

$$\ell_{\mu\theta} = \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\theta,\mu} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\theta}h_{t;\gamma} - \sum_{t=1}^{T} z_t \frac{1}{\sqrt{h_t}} h_{t;\gamma}$$

where

$$\begin{aligned} h_{t;\mu,\theta} &= \left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\theta} - \frac{1}{2}z_{t-1}h_{t-1;\mu} \\ &\quad + \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}h_{t-1;\mu} + \left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\theta}, \end{aligned}$$

with

$$Eh_{t;\mu,\theta} = \left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\theta} + \frac{1}{4}\gamma E\left|z\right|E_{;\theta;\mu} + \left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\mu,\theta} = E_{;\mu,\theta},$$

It follows that

$$E\left(\ell_{\mu\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\mu}\right).$$

Now, due to equation (1) and as

$$\begin{aligned} E\left(e^{q\ln h_t}h_{t;\theta}\right) &= E\left(e^{q\left[\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln\left(h_{t-1}\right)\right]}z_{t-1}\right) \\ &\quad + E\left(e^{q\left[\alpha+\theta z_{t-1}+\gamma\left|z_{t-1}\right|+\beta\ln\left(h_{t-1}\right)\right]}\left[z_{t-1}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\theta}\right]\right) \\ &\quad + e^{q\alpha}F_qE_{q\beta} + C_qE\left(e^{q\beta\ln\left(h_{t-1}\right)}h_{t-1;\theta}\right) \\ &= e^{q\alpha}F_qE_{q\beta} + e^{q\beta\alpha}C_qF_{q\beta}E_{q\beta^2} + C_qC_{q\beta}E\left(e^{q\beta^2\ln\left(h_{t-2}\right)}h_{t-2;\theta}\right) \\ &= \sum_{i=0}^{\infty}\left(e^{q\alpha\beta^i}F_{q\beta^i}E_{q\beta^{i+1}}\prod_{j=0}^{i-1}C_{q\beta^j}\right) = E_qE_{;\theta} \end{aligned}$$

with the convention that $\prod_{j=0}^{-1}C_{q\beta^j}=1$, again, we get

$$\begin{aligned} E\left(h_{t;\theta}h_{t;\mu}\right) &= -\gamma E\left|z\right|E_{-\frac{1}{2}} - \left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\theta} \\ &\quad -\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\mu} + \left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\mu} \\ &= E_{;\theta;\mu}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4 Third Order Likelihood Derivatives

In this section we present the third order derivatives, involving μ, and their expectation. For $\circ \in \{\alpha, \theta, \gamma, \beta\}$ we have that

$$\begin{aligned} \ell_{\mu\mu\circ} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu,\circ} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\circ}h_{t;\mu,\mu} + \sum_{t=1}^{T}\left(\frac{1}{h_t}h_{t;\circ} + \frac{1}{\sqrt{h_t}}z_t h_{t;\circ}h_{t;\mu}\right) \\ &\quad - \sum_{t=1}^{T}\left(2\frac{1}{\sqrt{h_t}}\left(z_t h_{t;\mu,\circ} - \frac{1}{2}z_t h_{t;\circ}h_{t;\mu}\right) - \frac{1}{2}z_t^2 h_{t;\circ}h_{t;\mu}^2 + z_t^2 h_{t;\mu}h_{t;\mu,\circ}\right), \end{aligned}$$

and it follows

$$\begin{aligned} E\ell_{\mu\mu\circ} &= -\frac{T}{2}Eh_{t;\circ}h_{t;\mu,\mu} - \frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\circ}\right) - 2E\left(\frac{1}{h_t}h_{t;\circ}\right) - E\left(h_{t;\circ}h_{t;\mu}^2\right)\right) \\ &= -\frac{T}{2}Eh_{t;\circ}h_{t;\mu,\mu} - \frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\circ}\right) - 2E_{-1}E_{;\circ} - E\left(h_{t;\circ}h_{t;\mu}^2\right)\right). \end{aligned}$$

for μ the derivative is given is the following subsection.

4.1 Miu-Miu-Miu

$$\begin{aligned} \ell_{\mu\mu\mu} &= -\sum_{t=1}^{T} z_t\frac{1}{\sqrt{h_t}}h_{t;\mu,\mu} - \frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\mu}h_{t;\mu,\mu} + \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu,\mu} + \sum_{t=1}^{T}\frac{1}{h_t}h_{t;\mu} \\ &\quad + \sum_{t=1}^{T} z_t\frac{1}{\sqrt{h_t}}\left(h_{t;\mu}\right)^2 + 2\sum_{t=1}^{T}\frac{1}{h_t}h_{t;\mu} + \sum_{t=1}^{T} z_t\frac{1}{\sqrt{h_t}}h_{t;\mu}h_{t;\mu} - 2\sum_{t=1}^{T} z_t\frac{1}{\sqrt{h_t}}\frac{\partial h_{t;\mu}}{\partial\mu} \\ &\quad + \sum_{t=1}^{T} z_t\frac{1}{\sqrt{h_t}}h_{t;\mu}^2 + \sum_{t=1}^{T}\frac{1}{2}z_t^2 h_{t;\mu}^3 - \sum_{t=1}^{T} z_t^2 h_{t;\mu}h_{t;\mu,\mu}, \end{aligned}$$

$$\begin{aligned} E\left(\ell_{\mu\mu\mu}\right) &= -\frac{3T}{2}E\left(h_{t;\mu}h_{t;\mu,\mu}\right) + 3TE\left(\frac{1}{h_t}h_{t;\mu}\right) + \frac{T}{2}Eh_{t;\mu}^3 \\ &= -\frac{3T}{2}E\left(h_{t;\mu}h_{t;\mu,\mu}\right) + 3TE_{-1}E_{;\mu} + \frac{T}{2}Eh_{t;\mu}^3. \end{aligned}$$

Now as

$$\begin{aligned} E\left(e^{q\ln h_t}h_{t;\mu}^2\right) &= \left(\theta^2+\gamma^2\right)A_q e^{q\alpha}E_{q\beta-1} + 2\gamma\theta S_q E_{q\beta-1} - 2Q_q E_{q\beta-\frac{1}{2}}E_{;\mu} \\ &\quad + R_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\mu}^2\right) \\ &= \left(\theta^2+\gamma^2\right)A_q e^{q\alpha}E_{q\beta-1} + 2\gamma\theta S_q E_{q\beta-1} - 2Q_q E_{q\beta-\frac{1}{2}}E_{;\mu} \\ &\quad + \left(\theta^2+\gamma^2\right)R_q A_{q\beta}e^{q\alpha\beta}E_{q\beta^2-1} + 2\gamma\theta R_q S_{q\beta}E_{q\beta^2-1} - 2R_q Q_{q\beta}E_{q\beta^2-\frac{1}{2}}E_{;\mu} \\ &\quad + R_q R_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}h_{t-2;\mu}^2\right) \\ &= \sum_{i=0}^{\infty}\left[\left(\theta^2+\gamma^2\right)A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}-1} + 2\gamma\theta S_{q\beta^i}E_{q\beta^{i+1}-1} - 2Q_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\mu}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\ &= E_q E_{(;\mu)^2} \end{aligned}$$

we get

$$
\begin{aligned}
Eh^3_{t;\mu} &= 3\left[\left(\theta^2+\gamma^2\right)\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+2\beta\gamma\theta E_I-\gamma\theta^2 E\left|z\right|\right]E_{-1}E_{;\mu} \\
&-\left\{\theta^3+\gamma\left[\left(3\theta^2+\gamma^2\right)E_I+3\theta\gamma\right]\right\}E_{-\frac{3}{2}} \\
&-3\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{(;\mu)^2} \\
&+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh^3_{t-1;\mu} \\
&= E_{(;\mu)^3},
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1$, as

$$
\begin{aligned}
E\left[\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma\left|z\right|\right)^2\right] &= \beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\
&+\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right) \\
&+\beta\gamma\left(E_I-\theta E\left|z\right|\right)
\end{aligned}
$$

$$E\left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)^2=\theta^2+2\gamma\theta E_I+\gamma^2$$

$$E\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]^2=\int_0^\infty f\left(z\right)dz+\int_{-\infty}^0 f\left(z\right)dz=1$$

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)^2 z=2\theta\gamma E\left|z\right|$$

$$
\begin{aligned}
E\left[\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)^2\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma\left|z\right|\right)\right] &= \left(\theta^2+\gamma^2\right)\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right) \\
&+2\beta\gamma\theta E_I-\gamma\theta^2 E\left|z\right|
\end{aligned}
$$

$$E\left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)^3=\theta^3+\gamma\left[\left(3\theta^2+\gamma^2\right)E_I+3\theta\gamma\right]$$

where

$$E\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]z^2=\int_0^\infty z^2 f\left(z\right)dz-\int_{-\infty}^0 z^2 f\left(z\right)dz=E_{Iz^2}$$

and

$$E\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]z\left|z\right|=\int_0^\infty z^2 f\left(z\right)dz+\int_{-\infty}^0 z^2 f\left(z\right)dz=1$$

$$E\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]^2 z=\int_0^\infty zf\left(z\right)dz+\int_{-\infty}^0 zf\left(z\right)dz=0$$

$$E\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]^2\left|z\right|=\int_0^\infty zf\left(z\right)dz-\int_{-\infty}^0 zf\left(z\right)dz=E\left|z\right|$$

Further, as

$$
\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\mu,\mu}\right) &= B_q E_{q\beta-\frac{1}{2}}E_{;\mu}+\frac{1}{4}U_q E_{q\beta}E_{(;\mu)^2} \\
&\quad +C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\mu,\mu}\right) \\
&= B_q E_{q\beta-\frac{1}{2}}E_{;\mu}+\frac{1}{4}U_q E_{q\beta}E_{(;\mu)^2} \\
&\quad +C_q B_{q\beta}E_{q\beta^2-\frac{1}{2}}E_{;\mu}+\frac{1}{4}C_q U_{q\beta}E_{q\beta^2}E_{(;\mu)^2} \\
&\quad +C_q C_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}h_{t-2;\mu,\mu}\right) \\
&= \sum_{i=0}^{\infty}\left[B_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}+\frac{1}{4}U_{q\beta^i}E_{q\beta^{i+1}}E_{(;\mu)^2}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_q E_{;\mu,\mu}
\end{aligned}
$$

we get

$$
\begin{aligned}
Eh_{t;\mu}h_{t;\mu,\mu} &= -\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-1}E_{;\mu}+\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|-\frac{1}{2}\gamma\theta E\left|z\right|z_{t-1}\right)E_{-\frac{1}{2}}E_{(;\mu)^2} \\
&\quad +\frac{1}{4}\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{(;\mu)^3} \\
&\quad -\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\mu,\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu,\mu}h_{t-1;\mu} \\
&= E_{;\mu;\mu,\mu},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$ and due to

$$
E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma\left|z\right|\right)z=\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)
$$

4.2 Miu-Miu-Alpha

We have that

$$
\begin{aligned}
E\ell_{\mu\mu\alpha} &= -\frac{T}{2}Eh_{t;\alpha}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\alpha}\right)-2E\left(\frac{1}{h_t}h_{t;\alpha}\right)-E\left(h_{t;\alpha}h_{t;\mu}^2\right)\right) \\
&= -\frac{T}{2}Eh_{t;\alpha}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\alpha}\right)-2E_{-1}E_{;\alpha}-E\left(h_{t;\alpha}h_{t;\mu}^2\right)\right).
\end{aligned}
$$

Now

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\mu}^{2} &= \left(\theta^{2}+2\gamma\theta E_{I}+\gamma^{2}\right)E_{-1}+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{(;\mu)^{2}} \\
&\quad+\left[\left(\theta^{2}+\gamma^{2}\right)\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\gamma\theta\left(2\beta E_{I}-\theta E\left|z\right|\right)\right]E_{-1}E_{;\alpha} \\
&\quad-2\left(\beta\theta+\beta\gamma E_{i}-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\mu} \\
&\quad-2\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right)\\+\frac{1}{4}\left(\theta^{2}+\gamma^{2}\right)\left(\theta+\gamma E_{Iz^{2}}\right)+\beta\gamma\left(E_{I}-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\alpha} \\
&\quad+\left(\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)-\frac{3}{2}\beta^{2}\gamma E\left|z\right|\\+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right)Eh_{t-1;\alpha}h_{t-1;\mu}^{2} \\
&= E_{;\alpha(;\mu)^{2}},
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^{3}+\frac{3}{4}\beta\theta^{2}+\frac{3}{4}\beta\gamma^{2}-\frac{1}{8}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right)-\frac{3}{2}\beta^{2}\gamma E\left|z\right|\\+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3}\end{array}\right|<1.$

Further, as

$$
\begin{aligned}
E\left(e^{q\ln h_{t}}h_{t;\alpha}h_{t;\mu}\right) &= -B_{q}E_{q\beta-\frac{1}{2}}-Q_{q}E_{q\beta-\frac{1}{2}}E_{;\alpha}+C_{q}E_{q\beta}E_{;\mu} \\
&\quad+R_{q}E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\mu}h_{t-1;\alpha}\right) \\
&= -B_{q}E_{q\beta-\frac{1}{2}}-Q_{q}E_{q\beta-\frac{1}{2}}E_{;\alpha}+C_{q}E_{q\beta}E_{;\mu} \\
&\quad-R_{q}B_{q\beta}E_{q\beta^{2}-\frac{1}{2}}-R_{q}Q_{q\beta}E_{q\beta^{2}-\frac{1}{2}}E_{;\alpha}+R_{q}C_{q\beta}E_{q\beta^{2}}E_{;\mu} \\
&\quad+R_{q}R_{q\beta}E\left(e^{q\beta^{2}\ln(h_{t-2})}h_{t-2;\mu}h_{t-2;\alpha}\right) \\
&= \sum_{i=0}^{\infty}\left[-B_{q\beta^{i}}E_{q\beta^{i+1}-\frac{1}{2}}-Q_{q\beta^{i}}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\alpha}+C_{q\beta^{i}}E_{q\beta^{i+1}}E_{;\mu}\right]\prod_{j=0}^{i-1}R_{q\beta^{j}} \\
&= E_{q}E_{;\alpha;\mu}
\end{aligned}
$$

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\mu,\mu} &= \left(\theta+\gamma E_{I}\right)E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\gamma E\left|z\right|E_{(;\mu)^{2}}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\mu,\mu} \\
&\quad+\left(\beta\theta+\beta\gamma E_{i}-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\mu} \\
&\quad+\frac{1}{4}\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^{2}}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{;\alpha(;\mu)^{2}} \\
&\quad+\left(\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\alpha}h_{t-1;\mu,\mu} \\
&= E_{;\alpha;\mu,\mu},
\end{aligned}
$$

if $\left|\beta^{2}+\frac{1}{4}\theta^{2}+\frac{1}{4}\gamma^{2}-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

Finally, as

$$
\begin{aligned}
h_{t;\mu,\alpha} &= \frac{1}{2}\left(\theta+\gamma\left[I\left(z_{t-1}\geq0\right)-I\left(z_{t-1}<0\right)\right]\right)\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}h_{t-1;\alpha}+\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}h_{t-1;\mu} \\
&\quad+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\mu,\alpha},
\end{aligned}
$$

we get

$$
\begin{aligned}
E\left(h_{t;\mu}h_{t;\mu,\alpha}\right) &= -\frac{1}{2}\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-1}E_{;\alpha}-\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\mu} \\
&+\frac{1}{2}\left(\beta\theta+\beta\gamma E_i-2\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\mu} \\
&+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)E_{;\alpha(;\mu)^2} \\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\mu,\alpha} \\
&= E_{;\mu;\mu,\alpha},
\end{aligned}
$$

as

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\theta z+\gamma\left|z\right|\right)=2\gamma\theta E\left|z\right|$$

and if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.3 Miu-Miu-Beta

Now

$$
\begin{aligned}
E\ell_{\mu\mu\beta} &= -\frac{T}{2}Eh_{t;\beta}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\beta}\right)-2E\left(\frac{1}{h_t}h_{t;\beta}\right)-E\left(h_{t;\beta}h_{t;\mu}^2\right)\right) \\
&= -\frac{T}{2}Eh_{t;\beta}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\beta}\right)-2E_{-1}E_{;\beta}-E\left(h_{t;\beta}h_{t;\mu}^2\right)\right).
\end{aligned}
$$

Now for the first term, notice that

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)^2\left(\theta z+\gamma\left|z\right|\right)=\gamma\left(3\theta^2+\gamma^2\right)E\left|z\right|$$

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\theta z+\gamma\left|z\right|\right)^2=\left(\theta+\gamma E_{Iz^2}\right)\left(\gamma^2+\theta^2\right)+2\gamma\theta\left(\theta Ez\left|z\right|+\gamma\right)$$

$$
\begin{aligned}
E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\theta z+\gamma\left|z\right|\right)\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma\left|z\right|\right) &= 2\beta\gamma\theta E\left|z\right|-\gamma\theta\left(\theta Ez\left|z\right|+\gamma\right) \\
&-\frac{1}{2}\left(\theta+\gamma E_{Iz^2}\right)\left(\gamma^2+\theta^2\right)
\end{aligned}
$$

and as

$$
\begin{aligned}
E\left(e^{q\ln h_t}\ln\left(h_t\right)h_{t;\mu}\right) &= -W_qE_{q\beta-\frac{1}{2}}+P_qE_{q\beta}E_{;\mu}-\beta B_qE_{q\beta-\frac{1}{2}}L \\
&-\beta C_qW_{q\beta}E_{q\beta^2-\frac{1}{2}}+\beta C_qP_{q\beta}E_{q\beta^2}E_{;\mu}-\beta C_qB_{q\beta}E_{q\beta^2-\frac{1}{2}}L \\
&+\beta^2C_qC_{q\beta}E\left(e^{q\beta^2\ln\left(h_{t-2}\right)}\ln\left(h_{t-2}\right)h_{t-2;\mu}\right) \\
&= \sum_{i=0}^{\infty}\beta^i\left(-W_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}+P_{q\beta^i}E_{q\beta^{i+1}}E_{;\mu}-B_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}L\right)\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qLE_{;\mu}
\end{aligned}
$$

it follows that

$$\begin{aligned}
E\ln h_t h_{t;\mu}^2 &= \left[\alpha\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)+\gamma\left(3\theta^2+\gamma^2\right)E\left|z\right|\right]E_{-1} \\
&\quad -\left[\begin{array}{c} 2\gamma\theta\left(2\beta E\left|z\right|-\theta Ez\left|z\right|-\gamma\right)-\left(\theta+\gamma E_{Iz^2}\right)\left(\gamma^2+\theta^2\right) \\ +2\alpha\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\mu} \\
&\quad +\left(\begin{array}{c} \alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) \\ -\beta\left(\theta^2+\gamma^2\right)+\beta^2\gamma E\left|z\right|+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E\left|z\right|^3 \\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E\left[z\left|z\right|\right]\end{array}\right)E_{(;\mu)^2} \\
&\quad +\beta\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-1}L-2\beta\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}LE_{;\mu} \\
&\quad +\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\ln\left(h_{t-1}\right)h_{t-1;\mu}^2 \\
&= LE_{(;\mu)^2},
\end{aligned}$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1.$

It follows that as

$$\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\beta}h_{t;\mu}\right) &= -B_qE_{q\beta-\frac{1}{2}}L+C_qE_{q\beta}LE_{;\mu}-Q_qE_{q\beta-\frac{1}{2}}E_{;\beta} \\
&\quad -R_qB_{q\beta}E_{q\beta^2-\frac{1}{2}}L+R_qC_{q\beta}E_{q\beta^2}LE_{;\mu}-R_qQ_{q\beta}E_{q\beta^2-\frac{1}{2}}E_{;\beta} \\
&\quad +R_qE\left(e^{q\beta\left[\alpha+\theta z_{t-2}+\gamma\left|z_{t-2}\right|+\beta\ln\left(h_{t-2}\right)\right]}\left(\beta-\frac{1}{2}\theta z_{t-2}-\frac{1}{2}\gamma\left|z_{t-2}\right|\right)^2h_{t-2;\beta}h_{t-2;\mu}\right) \\
&= \sum_{i=0}^{\infty}\left[-B_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}L+C_{q\beta^i}E_{q\beta^{i+1}}LE_{;\mu}-Q_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\beta}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_qE_{;\beta;\mu}
\end{aligned}$$

we get that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\mu}^2 &= \left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-1}L+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{(;\mu)^2} \\
&\quad -2\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}LE_{;\mu} \\
&\quad +\left[\left(\theta^2+\gamma^2\right)\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\gamma\theta\left(2\beta E_I-\theta E\left|z\right|\right)\right]E_{-1}E_{;\beta} \\
&\quad -2\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\beta;\mu} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\beta}h_{t-1;\mu}^2 \\
&= E_{;\beta(;\mu)^2},
\end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

Now for the first term, as

$$E\left(\theta+\gamma\left[I\left(z\geq0\right)-I\left(z<0\right)\right]\right)\left(\theta z+\gamma\left|z\right|\right)z=\theta\left(\theta+\gamma Ez\left|z\right|\right)+\gamma\left(\theta E_{Iz^2}+\gamma\right)$$

and

$$
\begin{aligned}
Eh_{t;\mu,\mu}\ln h_t &= \left[\alpha\left(\theta+\gamma E_I\right)+2\gamma\theta E\left|z\right|\right]E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{(;\mu)^2}\\
&+\frac{1}{4}\left[\theta\left(\theta+\gamma Ez\left|z\right|\right)+\gamma\left(\theta E_{Iz^2}+\gamma+\alpha E\left|z\right|\right)\right]E_{(;\mu)^2}\\
&+\left(\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\mu,\mu}\\
&+\beta\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\mu}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)Eh_{t-1;\mu,\mu}\ln h_{t-1}\\
&= LE_{;\mu,\mu},
\end{aligned}
$$

we get that

$$
\begin{aligned}
Eh_{t;\beta}h_{t;\mu,\mu} &= \left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\mu}+\frac{1}{4}\gamma E\left|z\right|LE_{;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\mu,\mu}\\
&+\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta;\mu}\\
&+\frac{1}{4}\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{;\beta(;\mu)^2}\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\mu,\mu}\\
&= E_{;\beta;\mu,\mu},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally for the second term $E\left(h_{t;\mu}h_{t;\mu,\beta}\right)$ and as

$$
\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\beta,\mu}\right) &= \frac{1}{2}B_qE_{q\beta}E_{;\beta}+A_qe^{q\alpha}E_{q\beta}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)M_qE_{q\beta}E_{;\beta;\mu}\\
&+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta,\mu}\right)\\
&= \sum_{i=0}^{\infty}\left[\frac{1}{2}B_{q\beta^i}E_{q\beta^{i+1}}E_{;\beta}+A_{q\beta^i}e^{\beta^iq\alpha}E_{q\beta^{i+1}}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)M_{q\beta^i}E_{q\beta^{i+1}}E_{;\beta;\mu}\right]\prod_{j=0}^{i-1}C_{q\beta^j}\\
&= E_qE_{;\beta,\mu}
\end{aligned}
$$

we get

$$
\begin{aligned}
Eh_{t;\mu}h_{t;\beta,\mu} &= -\frac{1}{2}\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-\frac{1}{2}}E_{;\beta}-\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\mu}\\
&-\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta,\mu}-\frac{1}{4}\left(\theta+\gamma\right)\left(\theta+\gamma E_{Iz^2}\right)E_{-\frac{1}{2}}E_{;\beta;\mu}\\
&+\frac{1}{2}\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{;\beta;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{(;\mu)^2}\\
&+\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\beta(;\mu)^2}h_{t-1}h_{t-1;\mu}\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\beta,\mu}\\
&= E_{;\mu;\beta,\mu}
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

4.4 Miu-Miu-Gama

We have that

$$\begin{aligned}E\ell_{\mu\mu\gamma} &= -\frac{T}{2}Eh_{t;\gamma}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\gamma}\right)-2E\left(\frac{1}{h_t}h_{t;\gamma}\right)-E\left(h_{t;\gamma}h_{t;\mu}^2\right)\right)\\ &= -\frac{T}{2}Eh_{t;\gamma}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\gamma}\right)-2E_{-1}E_{;\gamma}-E\left(h_{t;\gamma}h_{t;\mu}^2\right)\right).\end{aligned}$$

First, for the last term we have that, as

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)^2\left|z\right|=\left(\theta^2+\gamma^2\right)E\left|z\right|,$$

$$E\left(\theta+\gamma\left[I\left(z\geq 0\right)-I\left(z<0\right)\right]\right)\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma\left|z\right|\right)\left|z\right|=\theta\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|\right)-\gamma\left(\theta+\frac{1}{2}E_{Iz^2}\right)$$

$$\begin{aligned}E\left(e^{q\ln h_t}h_{t;\gamma}h_{t;\mu}\right) &= -X_qE_{q\beta-\frac{1}{2}}+Y_qE_{q\beta}E_{;\mu}-Q_qE_{q\beta-\frac{1}{2}}E_{;\gamma}\\ &\quad +R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}h_{t-1;\mu}\right)\\ &= \sum_{i=0}^{\infty}\left[-X_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}+Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\mu}-Q_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j}\\ &= E_qE_{;\gamma;\mu}\end{aligned}$$

we get

$$\begin{aligned}Eh_{t;\gamma}h_{t;\mu}^2 &= \left(\theta^2+\gamma^2\right)E\left|z\right|E\left(\frac{1}{h_{t-1}}\right)-2\left[\theta\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|\right)-\gamma\left(\theta+\frac{1}{2}E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\mu}\\ &\quad +\left(\begin{array}{c}\beta^2E\left|z\right|+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3\\ -\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^3\end{array}\right)E_{(;\mu)^2}\\ &\quad +\left[\left(\theta^2+\gamma^2\right)\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\gamma\theta\left(2\beta E_I-\theta E\left|z\right|\right)\right]E_{-1}E_{;\gamma}\\ &\quad -2\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right)\\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\gamma;\mu}\\ &\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)Ez^3\\ +\frac{3}{2}\beta\gamma\left(\theta E\left(z\left|z\right|\right)-\beta E\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\gamma}h_{t-1;\mu}^2\\ &= E_{;\gamma(;\mu)^2},\end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

For the first term

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\mu,\mu} &= \theta E\left|z\right|E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\left(\theta Ez\left|z\right|+\gamma\right)E_{(;\mu)^2}+\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\mu,\mu} \\
&\quad+\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma;\mu} \\
&\quad+\frac{1}{4}\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{;\gamma(;\mu)^2} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\mu,\mu} \\
&= E_{;\gamma;\mu,\mu},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, for the second term $E\left(h_{t;\mu}h_{t;\mu,\gamma}\right)$, as

$$\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\mu,\gamma}\right) &= \frac{1}{2}B_qE_{q\beta-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}G_qe^{q\alpha}E_{q\beta}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)M_qE_{q\beta}E_{;\gamma;\mu} \\
&\quad+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\mu,\gamma}\right) \\
&= \sum_{i=0}^{\infty}\left[\frac{1}{2}B_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}G_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^i}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)M_{q\beta^i}E_{q\beta^{i+1}}E_{;\gamma;\mu}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qE_{;\gamma,\mu}
\end{aligned}$$

we get

$$\begin{aligned}
Eh_{t;\mu}h_{t;\mu,\gamma} &= -\frac{1}{2}\left(\theta^2+2\gamma\theta E_I+\gamma^2\right)E_{-1}E_{;\gamma}-\frac{1}{2}E\left|z\right|E_{;\mu} \\
&\quad+\frac{1}{2}\left[\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|-\frac{1}{2}\left(\theta+\gamma\right)\left(\theta+\gamma E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\gamma;\mu} \\
&\quad-\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\mu,\gamma} \\
&\quad-\frac{1}{2}E\left|z\right|E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma\left|z\right|^3\right)E_{;\gamma(;\mu)^2} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\mu,\gamma} \\
&= E_{;\mu;\mu,\gamma},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.5 Miu-Miu-Theta

Now

$$\begin{aligned}
E\ell_{\mu\mu\theta} &= -\frac{T}{2}Eh_{t;\theta}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\theta}\right)-2E\left(\frac{1}{h_t}h_{t;\theta}\right)-E\left(h_{t;\theta}h_{t;\mu}^2\right)\right) \\
&= -\frac{T}{2}Eh_{t;\theta}h_{t;\mu,\mu}-\frac{T}{2}\left(2E\left(h_{t;\mu}h_{t;\mu,\theta}\right)-2E_{-1}E_{;\theta}-E\left(h_{t;\theta}h_{t;\mu}^2\right)\right).
\end{aligned}$$

For the last term we have that, as

$$
\begin{aligned}
E\left(e^{q[\alpha+\theta z_{t-1}+\gamma|z_{t-1}|+\beta\ln(h_{t-1})]}h_{t;\theta}h_{t;\mu}\right) &= -U_q E_{q\beta-\frac{1}{2}} + Z_q E_{q\beta}E_{;\mu} - Q_q E_{q\beta-\frac{1}{2}}E_{;\theta} \\
&\quad + R_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\theta}h_{t-1;\mu}\right) \\
&= \sum_{i=0}^{\infty}\left[\begin{array}{c} Z_{q\beta^i}E_{q\beta^{i+1}}E_{;\mu} - U_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}} \\ -Q_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\theta}\end{array}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_q E_{;\theta;\mu},
\end{aligned}
$$

$$
\begin{aligned}
Eh_{t;\theta}h_{t;\mu}^2 &= 2\theta\gamma E|z|E\left(\frac{1}{h_{t-1}}\right) - 2\left[\beta\gamma E|z| - \frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right) - \frac{1}{2}\gamma\left(\theta Ez|z|+\gamma\right)\right]E_{-\frac{1}{2}}E_{;\mu} \\
&\quad + \left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 \\ -\theta\beta - \gamma\beta E\left[z|z|\right] + \frac{1}{2}\gamma\theta E|z|^3\end{array}\right)E_{(;\mu)^2} \\
&\quad + \left[\left(\theta^2+\gamma^2\right)\left(\beta - \frac{1}{2}\gamma E|z|\right) + \gamma\theta\left(2\beta E_I - \theta E|z|\right)\right]E_{-1}E_{;\theta} \\
&\quad -2\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E|z|\right) + \frac{1}{2}\gamma\theta\left(\theta E\left(z|z|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right) + \beta\gamma\left(E_I - \theta E|z|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\theta;\mu} \\
&\quad + \left(\begin{array}{c}\beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E\left(z|z|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3\end{array}\right)Eh_{t-1;\theta}h_{t-1;\mu}^2 \\
&= E_{;\theta(;\mu)^2},
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E\left(z|z|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3\end{array}\right| < 1.$

For the second term, as

$$
\begin{aligned}
E\left(e^{q\ln h_t}h_{t;\mu,\theta}\right) &= B_q E_{q\beta-\frac{1}{2}}E_{;\theta} - \frac{1}{2}F_q e^{q\alpha}E_{q\beta}E_{;\mu} + \frac{1}{4}D_q E_{q\beta}E_{;\theta;\mu} \\
&\quad + C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\mu,\theta}\right) \\
&= \sum_{i=0}^{\infty}\left[B_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\theta} - \frac{1}{2}F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\mu} + \frac{1}{4}D_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta;\mu}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_q E_{;\mu,\theta}
\end{aligned}
$$

then

$$
\begin{aligned}
Eh_{t;\mu}h_{t;\mu,\theta} &= -\left(\theta^2 + 2\gamma\theta E_I + \gamma^2\right)E_{-1}E_{;\theta} + \frac{1}{2}E|z|E_{-\frac{1}{2}}E_{;\mu} \\
&\quad - \left(\beta\theta + \beta\gamma E_i - \gamma\theta E|z|\right)E_{-\frac{1}{2}}E_{;\mu,\theta} + \frac{1}{4}\left(\theta + \gamma Ez|z|\right)E_{(;\mu)^2} \\
&\quad + \left(\beta\theta + \beta\gamma E_i - \frac{3}{2}\gamma\theta E|z|\right)E_{-\frac{1}{2}}E_{;\theta;\mu} \\
&\quad + \frac{1}{4}\left(\beta\gamma E|z| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez|z|\right)h_{t-1;\mu}h_{t-1;\theta}h_{t-1;\mu} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E\left(z|z|\right)\right)Eh_{t-1;\mu}h_{t-1;\mu,\theta} \\
&= E_{;\mu;\mu,\theta},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

Finally, the first term

$$\begin{aligned} Eh_{t;\theta}h_{t;\mu,\mu} &= \gamma E\left|z\right|E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma E_{Iz^2}\right)E_{(;\mu)^2}-\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\mu,\mu} \\ &\quad +\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\theta;\mu}+\frac{1}{4}\left[\theta\left(\theta+\gamma Ez\left|z\right|\right)+\gamma\left(\theta E_{Iz^2}+\gamma\right)\right]E_{;\theta(;\mu)^2} \\ &\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\mu,\mu} \\ &= E_{;\theta;\mu,\mu}, \end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1.$

4.6 Miu-Alpha-Alpha

For $\circ\in\{\alpha,\theta,\gamma,\beta\}$ we have;

$$\begin{aligned} \ell_{\mu\alpha\circ} &= -\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;\circ}h_{t;\mu,\alpha}+\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\alpha,\circ}+\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\mu}h_{t;\alpha}h_{t;\circ}-h_{t;\mu}h_{t;\alpha,\circ}-h_{t;\alpha}h_{t;\mu,\circ}\right) \\ &\quad +\frac{1}{2}\sum_{t=1}^{T}z_t^2\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\circ}-\sum_{t=1}^{T}z_t\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha,\circ}-\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\circ}\right), \end{aligned}$$

with

$$E\ell_{\mu\alpha\circ}=-\frac{T}{2}h_{t;\mu,\alpha}Eh_{t;\circ}+\frac{T}{2}Eh_{t;\mu}h_{t;\alpha}h_{t;\circ}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\circ}+Eh_{t;\alpha}h_{t;\mu,\circ}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\circ}\right).$$

Hence, as

$$\begin{aligned} E\left(e^{q\ln h_t}h_{t;\alpha}^2\right) &= E_q+2C_qE_{q\beta}E_{;\alpha} \\ &\quad +R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\alpha}^2\right) \\ &= \sum_{i=0}^{\infty}\left[E_{q\beta^i}+2C_{q\beta^i}E_{q\beta^{i+1}}E_{;\alpha}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\ &= E_qE_{(;\alpha)^2} \end{aligned}$$

$$E\ell_{\mu\alpha\alpha}=-TEh_{t;\alpha}h_{t;\mu,\alpha}+\frac{T}{2}Eh_{t;\mu}h_{t;\alpha}^2-\frac{T}{2}Eh_{t;\mu}h_{t;\alpha,\alpha}+\frac{T}{2}E_{-\frac{1}{2}}E_{(;\alpha)^2}.$$

For the second term we have

$$\begin{aligned} Eh_{t;\mu}h_{t;\alpha}^2 &= -\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}-2\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha} \\ &\quad -\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{(;\alpha)^2} \\ &\quad +\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\mu}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\mu;\alpha} \\ &\quad +\left(\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|\right)Eh_{t-1;\mu}h_{t-1;\alpha}^2 \\ &= E_{;\mu(;\alpha)^2}, \end{aligned}$$

if $\left| \begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right) E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right) E\left|z\right|^3 \end{array} \right| < 1.$

For the third term, as

$$\begin{aligned} E\left(e^{q\ln h_t} h_{t;\alpha,\alpha}\right) &= \frac{1}{4}\left(D_q - \alpha e^{q\alpha} A_q\right) E_{q\beta} E_{(;\alpha)^2} + C_q E\left(e^{q\beta\ln(h_{t-1})} h_{t-1;\alpha,\alpha}\right) \\ &= \sum_{i=0}^{\infty} \frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha} A_{q\beta^i}\right) E_{q\beta^{i+1}} E_{(;\alpha)^2} \prod_{j=0}^{i-1} C_{q\beta^j} \\ &= E_q E_{;\alpha,\alpha} \end{aligned}$$

we get

$$\begin{aligned} Eh_{t;\mu} h_{t;\alpha,\alpha} &= -\frac{1}{2}\gamma\theta E\left|z\right| E_{-\frac{1}{2}} E_{(;\alpha)^2} - \left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right) E_{-\frac{1}{2}} E_{;\alpha,\alpha} \\ &\quad +\frac{1}{4}\left[\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E z\left|z\right|\right] E_{;\mu(;\alpha)^2} \\ &\quad +\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) Eh_{t-1;\mu} h_{t-1;\alpha,\alpha} \\ &= E_{;\mu;\alpha,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, for the first term

$$\begin{aligned} Eh_{t;\alpha} h_{t;\mu,\alpha} &= \frac{1}{2}\left(\theta + \gamma E_I\right) E_{-\frac{1}{2}} E_{;\alpha} + \frac{1}{4}\gamma E\left|z\right| E_{;\alpha;\mu} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E_{;\mu,\alpha} \\ &\quad +\frac{1}{2}\left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right) E_{-\frac{1}{2}} E_{(;\alpha)^2} \\ &\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta E z\left|z\right|\right) E_{;\mu(;\alpha)^2} \\ &\quad +\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) Eh_{t-1;\alpha} h_{t-1;\mu,\alpha} \\ &= E_{;\alpha;\mu,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

4.7 Miu-Alpha-Beta

As

$$\begin{aligned} E\left(e^{q\ln h_t} h_{t;\alpha} h_{t;\beta}\right) &= A_q e^{q\alpha} E_{q\beta} L + C_q E_{q\beta} L E_{;\alpha} + C_q E_{q\beta} E_{;\beta} \\ &\quad +R_q E\left(e^{q\beta\ln(h_{t-1})} h_{t-1;\beta} h_{t-1;\alpha}\right) \\ &= \sum_{i=0}^{\infty}\left[A_{q\beta^i} e^{q\alpha\beta^i} E_{q\beta^{i+1}} L + C_{q\beta^i} E_{q\beta^{i+1}} L E_{;\alpha} + C_{q\beta^i} E_{q\beta^{i+1}} E_{;\beta}\right] \prod_{j=0}^{i-1} R_{q\beta^j} \\ &= E_q E_{;\alpha;\beta} \end{aligned}$$

$$E\ell_{\mu\alpha\beta} = -\frac{T}{2}h_{t;\mu,\alpha}Eh_{t;\beta} + \frac{T}{2}Eh_{t;\mu}h_{t;\alpha}h_{t;\beta} - \frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\beta} + Eh_{t;\alpha}h_{t;\beta,\mu}\right) + \frac{T}{2}E_{-\frac{1}{2}}E_{;\alpha;\beta}.$$

For the second term, as

$$\begin{aligned}
E\left(e^{q\ln h_t}\ln\left(h_t\right)h_{t;\alpha}\right) &= D_qE_{q\beta} + \beta e^q A_q E_{q\beta}L + P_qE_{q\beta}E_{;\alpha} \\
&\quad + \beta C_q E\left(e^{q\beta\ln(h_{t-1})}\ln\left(h_{t-1}\right)h_{t-1;\alpha}\right) \\
&= D_qE_{q\beta} + \beta C_q D_{q\beta}E_{q\beta^2} + \beta A_q e^{q\alpha}E_{q\beta}L + \beta^2 C_q A_{q\beta}e^{q\alpha\beta}E_{q\beta^2}L \\
&\quad + P_qE_{q\beta}E_{;\alpha} + \beta C_q P_{q\beta}E_{q\beta^2}E_{;\alpha} \\
&\quad + \beta^2 C_q C_{q\beta}E\left(e^{q\beta^2\ln(h_{t-2})}\ln\left(h_{t-1}\right)h_{t-2;\alpha}\right) \\
&= \sum_{i=0}^{\infty}\left(\beta^i D_{q\beta^i}E_{q\beta^{i+1}} + \beta^{i+1}A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}L + \beta^i P_{q\beta^i}E_{q\beta^{i+1}}E_{;\alpha}\right)\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qLE_{;\alpha},
\end{aligned}$$

we have that

$$\begin{aligned}
E\ln h_t h_{t;\alpha}h_{t;\mu} &= -\left[2\gamma\theta E\left|z\right| + \alpha\left(\theta + \gamma E_I\right)\right]E_{-\frac{1}{2}} + \begin{pmatrix} \alpha\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) + \beta\gamma E\left|z\right| \\ -\frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right| \end{pmatrix}E_{;\mu} \\
&\quad - \left\{\begin{array}{c} \alpha\left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right) + 2\beta\gamma\theta E\left|z\right| \\ -\frac{1}{2}\left[\left(\theta + \gamma E_{Iz^2}\right)\left(\gamma^2 + \theta^2\right) + 2\gamma\theta\left(\theta Ez\left|z\right| + \gamma\right)\right] \end{array}\right\}E_{-\frac{1}{2}}E_{;\alpha} \\
&\quad + \begin{pmatrix} \alpha\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) \\ -\beta\left(\theta^2 + \gamma^2\right) + \beta^2\gamma E\left|z\right| + \frac{1}{4}\gamma\left(3\theta^2 + \gamma^2\right)E\left|z\right|^3 \\ +\frac{1}{4}\theta\left(\theta^2 + 3\gamma^2\right)Ez^3 - 2\beta\gamma\theta E\left[z\left|z\right|\right] \end{pmatrix}E_{;\alpha;\mu} - \beta\left(\theta + \gamma E_I\right)E_{-\frac{1}{2}}L \\
&\quad + \beta\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)LE_{;\mu} - \beta\left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}LE_{;\alpha} \\
&\quad + \beta\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E\ln\left(h_{t-1}\right)h_{t-1;\alpha}h_{t-1;\mu} \\
&= LE_{;\alpha;\mu},
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\mu}h_{t;\alpha}h_{t;\beta} &= -\left(\theta + \gamma E_I\right)E_{-\frac{1}{2}}L - \left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right)\left[E_{;\beta} + E_{-\frac{1}{2}}LE_{;\alpha}\right] \\
&\quad - \begin{pmatrix} \beta\theta\left(\beta - \gamma E\left|z\right|\right) + \frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right) + \frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2 + \gamma^2\right)\left(\theta + \gamma E_{Iz^2}\right) + \beta\gamma\left(E_I - \theta E\left|z\right|\right) \end{pmatrix}E_{-\frac{1}{2}}E_{;\alpha;\beta} \\
&\quad + \left(\beta - \frac{1}{2}\gamma\left|z\right|\right)LE_{;\mu} + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\beta;\mu} \\
&\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{;\alpha;\mu} \\
&\quad + \begin{pmatrix} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E\left|z\right|^3 \end{pmatrix}Eh_{t-1;\mu}h_{t-1;\alpha}h_{t-1;\beta} \\
&= E_{;\alpha;\beta;\mu},
\end{aligned}$$

if $\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E\left|z\right|^3 \end{array}\right| < 1.$

The forth term

$$\begin{aligned}
Eh_{t;\alpha}h_{t;\beta,\mu} &= \frac{1}{2}(\theta+\gamma E_I)E_{;\beta}+E_{;\mu}+\frac{1}{4}(\theta+\gamma)E_{;\beta;\mu}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\mu}\\
&+\frac{1}{2}(\beta\theta+\beta\gamma E_i-\gamma\theta E|z|)E_{;\alpha;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)h_{t-1;\alpha}h_{t-1;\mu}\\
&+\frac{1}{4}(\theta+\gamma)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E|z|^3\right)E_{;\alpha;\beta;\mu}\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)Eh_{t-1;\alpha}h_{t-1;\beta,\mu}\\
&= E_{;\alpha;\beta,\mu},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right|<1$.

The third term, as

$$\begin{aligned}
E\left(e^{q\ln h_t}h_{t-1;\alpha,\beta}\right) &= A_qe^{q\alpha}E_{q\beta}E_{;\alpha}+\frac{1}{4}(D_q-\alpha e^{q\alpha}A_q)E_{q\beta}E_{;\alpha;\beta}\\
&+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\alpha,\beta}\right)\\
&= \sum_{i=0}^{\infty}\left[A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\alpha}+\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\alpha;\beta}\right]\prod_{j=0}^{i-1}C_{q\beta^j}\\
&= E_qE_{;\alpha,\beta},
\end{aligned}$$

is given by:

$$\begin{aligned}
Eh_{t;\mu}h_{t;\alpha,\beta} &= -(\theta+\gamma E_I)E_{-\frac{1}{2}}E_{;\alpha}-\frac{1}{2}\gamma\theta E|z|E_{-\frac{1}{2}}E_{;\alpha;\beta}\\
&-(\beta\theta+\beta\gamma E_i-\gamma\theta E|z|)E_{-\frac{1}{2}}E_{;\alpha,\beta}\\
&+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\mu}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez|z|\right)E_{;\alpha;\beta;\mu}\\
&+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)Eh_{t-1;\mu}h_{t-1;\alpha,\beta}\\
&= E_{;\mu;\alpha,\beta},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right|<1$.

Finally, the first term

$$\begin{aligned}
\ln(h_t)h_{t;\mu,\alpha} &= \left(\frac{1}{2}\alpha(\theta+\gamma E_I)+\gamma\theta E|z|\right)E_{-\frac{1}{2}}E_{;\alpha}\\
&+\frac{1}{4}\left(\theta^2+2\gamma\theta Ez|z|+\gamma^2+\alpha\gamma E|z|\right)E_{;\alpha;\mu}+\begin{pmatrix}\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)+\beta\gamma E|z|\\ -\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez|z|\end{pmatrix}E_{;\mu,\alpha}\\
&+\frac{1}{2}\beta(\theta+\gamma E_I)E_{-\frac{1}{2}}LE_{;\alpha}+\frac{1}{4}\beta\gamma E|z|LE_{;\alpha;\mu}\\
&+\beta\left(\beta-\frac{1}{2}\gamma E|z|\right)E\ln(h_{t-1})h_{t-1;\mu,\alpha}\\
&= LE_{;\mu,\alpha},
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\mu,\alpha} &= \frac{1}{2}\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\alpha}+\frac{1}{4}\gamma E\left|z\right|LE_{;\alpha;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\mu,\alpha} \\
&\quad +\frac{1}{2}\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\beta} \\
&\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)h_{t-1;\beta}h_{t-1;\alpha}h_{t-1;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\mu,\alpha} \\
&= E_{;\beta;\mu,\alpha},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.8 Miu-Alpha-Gamma

As

$$\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\alpha}h_{t;\gamma}\right) &= G_qe^{q\alpha}E_{q\beta}+Y_qE_{q\beta}E_{;\alpha}+C_qE_{q\beta}E_{;\gamma} \\
&\quad +R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}h_{t-1;\alpha}\right) \\
&= \sum_{i=0}^{\infty}\left[G_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}+Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\alpha}+C_{q\beta^i}E_{q\beta^{i+1}}E_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_qE_{;\alpha;\gamma}
\end{aligned}$$

$$\begin{aligned}
E\ell_{\mu\alpha\gamma} &= -\frac{T}{2}Eh_{t;\mu,\alpha}h_{t;\gamma}+\frac{T}{2}Eh_{t;\alpha}h_{t;\gamma}h_{t;\mu}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\gamma}+Eh_{t;\alpha}h_{t;\gamma,\mu}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\gamma}\right) \\
&= -\frac{T}{2}Eh_{t;\mu,\alpha}h_{t;\gamma}+\frac{T}{2}Eh_{t;\alpha}h_{t;\gamma}h_{t;\mu}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\gamma}+Eh_{t;\alpha}h_{t;\gamma,\mu}\right)+\frac{T}{2}E_{-\frac{1}{2}}E_{;\alpha;\gamma}.
\end{aligned}$$

For the second term we have that

$$\begin{aligned}
Eh_{t;\alpha}h_{t;\gamma}h_{t;\mu} &= -\theta E\left|z\right|E_{-\frac{1}{2}}+\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\mu} \\
&\quad -\left[\theta\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|\right)-\gamma\left(\theta+\frac{1}{2}E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\alpha} \\
&\quad +\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\alpha;\mu}-\left(\beta\theta+\beta\gamma E_i-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\gamma;\mu} \\
&\quad -\left(\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right)E_{-\frac{1}{2}}E_{;\alpha;\gamma} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\gamma}h_{t-1;\alpha}h_{t-1;\mu} \\
&= E_{;\alpha;\gamma;\mu},
\end{aligned}$$

if $\left| \begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2 + 3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2 + 3\theta^2\right)E\left|z\right|^3 \end{array} \right| < 1.$

For the fourth term

$$\begin{aligned} Eh_{t;\alpha}h_{t;\mu,\gamma} &= \frac{1}{2}\left(\theta + \gamma E_I\right)E_{-\frac{1}{2}}E_{;\gamma} - \frac{1}{2}E\left|z\right|h_{;\mu} + \frac{1}{4}\left(\theta + \gamma\right)E_{;\gamma;\mu} + \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)E_{;\mu,\gamma} \\ &\quad + \frac{1}{2}\left(\beta\theta + \beta\gamma E_i - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\gamma} - \frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\alpha;\mu} \\ &\quad + \frac{1}{4}\left(\theta + \gamma\right)\left(\beta - \frac{1}{2}\theta Ez^3 - \frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\alpha;\gamma;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\alpha}h_{t-1;\mu,\gamma} \\ &= E_{;\alpha;\mu,\gamma}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

For the third term, as

$$\begin{aligned} E\left(e^{q\ln(h_t)}h_{t;\alpha,\gamma}\right) &= -\frac{1}{2}G_q e^{q\alpha}E_{q\beta}E_{;\alpha} + \frac{1}{4}\left(D_q - \alpha e^{q\alpha}A_q\right)E_{q\beta}E_{;\alpha;\gamma} \\ &\quad + C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma,\alpha}\right) \\ &= \sum_{i=0}^{\infty}\left[-\frac{1}{2}G_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\alpha} + \frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\alpha;\gamma}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\ &= E_q E_{;\alpha,\gamma} \end{aligned}$$

we get

$$\begin{aligned} h_{t;\mu}h_{t;\gamma,\alpha} &= \frac{1}{2}\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha} - \frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha;\gamma} - \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha,\gamma} \\ &\quad - \frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\alpha;\mu} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\alpha;\gamma;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\gamma,\alpha} \\ &= E_{;\mu;\gamma,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, for the first term

$$\begin{aligned} Eh_{t;\gamma}h_{t;\mu,\alpha} &= \frac{1}{2}\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha} + \frac{1}{4}\left(\theta Ez\left|z\right| + \gamma\right)E_{;\alpha;\mu} + \left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\mu,\alpha} \\ &\quad - \frac{1}{2}\left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\gamma} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\alpha;\gamma;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\mu,\alpha} \\ &= E_{;\gamma;\mu,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

4.9 Miu-Alpha-Theta

As now

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\alpha}h_{t;\theta}\right) &= F_q e^{q\alpha}E_{q\beta}+Z_qE_{q\beta}E_{;\alpha}+C_qE_{q\beta}E_{;\theta} \\
&\quad +R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\theta}h_{t-1;\alpha}\right) \\
&= \sum_{i=0}^{\infty}\left[F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}+Z_{q\beta^i}E_{q\beta^{i+1}}E_{;\alpha}+C_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_qE_{;\alpha;\theta},
\end{aligned}
$$

then

$$
\begin{aligned}
E\ell_{\mu\alpha\theta} &= -\frac{T}{2}Eh_{t;\mu,\alpha}h_{t;\theta}+\frac{T}{2}Eh_{t;\alpha}h_{t;\theta}h_{t;\mu}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\theta}+Eh_{t;\alpha}h_{t;\theta,\mu}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\theta}\right) \\
&= -\frac{T}{2}Eh_{t;\mu,\alpha}h_{t;\theta}+\frac{T}{2}Eh_{t;\alpha}h_{t;\theta}h_{t;\mu}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\alpha,\theta}+Eh_{t;\alpha}h_{t;\theta,\mu}\right)+E_{-\frac{1}{2}}E_{;\alpha;\theta}.
\end{aligned}
$$

For the second term we have that

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\theta}h_{t;\mu} &= -\gamma E\left|z\right|E_{-\frac{1}{2}}-\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\theta} \\
&\quad -\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{-\frac{1}{2}}E_{;\alpha} \\
&\quad -\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\alpha;\theta} \\
&\quad -\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\mu}+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)E_{;\theta;\mu} \\
&\quad +\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 \\ -\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\end{array}\right)E_{;\alpha;\mu} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\alpha}h_{t-1;\theta} \\
&= E_{;\alpha;\theta;\mu}
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

For the fourth term we have that

$$
\begin{aligned}
Eh_{t;\alpha}h_{t;\mu,\theta} &= \left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{4}\gamma E\left|z\right|E_{;\theta;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\mu,\theta} \\
&\quad +\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\theta}+\frac{1}{4}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\alpha;\mu} \\
&\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\alpha;\theta;\mu} \\
&\quad +\left(\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|\right)Eh_{t-1;\alpha}h_{t-1;\mu,\theta} \\
&= E_{;\alpha;\mu,\theta},
\end{aligned}
$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

For the third term, as

$$\begin{aligned} E\left(e^{q\ln(h_t)}h_{t;\alpha,\theta}\right) &= -\frac{1}{2}F_q e^{q\alpha}E_{q\beta}E_{;\alpha} + \frac{1}{4}\left(D_q - \alpha e^{q\alpha}A_q\right)E_{q\beta}E_{;\alpha;\theta} \\ &\quad + C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\alpha,\theta}\right) \\ &= \sum_{i=0}^{\infty}\left[-\frac{1}{2}F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\alpha} + \frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\alpha;\theta}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\ &= E_q E_{;\alpha,\theta}, \end{aligned}$$

we get

$$\begin{aligned} Eh_{t;\mu}h_{t;\alpha,\theta} &= \frac{1}{2}\gamma E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha} - \frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha;\theta} - \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha,\theta} \\ &\quad + \frac{1}{4}\left(\theta + \gamma Ez\left|z\right|\right)E_{;\alpha;\mu} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\alpha;\theta;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\alpha,\theta} \\ &= E_{;\mu;\alpha,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, for the first term we have

$$\begin{aligned} Eh_{t;\theta}h_{t;\mu,\alpha} &= \frac{1}{2}\gamma E\left|z\right|E_{-\frac{1}{2}}E_{;\alpha} + \frac{1}{4}\left(\theta + \gamma Ez\left|z\right|\right)E_{;\alpha;\mu} - \frac{1}{2}\left(\theta + \gamma Ez\left|z\right|\right)E_{;\mu,\alpha} \\ &\quad + \frac{1}{2}\left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\alpha;\theta} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\alpha;\theta;\mu} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\theta}h_{t-1;\mu,\alpha}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

4.10 Miu-Beta-Beta

For $\circ \in \{\beta, \gamma, \theta\}$ we have;

$$\begin{aligned} \ell_{\mu\beta\circ} &= -\frac{1}{2}\sum_{t=1}^{T} z_t^2 h_{t;\circ}h_{t;\mu,\beta} + \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2 - 1\right)h_{t;\mu,\beta,\circ} + \frac{1}{2}\sum_{t=1}^{T} z_t^2\left(h_{t;\mu}h_{t;\beta}h_{t;\circ} - \left(h_{t;\mu}h_{t;\beta,\circ} + h_{t;\beta}h_{t;\mu,\circ}\right)\right) \\ &\quad + \frac{1}{2}\sum_{t=1}^{T} z_t^2\frac{1}{\sqrt{h_t}}h_{t;\beta}h_{t;\circ} - \sum_{t=1}^{T} z_t\left(\frac{1}{\sqrt{h_t}}h_{t;\beta,\circ} - \frac{1}{\sqrt{h_t}}h_{t;\beta}h_{t;\circ}\right), \end{aligned}$$

with

$$E\ell_{\mu\beta\circ} = -\frac{T}{2}Eh_{t;\mu,\beta}h_{t;\circ} + \frac{T}{2}Eh_{t;\mu}h_{t;\beta}h_{t;\circ} - \frac{T}{2}\left(Eh_{t;\mu}h_{t;\beta,\circ} + Eh_{t;\beta}h_{t;\mu,\circ}\right) + \frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\beta}h_{t;\circ}\right).$$

Hence, as

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}\ln(h_t)\,h_{t;\beta}\right) &= D_qE_{q\beta}L+P_qE_{q\beta}E_{;\beta}+\beta A_qe^{q\alpha}E_{q\beta}L^2\\
&\quad+\beta C_qE\left(e^{q\beta\ln(h_{t-1})}\ln(h_{t-1})\,h_{t-1;\beta}\right)\\
&= \sum_{i=0}^{\infty}\left[D_{q\beta^i}E_{q\beta^{i+1}}L+P_{q\beta^i}E_{q\beta^{i+1}}E_{;\beta}+\beta^{i+1}A_{q\beta^i}e^{q\alpha\beta^i}2C_{q\beta^i}E_{q\beta^{i+1}}L^2\right]\beta^i\prod_{j=0}^{i-1}C_{q\beta^j}\\
&= E_qLE_{;\beta}
\end{aligned}
$$

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta}^2\right) &= A_qe^{q\alpha}E_{q\beta}L^2+2C_qE_{q\beta}LE_{;\beta}\\
&\quad+R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta}^2\right)\\
&= \sum_{i=0}^{\infty}\left[A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}L^2+2C_{q\beta^i}E_{q\beta^{i+1}}LE_{;\beta}\right]\prod_{j=0}^{i-1}R_{q\beta^j}\\
&= E_qE_{(;\beta)^2}
\end{aligned}
$$

we get

$$
\begin{aligned}
E\ell_{\mu\beta\beta} &= -TEh_{t;\mu,\beta}h_{t;\beta}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}^2-\frac{T}{2}Eh_{t;\mu}h_{t;\beta,\beta}+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\beta}^2\right)\\
&= =-TEh_{t;\mu,\beta}h_{t;\beta}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}^2-\frac{T}{2}Eh_{t;\mu}h_{t;\beta,\beta}+\frac{T}{2}E_{-\frac{1}{2}}E_{(;\beta)^2}.
\end{aligned}
$$

For the second term we have that as

$$
\begin{aligned}
E\left[e^{q\ln h_t}\ln^2(h_t)\right] &= N_qE_{q\beta}+2\beta D_qE_{q\beta}L+\beta^2e^{q\alpha}A_qE\left(e^{q\beta\ln(h_{t-1})}\ln^2(h_{t-1})\right)\\
&= N_qE_{q\beta}+\beta^2e^{q\alpha}A_qN_{q\beta}E_{q\beta^2}+2\beta D_qE_{q\beta}L+2\beta^3e^{q\alpha}A_qD_{q\beta}E_{q\beta^2}L\\
&\quad+\beta^4e^{q\alpha}A_qA_{q\beta}e^{q\alpha\beta}E\left(e^{q\beta^2\ln(h_{t-2})}\ln^2(h_{t-2})\right)\\
&= \sum_{i=0}^{\infty}\beta^{2i}\left(N_{q\beta^i}E_{q\beta^{i+1}}+2\beta D_{q\beta^i}E_{q\beta^{i+1}}L\prod_{j=0}^{i-1}A_{q\beta^j}\right)\prod_{j=0}^{i-1}e^{q\alpha\beta^j}\\
&= E_qL^2,
\end{aligned}
$$

where by convention $\prod_{j=0}^{-1}A_{q\beta^j}=1$ and $\prod_{j=0}^{-1}e^{q\alpha\beta^j}=1$,

$$
\begin{aligned}
E\ln^2(h_t)\,h_{t;\mu} &= -\left[\alpha^2(\theta+\gamma E_I)+4\alpha\gamma\theta E|z|+(\theta+\gamma E_{Iz^2})\left(\theta^2+\gamma^2\right)+2\gamma\theta(\theta Ez|z|+\gamma)\right]E_{-\frac{1}{2}}\\
&\quad-2\beta\left[\alpha(\theta+\gamma E_I)+2\gamma\theta E|z|\right]E_{-\frac{1}{2}}L-\beta^2(\theta+\gamma E_I)E_{-\frac{1}{2}}L^2\\
&\quad+\left\{\begin{array}{c}\alpha^2\left(\beta-\frac{1}{2}\gamma E|z|\right)+\left(\theta^2+\gamma^2\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E|z|^3\right)-\alpha\theta(\theta+\gamma Ez|z|)\\+2\alpha\gamma\left(\beta E|z|-\frac{1}{2}\theta Ez|z|-\frac{1}{2}\gamma\right)+2\gamma\theta\left(\beta Ez|z|-\frac{1}{2}\theta E|z|^3-\frac{1}{2}\gamma Ez^3\right)\end{array}\right\}E_{;\mu}\\
&\quad+2\beta\left[\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)+\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez|z|\right]LE_{;\mu}\\
&\quad+\beta^2\left(\beta-\frac{1}{2}\gamma E|z|\right)E\ln^2(h_{t-1})\,h_{t-1;\mu}\\
&= L^2E_{;\mu}
\end{aligned}
$$

and

$$
\begin{aligned}
E\ln(h_t)\,h_{t;\beta}h_{t;\mu} &= -\left[\alpha(\theta+\gamma E_I)+2\gamma\theta E|z|\right]E_{-\frac{1}{2}}L+\left[\alpha\left(\beta-\frac{1}{2}\gamma E|z|\right)+2\gamma\theta E|z|\right]LE_{;\mu} \\
&\quad -\left\{\begin{array}{c}\alpha\left[\beta(\theta+\gamma E_I)-\gamma\theta E|z|\right]+2\beta\gamma\theta E|z| \\ -\frac{1}{2}\left[(\theta+\gamma E_{Iz^2})\left(\gamma^2+\theta^2\right)+2\gamma\theta(\theta Ez|z|+\gamma)\right]\end{array}\right\}E_{-\frac{1}{2}}E_{;\beta} \\
&\quad +\left(\begin{array}{c}\alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right) \\ -\beta\left(\theta^2+\gamma^2\right)+\beta^2\gamma E|z|+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E|z|^3 \\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E[z|z|]\end{array}\right)E_{;\beta;\mu} \\
&\quad -\beta(\theta+\gamma E_I)E_{-\frac{1}{2}}L^2-\beta\left[\beta(\theta+\gamma E_I)-\gamma\theta E|z|\right]E_{-\frac{1}{2}}LE_{;\beta} \\
&\quad +\beta\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2E_{;\mu} \\
&\quad +\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E\ln(h_{t-1})\,h_{t-1;\mu}h_{t-1;\beta} \\
&= LE_{;\mu;\beta}
\end{aligned}
$$

if $\beta\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right|<1$, we get

$$
\begin{aligned}
Eh_{t;\mu}h_{t;\beta}^2 &= -(\theta+\gamma E_I)E_{-\frac{1}{2}}L^2-2(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|)E_{-\frac{1}{2}}LE_{;\beta} \\
&\quad -\left[\begin{array}{c}\beta\theta(\beta-\gamma E|z|)+\frac{1}{2}\gamma\theta\left(\theta E(z|z|)+\frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)(\theta+\gamma E_{Iz^2})+\beta\gamma(E_I-\theta E|z|)\end{array}\right]E_{-\frac{1}{2}}E_{(;\beta)^2} \\
&\quad +\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2E_{;\mu}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)LE_{;\beta;\mu} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E(z|z|)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\beta}^2 \\
&= E_{(;\beta)^2;\mu},
\end{aligned}
$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E(z|z|)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3\end{array}\right|<1.$

For the third term, as

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta,\beta}\right) &= 2A_qe^{q\alpha}E_{q\beta}E_{;\beta}+\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta}E_{(;\beta)^2} \\
&\quad +C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta,\beta}\right) \\
&= \sum_{i=0}^{\infty}\left[A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\beta}+\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{(;\beta)^2}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qE_{;\beta,\beta},
\end{aligned}
$$

we get

$$\begin{aligned}
h_{t;\mu}h_{t;\beta,\beta} &= -2\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\beta}-\frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{(;\beta)^2}-\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta,\beta} \\
&\quad+2\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\beta;\mu}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\mu(;\beta)^2} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\beta,\beta} \\
&= E_{;\mu;\beta,\beta}
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally the first term, as

$$\begin{aligned}
E\ln\left(h_t\right)h_{t;\beta,\mu} &= \left[\frac{1}{2}\alpha\left(\theta+\gamma E_I\right)+\gamma\theta E\left|z\right|\right]E_{;\beta}+\left(\alpha+\gamma E\left|z\right|\right)E_{;\mu} \\
&\quad+\frac{1}{4}\left[\left(\theta+\gamma\right)\left(\theta Ez^3+\gamma E\left|z\right|^3\right)+\alpha\left(\theta+\gamma\right)\right]E_{;\beta;\mu} \\
&\quad+\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right]E_{;\beta,\mu} \\
&\quad+\frac{1}{2}\beta\left(\theta+\gamma E_I\right)LE_{;\beta}+\beta LE_{;\mu}+\frac{1}{4}\beta\left(\theta+\gamma\right)LE_{;\beta;\mu} \\
&\quad+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\ln\left(h_{t-1}\right)h_{t-1;\beta,\mu} \\
&= LE_{;\beta,\mu}
\end{aligned}$$

it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\beta,\mu} &= \frac{1}{2}\left(\theta+\gamma E_I\right)LE_{;\beta}+LE_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)LE_{;\beta;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\beta,\mu} \\
&\quad+\frac{1}{2}\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{(;\beta)^2}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\beta;\mu} \\
&\quad+\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{(;\beta)^2;\mu} \\
&\quad+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\beta,\mu} \\
&= E_{;\beta;\beta,\mu},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.11 Miu-Beta-Gamma

As

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}\ln(h_t)\,h_{t;\gamma}\right) &= \left[\alpha G_q e^{q\alpha}+\theta H_q+\gamma M_q\right]E_{q\beta}+\beta G_q e^{q\alpha}E_{q\beta}L+P_qE_{q\beta}E_{;\gamma}\\
&\quad+\beta C_q E\left(e^{q\beta\ln(h_{t-1})}\ln(h_{t-1})\,h_{t-1;\gamma}\right)\\
&= \sum_{i=0}^{\infty}\left\{\begin{array}{c}\left[\alpha G_{q\beta^i}e^{q\alpha\beta^i}+\theta H_{q\beta^i}+\gamma M_{q\beta^i}\right]E_{q\beta^{i+1}}\\ +\beta^{i+1}G_{q\beta^i}e^{\alpha q\beta^i}E_{q\beta^{i+1}}E_{;\gamma}\end{array}\right\}\beta^i\prod_{j=0}^{i-1}C_{q\beta^j}\\
&= E_qLE_{;\gamma}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta}h_{t;\gamma}\right) &= G_qe^{\alpha q}E_{q\beta}L+Y_qE_{q\beta}E_{;\beta}+D_qE_{q\beta}LE_{;\gamma}\\
&\quad+R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}h_{t-1;\beta}\right)\\
&= \sum_{i=0}^{\infty}\left[G_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}L+Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\beta}+D_{q\beta^i}E_{q\beta^{i+1}}LE_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j}\\
&= E_qE_{;\beta;\gamma}
\end{aligned}
$$

we get

$$
\begin{aligned}
E\ell_{\mu\beta\gamma} &= -\frac{T}{2}Eh_{t;\mu,\beta}h_{t;\gamma}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}h_{t;\gamma}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\beta,\gamma}+Eh_{t;\beta}h_{t;\mu,\gamma}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\beta}h_{t;\gamma}\right)\\
&= -\frac{T}{2}Eh_{t;\mu,\beta}h_{t;\gamma}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}h_{t;\gamma}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\beta,\gamma}+Eh_{t;\beta}h_{t;\mu,\gamma}\right)+\frac{T}{2}E_{-\frac{1}{2}}E_{;\beta;\gamma}.
\end{aligned}
$$

For the second term, as

$$
\begin{aligned}
E\ln(h_t)\,h_{t;\gamma}h_{t;\mu} &= -\left(\alpha\theta E|z|+\theta^2Ez|z|+2\gamma\theta+\gamma^2E_{Iz^2}\right)E_{-\frac{1}{2}}+\left[\begin{array}{c}\alpha\left(\beta E|z|-\frac{1}{2}\theta Ez|z|-\frac{1}{2}\gamma\right)\\ +\theta^2Ez|z|+2\gamma\theta+\gamma^2E_{Iz^2}\end{array}\right]E_{;\mu}\\
&\quad-\left\{\begin{array}{c}\alpha\left(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|\right)+2\beta\gamma\theta E|z|\\ -\frac{1}{2}\left[\left(\theta+\gamma E_{Iz^2}\right)\left(\gamma^2+\theta^2\right)+2\gamma\theta\left(\theta Ez|z|+\gamma\right)\right]\end{array}\right\}E_{-\frac{1}{2}}E_{;\gamma}\\
&\quad+\left(\begin{array}{c}\alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right)h_{t-1;\gamma}h_{t-1;\mu}\\ -\beta\left(\theta^2+\gamma^2\right)+\beta^2\gamma E|z|+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E|z|^3\\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E\left[z|z|\right]\end{array}\right)E_{;\gamma;\mu}\\
&\quad-\beta\theta E|z|E_{-\frac{1}{2}}L+\beta\left(\beta E|z|-\frac{1}{2}\theta Ez|z|-\frac{1}{2}\gamma\right)LE_{;\mu}\\
&\quad-\beta\left(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|\right)E_{-\frac{1}{2}}LE_{;\gamma}\\
&\quad+\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right)E\ln(h_{t-1})\,h_{t-1;\gamma}h_{t-1;\mu}\\
&= LE_{;\gamma;\mu},
\end{aligned}
$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$ and it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\gamma}h_{t;\mu} &= -\theta E\left|z\right|E_{-\frac{1}{2}}L-\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}LE_{;\gamma} \\
&\quad -\left[\theta\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|\right)-\gamma\left(\theta+\frac{1}{2}E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\beta} \\
&\quad -\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right)\\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\beta;\gamma} \\
&\quad +\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)LE_{;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{;\gamma;\mu} \\
&\quad +\left(\begin{array}{c}\beta^2E\left|z\right|+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3\\ -\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^3\end{array}\right)E_{;\beta;\mu} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\beta}h_{t-1;\gamma} \\
&= E_{;\beta;\gamma;\mu}
\end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

For the fourth term, as

$$\begin{aligned}
E\ln\left(h_t\right)h_{t;\mu,\gamma} &= \left[\frac{1}{2}\alpha\left(\theta+\gamma E_I\right)+\gamma\theta E\left|z\right|\right]E_{-\frac{1}{2}}E_v-\frac{1}{2}\left(\alpha E\left|z\right|+\theta Ez\left|z\right|+\gamma\right)E_{;\mu} \\
&\quad +\frac{1}{4}\left(\theta+\gamma\right)\left(\alpha+\theta Ez^3+\gamma E\left|z\right|^3\right)E_{;\gamma;\mu} \\
&\quad +\left[\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right]E_{;\mu,\gamma} \\
&\quad +\frac{1}{2}\beta\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\gamma}-\frac{1}{2}\beta E\left|z\right|LE_{;\mu}+\frac{1}{4}\beta\left(\theta+\gamma\right)LE_{;\gamma;\mu} \\
&\quad +\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E\ln\left(h_{t-1}\right)h_{t-1;\mu,\gamma} \\
&= LE_{;\gamma,\mu},
\end{aligned}$$

it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\mu,\gamma} &= \frac{1}{2}\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\gamma}-\frac{1}{2}E\left|z\right|LE_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)LE_{;\gamma;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\gamma,\mu} \\
&\quad +\frac{1}{2}\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E\left(\frac{1}{\sqrt{h_{t-1}}}h_{t-1;\beta}h_{t;\gamma}\right)-\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\beta;\mu} \\
&\quad +\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\beta;\gamma;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\mu,\gamma} \\
&= E_{;\beta;\gamma,\mu},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

The third term, as

$$\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta,\gamma}\right) &= A_qe^{q\alpha}E_{q\beta}E_{;\gamma}-\frac{1}{2}G_qe^{q\alpha}E_{q\beta}E_{;\beta}+\frac{1}{4}\left(D_q-\alpha e^{q\alpha}A_q\right)E_{q\beta}E_{;\beta;\gamma} \\
&\quad +C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta,\gamma}\right) \\
&= \sum_{i=0}^{\infty}\left[\begin{array}{c} A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\gamma}-\frac{1}{2}G_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\beta} \\ +\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\beta;\gamma}\end{array}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qE_{;\beta,\gamma},
\end{aligned}$$

becomes

$$\begin{aligned}
Eh_{t;\mu}h_{t;\beta,\gamma} &= -\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\gamma}+\frac{1}{2}\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\beta}-\frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\beta;\gamma} \\
&\quad -\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta,\gamma} \\
&\quad +\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\gamma;\mu}-\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\beta;\mu} \\
&\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\beta;\gamma;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\beta,\gamma} \\
&= Eh_{t-1;\mu}h_{t-1;\beta,\gamma},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, the first term

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\beta,\mu} &= \frac{1}{2}\theta E\left|z\right|E_{;\beta}+E\left|z\right|E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)E\left|z\right|^3E_{;\beta;\mu}+\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\beta,\mu} \\
&\quad +\frac{1}{2}\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{;\beta;\gamma}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\gamma;\mu} \\
&\quad +\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\beta;\gamma;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\gamma}h_{t-1;\beta,\mu} \\
&= E_{;\gamma;\beta,\mu}
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.12 Miu-Beta-Theta

As

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}\ln(h_t)\,h_{t;\theta}\right) &= \left[\alpha A_q e^{q\alpha}+\theta M_q+\gamma H_q\right]E_q+P_qE_{q\beta}E_{;\theta}+\beta F_q e^{q\alpha}E_{q\beta}L \\
&\quad +\beta C_q E\left(e^{q\beta\ln(h_{t-1})}\ln(h_{t-1})\,h_{t-1;\theta}\right) \\
&= \sum_{i=0}^{\infty}\left[\begin{array}{c}\left(\alpha A_{q\beta^i}e^{q\alpha\beta^i}+\theta M_{q\beta^i}+\gamma H_{q\beta^i}\right)E_{q\beta^{i+1}}+P_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta} \\ +\beta^{i+1}F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}L\end{array}\right]\beta^i\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_qLE_{;\theta}
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta}h_{t;\theta}\right) &= F_q^{q\alpha}E_{q\beta}L+Z_qE_{q\beta}E_{;\beta}+C_qE_{q\beta}LE_{;\theta} \\
&\quad +R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta}h_{t-1;\theta}\right) \\
&= \sum_{i=0}^{\infty}\left[F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}L+Z_{q\beta^i}E_{q\beta^{i+1}}E_{;\beta}+C_{q\beta^i}E_{q\beta^{i+1}}LE_{;\theta}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_qE_{;\beta;\theta}
\end{aligned}
$$

we get

$$
\begin{aligned}
E\ell_{\mu\beta\theta} &= -\frac{T}{2}Eh_{t;\mu,\beta}h_{t;\theta}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}h_{t;\theta}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\beta,\theta}+Eh_{t;\beta}h_{t;\mu,\theta}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\beta}h_{t;\theta}\right) \\
&= -\frac{T}{2}Eh_{t;\mu,\beta}h_{t;\theta}+\frac{T}{2}Eh_{t;\mu}h_{t;\beta}h_{t;\theta}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\beta,\theta}+Eh_{t;\beta}h_{t;\mu,\theta}\right)+\frac{T}{2}E_{-\frac{1}{2}}E_{;\beta;\theta}.
\end{aligned}
$$

For the second term, as

$$
\begin{aligned}
E\ln(h_t)\,h_{t;\theta}h_{t;\mu} &= -\left[\alpha\gamma E|z|+\theta(\theta+\gamma Ez|z|)+\gamma(\theta E_{Iz^2}+\gamma)\right]E_{-\frac{1}{2}} \\
&\quad +\left[\theta(\theta+\gamma Ez|z|)+\gamma(\theta E_{Iz^2}+\gamma)-\frac{1}{2}\alpha(\theta+\gamma Ez|z|)\right]E_{;\mu} \\
&\quad -\left\{\begin{array}{c}\alpha(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|)+2\beta\gamma\theta E|z| \\ -\frac{1}{2}\left[(\theta+\gamma E_{Iz^2})\left(\gamma^2+\theta^2\right)+2\gamma\theta(\theta Ez|z|+\gamma)\right]\end{array}\right\}E_{-\frac{1}{2}}E_{;\theta} \\
&\quad +\left(\begin{array}{c}\alpha\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right) \\ -\beta\left(\theta^2+\gamma^2\right)+\beta^2\gamma E|z|+\frac{1}{4}\gamma\left(3\theta^2+\gamma^2\right)E|z|^3 \\ +\frac{1}{4}\theta\left(\theta^2+3\gamma^2\right)Ez^3-2\beta\gamma\theta E[z|z|]\end{array}\right)E_{;\theta;\mu} \\
&\quad -\beta\gamma E|z|E_{-\frac{1}{2}}L-\beta(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|)E_{-\frac{1}{2}}LE_{;\theta}-\frac{1}{2}\beta(\theta+\gamma Ez|z|)LE_{;\mu} \\
&\quad +\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E\ln(h_{t-1})\,h_{t-1;\mu}h_{t-1;\theta} \\
&= LE_{;\theta;\mu}
\end{aligned}
$$

if $\left|\beta\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)\right|<1$, and it follows that

$$\begin{aligned}
Eh_{t;\mu}h_{t;\theta}h_{t;\beta} &= -\gamma E\left|z\right|E_{-\frac{1}{2}}L-\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{-\frac{1}{2}}E_{;\beta} \\
&\quad -\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}LE_{;\theta} \\
&\quad -\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right)\\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\beta;\theta} \\
&\quad -\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)LE_{;\mu}+\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3\\ -\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\end{array}\right)E_{;\beta;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)LE_{;\beta;\theta} \\
&\quad +\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)\\ -\frac{3}{2}\beta^2\gamma E\left|z\right|+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\theta}h_{t-1;\beta} \\
&= E_{;\beta;\theta;\mu},
\end{aligned}$$

if $\left|\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right|<1.$

The fourth term, as

$$\begin{aligned}
E\ln\left(h_t\right)h_{t;\mu,\theta} &= \left[\alpha\left(\theta+\gamma E_I\right)+2\gamma\theta E\left|z\right|\right]E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{4}\left(\alpha\gamma E\left|z\right|+\theta^2+\gamma^2+2\gamma\theta Ez\left|z\right|\right)E_{;\theta;\mu} \\
&\quad +\left(\begin{array}{c}\alpha\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)+\beta\gamma E\left|z\right|\\ -\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\end{array}\right)E_{;\mu,\theta}-\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\mu} \\
&\quad +\beta\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\theta}+\frac{1}{4}\beta\gamma E\left|z\right|LE_{;\theta;\mu}+\beta\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)\ln\left(h_{t-1}\right)h_{t-1;\mu,\theta} \\
&= LE_{;\mu,\theta},
\end{aligned}$$

and it follows that

$$\begin{aligned}
Eh_{t;\beta}h_{t;\mu,\theta} &= \left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}LE_{;\theta}+\frac{1}{4}\gamma E\left|z\right|LE_{;\theta;\mu}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)LE_{;\mu,\theta} \\
&\quad +\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta;\theta} \\
&\quad +\frac{1}{4}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\beta;\mu}+\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\beta;\theta;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\beta}h_{t-1;\mu,\theta} \\
&= E_{;\beta;\mu,\theta},
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$, The third term, as

$$\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\beta,\theta}\right) &= A_q e^{q\alpha}E_{q\beta}E_{;\theta}-\frac{1}{2}F_q e^{q\alpha}E_{q\beta}E_{;\beta}+\frac{1}{4}\left(D_q-\alpha e^{q\alpha}A_q\right)E_{q\beta}E_{;\beta;\theta} \\
&\quad +C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\beta,\theta}\right) \\
&= \sum_{i=0}^{\infty}\left[\begin{array}{c} A_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\theta}-\frac{1}{2}F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\beta} \\ +\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\beta;\theta}\end{array}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\
&= E_q E_{;\beta,\theta}
\end{aligned}$$

becomes

$$\begin{aligned}
Eh_{t;\mu}h_{t;\beta,\theta} &= -\left(\theta+\gamma E_I\right)E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{2}\gamma E\left|z\right|E_{-\frac{1}{2}}E_{;\beta}-\frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\beta;\theta} \\
&\quad -\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\beta,\theta} \\
&\quad +\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\theta;\mu}+\frac{1}{4}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\beta;\mu} \\
&\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez\left|z\right|\right)E_{;\beta;\theta;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\beta,\theta} \\
&= E_{;\mu;\beta,\theta}
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally, the first term

$$\begin{aligned}
Eh_{t;\theta}h_{t;\beta,\mu} &= \frac{1}{2}\gamma E\left|z\right|E_{;\beta}+\frac{1}{4}\left(\theta+\gamma\right)Ez^3E_{;\beta;\mu}-\frac{1}{2}\left(\theta+\gamma Ez\left|z\right|\right)E_{;\beta,\mu} \\
&\quad +\frac{1}{2}\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{;\beta;\theta}+\left(\beta-\frac{1}{2}\gamma E\left|z\right|\right)E_{;\theta;\mu} \\
&\quad +\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\beta;\theta;\mu} \\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\theta}h_{t-1;\beta,\mu} \\
&= E_{;\theta;\beta,\mu}
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.13 Miu-Gamma-Gamma

Now, as

$$\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\gamma}^2\right) &= M_q E_{q\beta}+2Y_q E_{q\beta}E_{;\gamma}+R_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}^2\right) \\
&= \sum_{i=0}^{\infty}\left[M_{q\beta^i}E_{q\beta^{i+1}}+2Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\
&= E_q E_{(;\gamma)^2}
\end{aligned}$$

we get

$$E\ell_{\mu\gamma\gamma} = -TEh_{t;\mu,\gamma}h_{t;\gamma} + \frac{T}{2}Eh_{t;\mu}h_{t;\gamma}^2 - \frac{T}{2}Eh_{t;\mu}h_{t;\gamma,\gamma} + \frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\gamma}^2\right)$$
$$= -TEh_{t;\mu,\gamma}h_{t;\gamma} + \frac{T}{2}Eh_{t;\mu}h_{t;\gamma}^2 - \frac{T}{2}Eh_{t;\mu}h_{t;\gamma,\gamma} + \frac{T}{2}E_{-\frac{1}{2}}E_{(;\gamma)^2}.$$

For the second term, as

$$E\left(e^{q\ln(h_t)}h_{t;\gamma}h_{t;\mu}\right) = -X_qE_{q\beta-\frac{1}{2}} + Y_qE_{q\beta}E_{;\mu} - Q_qE_{q\beta-\frac{1}{2}}E_{;\gamma}$$
$$+R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}h_{t-1;\mu}\right)$$
$$= \sum_{i=0}^{\infty}\left[-X_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}} + Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\mu} - Q_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}}E_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j}$$
$$= E_qE_{;\gamma;\mu},$$

we get

$$Eh_{t;\mu}h_{t;\gamma}^2 = -\left(\theta+\gamma E_{Iz^2}\right)E_{-\frac{1}{2}} - 2\left[\theta\left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right|\right) - \gamma\left(\theta + \frac{1}{2}E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\gamma;\mu}$$
$$-\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right) + \frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right) + \frac{1}{2}\gamma\right)\\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right) + \beta\gamma\left(E_I - \theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{(;\gamma)^2}$$
$$+\left(\beta - \frac{1}{2}\theta Ez^3 - \frac{1}{2}\gamma E\left|z\right|^3\right)E_{;\mu} + 2\left(\begin{array}{c}\beta^2E\left|z\right| + \frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3\\ -\theta\beta E\left[z\left|z\right|\right] - \gamma\beta + \frac{1}{2}\gamma\theta Ez^3\end{array}\right)E_{;\gamma;\mu}$$
$$+\left(\begin{array}{c}\beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\gamma}^2$$
$$= E_{(;\gamma)^2;\mu},$$

if $\left|\begin{array}{c}\beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right|\\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right| < 1.$

For the third term, as

$$E\left(e^{q\ln(h_t)}h_{t;\gamma,\gamma}\right) = -G_qe^{q\alpha}E_{q\beta}E_{;\gamma} + \frac{1}{4}\left(D_q - \alpha e^{q\alpha}A_{q^i}\right)E_{q\beta}E_{(;\gamma)^2}$$
$$+C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma,\gamma}\right)$$
$$= \sum_{i=0}^{\infty}\left[-X_{q\beta^i}E_{q\beta^{i+1}-\frac{1}{2}} + \frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{(;\gamma)^2}\right]\prod_{j=0}^{i-1}C_{q\beta^j}$$
$$= E_qE_{;\gamma,\gamma}$$

we get

$$Eh_{t;\mu}h_{t;\gamma,\gamma} = \theta E\left|z\right|E_{-\frac{1}{2}}E_{;\gamma} - \frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{(;\gamma)^2} - \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma,\gamma}$$
$$-\left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\gamma;\mu} + \frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{(;\gamma)^2;\mu}$$
$$+\left(\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|\right)Eh_{t-1;\mu}h_{t-1;\gamma,\gamma}$$
$$= E_{;\mu;\gamma,\gamma}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

Finally the first term

$$\begin{aligned}
Eh_{t;\gamma}h_{t;\mu,\gamma} &= \frac{1}{2}\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma\right)E\left|z\right|^3E_{;\gamma;\mu}+\frac{1}{2}\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|-\frac{1}{2}\gamma\right)E_{;\mu,\gamma}\\
&+\frac{1}{2}\left(\beta\theta+\beta\gamma E_I-\gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{(;\gamma)^2}+\frac{1}{4}\left(\theta+\gamma\right)\left(\beta-\frac{1}{2}\theta Ez^3-\frac{1}{2}\gamma E\left|z\right|^3\right)E_{(;\gamma)^2;\mu}\\
&+\left(\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|\right)Eh_{t-1;\gamma}h_{t-1;\mu,\gamma}\\
&= E_{;\gamma;\mu,\gamma}
\end{aligned}$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|<1$.

4.14 Miu-Gamma-Theta

As

$$\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\gamma}h_{t;\theta}\right) &= H_qE_{q\beta}+Y_qE_{q\beta}E_{;\theta}+Z_qE_{q\beta}E_{;\gamma}\\
&\quad+R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma}h_{t-1;\theta}\right)\\
&= \sum_{i=0}^{\infty}\left[H_{q\beta^i}E_{q\beta^{i+1}}+Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta}+Y_{q\beta^i}E_{q\beta^{i+1}}E_{;\gamma}\right]\prod_{j=0}^{i-1}R_{q\beta^j}\\
&= E_qE_{;\gamma;\theta}
\end{aligned}$$

we get

$$\begin{aligned}
E\ell_{\mu\gamma\theta} &= -\frac{T}{2}Eh_{t;\mu,\gamma}h_{t;\theta}+\frac{T}{2}Eh_{t;\mu}h_{t;\gamma}h_{t;\theta}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\gamma,\theta}+Eh_{t;\gamma}h_{t;\mu,\theta}\right)+\frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\gamma}h_{t;\theta}\right)\\
&= -\frac{T}{2}Eh_{t;\mu,\gamma}h_{t;\theta}+\frac{T}{2}Eh_{t;\mu}h_{t;\gamma}h_{t;\theta}-\frac{T}{2}\left(Eh_{t;\mu}h_{t;\gamma,\theta}+Eh_{t;\gamma}h_{t;\mu,\theta}\right)+\frac{T}{2}E_{-\frac{1}{2}}E_{;\gamma;\theta}.
\end{aligned}$$

Hence, for the second term we have that

$$\begin{aligned}
Eh_{t;\mu}h_{t;\gamma}h_{t;\theta} &= -\left(\theta Ez\left|z\right|+\gamma\right)E_{-\frac{1}{2}}-\left[\theta\left(\beta E\left|z\right|-\frac{1}{2}\theta Ez\left|z\right|\right)-\gamma\left(\theta+\frac{1}{2}E_{Iz^2}\right)\right]E_{-\frac{1}{2}}E_{;\theta}\\
&-\left[\beta\gamma E\left|z\right|-\frac{1}{2}\theta\left(\theta+\gamma E_{Iz^2}\right)-\frac{1}{2}\gamma\left(\theta Ez\left|z\right|+\gamma\right)\right]E_{-\frac{1}{2}}E_{;\gamma}\\
&-\left[\begin{array}{c}\beta\theta\left(\beta-\gamma E\left|z\right|\right)+\frac{1}{2}\gamma\theta\left(\theta E\left(z\left|z\right|\right)+\frac{1}{2}\gamma\right)\\+\frac{1}{4}\left(\theta^2+\gamma^2\right)\left(\theta+\gamma E_{Iz^2}\right)+\beta\gamma\left(E_I-\theta E\left|z\right|\right)\end{array}\right]E_{-\frac{1}{2}}E_{;\gamma;\theta}\\
&+\left(\beta z_{t-1}\left|z_{t-1}\right|-\frac{1}{2}\theta E\left|z\right|^3-\frac{1}{2}\gamma Ez^3\right)E_{;\mu}+\left(\begin{array}{c}\beta^2E\left|z\right|+\frac{1}{4}\left(\theta^2+\gamma^2\right)E\left|z\right|^3\\-\theta\beta E\left[z\left|z\right|\right]-\gamma\beta+\frac{1}{2}\gamma\theta Ez^3\end{array}\right)E_{;\theta;\mu}\\
&+\left(\begin{array}{c}\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3\\-\theta\beta-\gamma\beta E\left[z\left|z\right|\right]+\frac{1}{2}\gamma\theta E\left|z\right|^3\end{array}\right)E_{;\gamma;\mu}\\
&+\left(\begin{array}{c}\beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right|\\+\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3\end{array}\right)Eh_{t-1;\mu}h_{t-1;\gamma}h_{t-1;\theta}\\
&= E_{;\gamma;\theta;\mu},
\end{aligned}$$

if $\left| \begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right) - \frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3 \end{array} \right| < 1.$

The fourth term

$$\begin{aligned} Eh_{t;\gamma}h_{t;\mu,\theta} &= \theta E\left|z\right|E_{-\frac{1}{2}}E_{;\theta} - \frac{1}{2}Ez\left|z\right|E_{;\mu} + \frac{1}{4}\left(\theta Ez\left|z\right| + \gamma\right)E_{;\theta;\mu} + \left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\mu,\theta} \\ &\quad + \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma;\theta} + \frac{1}{4}\left(\theta + \gamma Ez\left|z\right|\right)E_{;\gamma;\mu} \\ &\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\gamma;\theta;\mu} \\ &\quad + \left(\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right|\right)Eh_{t-1;\gamma}h_{t-1;\mu,\theta} \\ &= E_{;\gamma;\mu,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

The third term, as

$$\begin{aligned} E\left(e^{q\ln(h_t)}h_{t;\gamma,\theta}\right) &= -\frac{1}{2}G_q e^{q\alpha}E_{q\beta}E_{;\theta} - \frac{1}{2}F_q e^{q\alpha}E_{q\beta}E_{;\gamma} + \frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta}E_{;\gamma;\theta} \\ &\quad +C_q E\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\gamma,\theta}\right) \\ &= \sum_{i=0}^{\infty}\left[\begin{array}{c} -\frac{1}{2}G_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta} - \frac{1}{2}F_{q\beta^i}E_{q\beta^{i+1}}E_{;\gamma} \\ +\frac{1}{4}\left(D_{q\beta^i} - \alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{;\gamma;\theta} \end{array}\right]\prod_{j=0}^{i-1}C_{q\beta^j} \\ &= E_q E_{;\gamma,\theta} \end{aligned}$$

we get

$$\begin{aligned} Eh_{t;\mu}h_{t;\gamma,\theta} &= \frac{1}{2}\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\theta} + \frac{1}{2}\gamma E\left|z\right|E_{-\frac{1}{2}}E_{;\gamma} - \frac{1}{2}\gamma\theta E\left|z\right|E_{-\frac{1}{2}}E_{;\gamma;\theta} \\ &\quad - \left(\beta\theta + \beta\gamma E_I - \gamma\theta E\left|z\right|\right)E_{-\frac{1}{2}}E_{;\gamma,\theta} \\ &\quad -\frac{1}{2}\left(\beta E\left|z\right| - \frac{1}{2}\theta Ez\left|z\right| - \frac{1}{2}\gamma\right)E_{;\theta;\mu} + \frac{1}{4}\left(\theta + \gamma Ez\left|z\right|\right)h_{t-1;\mu}h_{t-1;\gamma} \\ &\quad +\frac{1}{4}\left(\beta\gamma E\left|z\right| - \frac{1}{2}\gamma^2 - \frac{1}{2}\theta^2 - \gamma\theta Ez\left|z\right|\right)E_{;\gamma;\theta;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right)Eh_{t-1;\mu}h_{t-1;\gamma,\theta} \\ &= E_{;\mu;\gamma,\theta}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1.$

Finally, the first term

$$\begin{aligned} Eh_{t;\theta}h_{t;\mu,\gamma} &= \frac{1}{2}\gamma E|z|E_{-\frac{1}{2}}E_{;\gamma} - \frac{1}{2}Ez|z|E_{;\mu} + \frac{1}{4}(\theta+\gamma)Ez^3E_{;\gamma;\mu} - \frac{1}{2}(\theta+\gamma Ez|z|)E_{;\mu,\gamma} \\ &\quad + \frac{1}{2}(\beta\theta+\beta\gamma E_I - \gamma\theta E|z|)E_{-\frac{1}{2}}E_{;\gamma;\theta} - \frac{1}{2}\left(\beta E|z| - \frac{1}{2}\theta Ez|z| - \frac{1}{2}\gamma\right)E_{;\theta;\mu} \\ &\quad + \frac{1}{4}(\theta+\gamma)\left(\beta - \frac{1}{2}\theta Ez^3 - \frac{1}{2}\gamma E|z|^3\right)E_{;\gamma;\theta;\mu} \\ &\quad + \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)Eh_{t-1;\theta}h_{t-1;\mu,\gamma} \\ &= E_{;\theta;\mu,\gamma}, \end{aligned}$$

if $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right| < 1$.

4.15 Miu-Theta-Theta

Finally, we have that as

$$\begin{aligned} E\left(e^{q\ln(h_t)}h_{t;\theta}^2\right) &= M_qE_{q\beta} + 2Z_qE_{q\beta}E_{;\theta} + R_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\theta}^2\right) \\ &= \sum_{i=0}^{\infty}\left[M_{q\beta^i}E_{q\beta^{i+1}} + 2Z_{q\beta^i}E_{q\beta^{i+1}}E_{;\theta}\right]\prod_{j=0}^{i-1}R_{q\beta^j} \\ &= E_qE_{(;\theta)^2} \end{aligned}$$

$$\begin{aligned} E\ell_{\mu\theta\theta} &= -TEh_{t;\mu,\theta}h_{t;\theta} + \frac{T}{2}Eh_{t;\mu}h_{t;\theta}^2 - \frac{T}{2}Eh_{t;\mu}h_{t;\theta,\theta} + \frac{T}{2}E\left(\frac{1}{\sqrt{h_t}}h_{t;\theta}^2\right) \\ &= -TEh_{t;\mu,\theta}h_{t;\theta} + \frac{T}{2}Eh_{t;\mu}h_{t;\theta}^2 - \frac{T}{2}Eh_{t;\mu}h_{t;\theta,\theta} + \frac{T}{2}E_{-\frac{1}{2}}E_{(;\theta)^2}. \end{aligned}$$

For the second term

$$\begin{aligned} Eh_{t;\mu}h_{t;\theta}^2 &= -(\theta+\gamma E_{Iz^2})E_{-\frac{1}{2}} - 2\left[\beta\gamma E|z| - \frac{1}{2}\theta(\theta+\gamma E_{Iz^2}) - \frac{1}{2}\gamma(\theta Ez|z|+\gamma)\right]E_{-\frac{1}{2}}E_{;\theta} \\ &\quad - \left[\begin{array}{c} \beta\theta(\beta-\gamma E|z|) + \frac{1}{2}\gamma\theta\left(\theta E(z|z|) + \frac{1}{2}\gamma\right) \\ +\frac{1}{4}\left(\theta^2+\gamma^2\right)(\theta+\gamma E_{Iz^2}) + \beta\gamma(E_I - \theta E|z|) \end{array}\right]E_{-\frac{1}{2}}E_{(;\theta)^2} \\ &\quad + \left(\beta - \frac{1}{2}\theta Ez^3 - \frac{1}{2}\gamma E|z|^3\right)E_{;\mu} + 2\left(\begin{array}{c} \frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3 \\ -\theta\beta - \gamma\beta E[z|z|] + \frac{1}{2}\gamma\theta E|z|^3 \end{array}\right)E_{;\theta;\mu} \\ &\quad + \left(\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E(z^3) - \frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E(z|z|) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3 \end{array}\right)Eh_{t-1;\mu}h_{t-1;\theta}^2 \\ &= E_{(;\theta)^2;\mu}, \end{aligned}$$

$$\text{if }\left|\begin{array}{c} \beta^3 + \frac{3}{4}\beta\theta^2 + \frac{3}{4}\beta\gamma^2 - \frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E(z^3) - \frac{3}{2}\beta^2\gamma E|z| \\ +\frac{3}{2}\beta\gamma\theta E(z|z|) - \frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E|z|^3 \end{array}\right| < 1.$$

The third term, as

$$
\begin{aligned}
E\left(e^{q\ln(h_t)}h_{t;\theta,\theta}\right) &= -F_q e^{q\alpha}E_{q\beta}E_{;\theta}+\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta}E_{(;\theta)^2}\\
&\quad +C_qE\left(e^{q\beta\ln(h_{t-1})}h_{t-1;\theta,\theta}\right)\\
&= \sum_{i=0}^{\infty}\left[-F_{q\beta^i}e^{q\alpha\beta^i}E_{q\beta^{i+1}}E_{;\theta}+\frac{1}{4}\left(D_{q\beta^i}-\alpha e^{q\beta^i\alpha}A_{q\beta^i}\right)E_{q\beta^{i+1}}E_{(;\theta)^2}\right]\prod_{j=0}^{i-1}C_{q\beta^j}\\
&= E_qE_{;\theta,\theta}
\end{aligned}
$$

becomes

$$
\begin{aligned}
Eh_{t;\mu}h_{t;\theta,\theta} &= \gamma E|z|E_{-\frac{1}{2}}E_{;\theta}-\frac{1}{2}\gamma\theta E|z|E_{-\frac{1}{2}}E_{(;\theta)^2}\\
&\quad -\left(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|\right)E_{-\frac{1}{2}}E_{;\theta,\theta}\\
&\quad +\frac{1}{2}\left(\theta+\gamma Ez|z|\right)E_{;\theta;\mu}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez|z|\right)E_{(;\theta)^2;\mu}\\
&\quad +\left(\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right|\right)Eh_{t-1;\mu}h_{t-1;\theta,\theta}\\
&= E_{;\mu;\theta,\theta}
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right|<1$.

Finally, the first term equals

$$
\begin{aligned}
Eh_{t;\theta}h_{t;\mu,\theta} &= \gamma E|z|E_{-\frac{1}{2}}E_{;\theta}-\frac{1}{2}E_{;\mu}+\frac{1}{4}\left(\theta+\gamma Ez|z|\right)E_{;\theta;\mu}-\frac{1}{2}\left(\theta+\gamma Ez|z|\right)E_{;\mu,\theta}\\
&\quad +\left(\beta\theta+\beta\gamma E_I-\gamma\theta E|z|\right)E_{-\frac{1}{2}}E_{(;\theta)^2}+\frac{1}{4}\left(\theta+\gamma Ez|z|\right)E_{;\theta;\mu}\\
&\quad +\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta Ez|z|\right)E_{(;\theta)^2;\mu}\\
&\quad +\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right)Eh_{t-1;\theta}h_{t-1;\mu,\theta}\\
&= E_{;\theta;\mu,\theta},
\end{aligned}
$$

if $\left|\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E\left(z|z|\right)\right|<1$.

Appendix

The following are a collection of helpful expectations that are frequently employed in the proofs of the main results. Notice that these expectations are analytical functions of the parameters only under the assumption of normality. In the general case these can be estimated from their sample counterparts, e.g. $E\exp\left[q\beta^i\left(\theta z+\gamma\left|z\right|\right)\right]$ can be estimated by $\frac{1}{T}\sum_{t=1}^{T}\exp\left[q\beta^i\left(\theta z_t+\gamma\left|z_t\right|\right)\right]$, where z_t is the standardized residual.

A Various Expected Values

First, notice that $E\exp\left[q\beta^i\left(\theta z+\gamma\left|z\right|\right)\right]$ depends on the distribution of z. For example, under normality, i.e. $z_t\overset{iid}{\sim}N\left(0,1\right)$, we get that, for $\kappa_1=q\theta\beta^i$ and $\kappa_2=q\gamma\beta^i$,

$$
\begin{aligned}
E\exp\left[q\beta^i\left(\theta z+\gamma\left|z\right|\right)\right] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\kappa_1 z+\kappa_2\left|z\right|-\frac{1}{2}z^2\right)dz\\
&= \exp\left(\frac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(z-\left(\kappa_1-\kappa_2\right)\right)^2\right)dz\\
&\quad+\exp\left(\frac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\frac{1}{2}\left(z-\left(\kappa_1+\kappa_2\right)\right)^2\right)dz\\
&= \exp\left(\frac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\left(\kappa_1-\kappa_2\right)}\exp\left(-\frac{1}{2}u^2\right)du\\
&\quad+\exp\left(\frac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\left(\kappa_1+\kappa_2\right)}\exp\left(-\frac{1}{2}u^2\right)du\\
&= \exp\left(\frac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\Phi\left(-\left(\kappa_1-\kappa_2\right)\right)+\exp\left(\frac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\Phi\left(\kappa_1+\kappa_2\right)\\
&= \exp\left(\frac{q^2\beta^{2i}\left(\gamma-\theta\right)^2}{2}\right)\Phi\left(q\beta^i\left(\gamma-\theta\right)\right)\\
&\quad+\exp\left(\frac{q^2\beta^{2i}\left(\theta+\gamma\right)^2}{2}\right)\Phi\left(q\beta^i\left(\theta+\gamma\right)\right)\\
&= A_{q\beta^i}
\end{aligned}
$$

where $\Phi\left(\bullet\right)$ the standard normal cumulative distribution.

Moreover, consider

$$E\left(z\exp\left(\kappa_1 z+\kappa_2\left|z\right|\right)\right)=\int_{-\infty}^{\infty}z\exp\left(\kappa_1 z+\kappa_2\left|z\right|\right)f\left(z\right)dz=F_{q\beta^i},$$

where $\kappa_1=q\theta\beta^i$, $\kappa_2=q\gamma\beta^i$ and $f\left(z\right)$ the density function of the standardized error.

Under normality, i.e. $z_t \overset{iid}{\sim} N(0,1)$, we get that

$$\begin{aligned}
E\left(z \exp q\beta^i\left(\theta z+\gamma|z|\right)\right) &= E\left(z\exp\left(\kappa_1 z+\kappa_2|z|\right)\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z\exp\left(\kappa_1 z+\kappa_2|z|-\frac{1}{2}z^2\right)dz \\
&= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-(\kappa_1-\kappa_2)}(u+(\kappa_1-\kappa_2))\exp\left(-\frac{1}{2}u^2\right)du \\
&\quad +\exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\int_{-(\kappa_1+\kappa_2)}^{\infty}(x+(\kappa_1+\kappa_2))\exp\left(-\frac{1}{2}x^2\right)dx
\end{aligned}$$

where $u=z-(\kappa_1-\kappa_2)$ and $x=z-(\kappa_1+\kappa_2)$, and it follows

$$\begin{aligned}
E\left(z\exp\left(\kappa_1 z+\kappa_2|z|\right)\right) &= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\left[\begin{array}{c}-\exp\left(-\frac{(\kappa_1-\kappa_2)^2}{2}\right)\\ +(\kappa_1-\kappa_2)\sqrt{2\pi}\Phi\left(-(\kappa_1-\kappa_2)\right)\end{array}\right] \\
&\quad +\exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\frac{1}{\sqrt{2\pi}}\left[\begin{array}{c}\exp\left(-\frac{(\kappa_1+\kappa_2)^2}{2}\right)\\ +(\kappa_1+\kappa_2)\sqrt{2\pi}\left(1-\Phi\left(-(\kappa_1+\kappa_2)\right)\right)\end{array}\right] \\
&= (\kappa_1-\kappa_2)\Phi\left(-(\kappa_1-\kappa_2)\right)\exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right) \\
&\quad +(\kappa_1+\kappa_2)\Phi(\kappa_1+\kappa_2)\exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right) \\
&= q\beta^i(\theta-\gamma)\Phi\left(q\beta^i(\gamma-\theta)\right)\exp\left(\frac{q^2\beta^{2i}(\gamma-\theta)^2}{2}\right) \\
&\quad +q\beta^i(\gamma+\theta)\Phi\left(q\beta^i(\gamma+\theta)\right)\exp\left(\frac{q^2\beta^{2i}(\gamma+\theta)^2}{2}\right) \\
&= F_{q\beta^i},
\end{aligned}$$

where, again, $\Phi(\bullet)$ the standard normal cumulative distribution.

Further,

$$E\left(|z|\exp q\beta^i\left(\theta z+\gamma|z|\right)\right)=E\left(|z|\exp\left(\kappa_1 z+\kappa_2|z|\right)\right)=\int_{-\infty}^{\infty}|z|\exp\left(\kappa_1 z+\kappa_2|z|\right)f(z)\,dz=G_{q\beta^i},$$

where, again, $\kappa_1=q\theta\beta^i$, $\kappa_2=q\gamma\beta^i$ and $f(z)$ the density function of the standardized error. Under normality, i.e. $z_t \overset{iid}{\sim} N(0,1)$, we get that

$$\begin{aligned}
E\left(|z|\exp\left(\kappa_1 z+\kappa_2|z|\right)\right) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|z|\exp\left(\kappa_1 z+\kappa_2|z|-\frac{1}{2}z^2\right)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}-z\exp\left(-\frac{1}{2}\left(-2(\kappa_1-\kappa_2)z+z^2\pm(\kappa_1-\kappa_2)^2\right)\right)dz \\
&\quad +\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}z\exp\left(-\frac{1}{2}\left(-2(\kappa_1+\kappa_2)z+z^2\pm(\kappa_1+\kappa_2)^2\right)\right)dz
\end{aligned}$$

and applying the same technique we get

$$
\begin{aligned}
E\left(|z|\exp\left(\kappa_1 z+\kappa_2|z|\right)\right) &= \frac{2}{\sqrt{2\pi}}+q\beta^i(\theta-\gamma)\,\Phi\left(q\beta^i(\gamma-\theta)\right)\exp\left(\frac{q^2\beta^{2i}(\gamma-\theta)^2}{2}\right)\\
&\quad +q\beta^i(\gamma+\theta)\,\Phi\left(q\beta^i(\gamma+\theta)\right)\exp\left(\frac{q^2\beta^{2i}(\gamma+\theta)^2}{2}\right)\\
&= G_{q\beta^i}.
\end{aligned}
$$

Additionally,

$$
\begin{aligned}
E\left\{(\alpha+\theta z+\gamma|z|)^2\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= \alpha^2 A_{q\beta^i}\exp\left(q\alpha\beta^i\right)+2\alpha D_{q\beta^i}\exp(-\alpha)\\
&\quad +2\gamma\theta H_{q\beta^i}+\left(\theta^2+\gamma^2\right)M_{q\beta^i}\\
&= N_{q\beta^i},
\end{aligned}
$$

where

$$
\begin{aligned}
E\left\{z|z|\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= H_{q\beta^i}\quad \textit{and}\\
E\left\{z^2\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= M_{q\beta^i}
\end{aligned}
$$

and they are evaluated as the above expectations.

Furthermore we employ the following symbols to manage the length of various expressions, where z is a zero mean unit variance *iid* random variable

$$
\begin{aligned}
E\left\{\left((\alpha+\theta z+\gamma|z|)\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma|z|\right)\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right)\right\} &= P_{q\beta^i}\\
E\left\{\left[(\theta+\gamma[I(z\geq 0)-I(z<0)])\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma|z|\right)\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right]\right\} &= Q_{q\beta^i}\\
E\left\{\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma|z|\right)^2\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= R_{q\beta^i}\\
E\left\{([I(z\geq 0)-I(z<0)])\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= S_{q\beta^i}\\
E\left\{z(\theta+\gamma[I(z\geq 0)-I(z<0)])\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= U_{q\beta^i}\\
E\left\{(\alpha+\theta z+\gamma|z|)(\theta+\gamma[I(z\geq 0)-I(z<0)])\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= W_{q\beta^i}\\
E\left\{|z|(\theta+\gamma[I(z\geq 0)-I(z<0)])\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= X_{q\beta^i}\\
E\left\{|z|\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma|z|\right)\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= Y_{q\beta^i}\\
E\left\{z\left(\beta-\frac{1}{2}\theta z-\frac{1}{2}\gamma|z|\right)\exp\left[q\beta^i(\alpha+\theta z+\gamma|z|)\right]\right\} &= Z_{q\beta^i}
\end{aligned}
$$

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