On the Optimal Taxation of Common-Pool Resources

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Abstract

Recent research developments in common-pool resource models emphasize the importance of links with ecological systems and the presence of non-linearities, thresholds and multiple steady states. In a recent paper Kossioris et al. (2008) develop a methodology for deriving feedback Nash equilibria for non-linear differential games and apply this methodology to a common-pool resource model of a lake where pollution corresponds to benefits and at the same time affects the ecosystem services. This paper studies the structure of optimal state-dependent taxes that steer the combined economic-ecological system towards the trajectory of optimal management, and provides an algorithm for calculating such taxes.

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1 Introduction

Recent developments in the study of common-pool resources embedded in ecological systems stress the presence of thresholds, multiple steady states and hysteresis effects, which are empirically observable features in these ecosystems, as opposed to the traditional approach of simple linear dynamics. Good examples are lakes, grasslands and coral reef systems.[1] In these systems many agents with potentially diverse objectives interact within the ecosystem and take actions which affect its dynamic behavior and long-run equilibrium. It follows that in the presence of complex dynamics, the socially optimal management of these ecosystems which is attained through cooperative behavior, and the characterization of outcomes under non-cooperative behavior have to be reconsidered. The basic methodology for this must be able to handle complex dynamics as well as strategic interaction between users of the resource. In recent papers, Brock and Starrett (2003), Mäler et al. (2003) and Kossioris et al. (2008) develop methodologies for analyzing the socially optimal (or cooperative) and the non-cooperative outcomes of ecosystems characterized by non-linear dynamics. In particular Kossioris et al. (2008) derived feedback Nash equilibria (FBNE) for non-linear differential games and applied this methodology to a model for lakes where pollution affects the ecosystem services. The main result of this analysis was that the socially optimal management solution and the best non-cooperative solution differ in terms of steady-state outcomes, paths towards the steady state and total welfare. The FBNE outcome is inferior to the optimal management outcome, in the sense that the steady-state accumulation of pollution is higher and the welfare lower at the FBNE relative to the optimal management. This result suggests that regulation is required in order to improve upon the unregulated FBNE outcome. This however raises the methodological issue of designing regulation of non-linear differential games. This problem has been analyzed in Mäler et al. (2003) for non-linear differential games with open-loop information structure, but has not been studied, as far as we know, for

[1]See for example Scheffer (1997), Carpenter and Cottingham (1997), Crépin (2006), Crépin and Lindahl (2008).

non-linear differential games with feedback information structure.

The purpose of this paper is to consider whether optimal regulation in the form of taxation can be designed for non-linear differential games with feedback information structure, which models interactions in economic-ecological systems of common-pool resources characterized by complex dynamics. We study optimal taxation in the sense of deriving a tax scheme which depends on the state of the system (i.e. the stock of pollution) at each point in time and which has the property of steering the regulated system to the socially optimal steady state. The structure of this general state-dependent tax rate is determined in the context of a differential game with non-linear objective, with state dynamics characterized by convex-concave feedbacks, and with solutions defined in terms of non-linear feedback Nash equilibrium strategies. We show that the non-linear feedback Nash equilibrium strategies and the optimal state-dependent tax should jointly satisfy a non-linear differential equation which is derived from the Hamilton-Jacobi-Bellman (HJB) equation of the dynamic programming representation of the non-linear differential game. Furthermore, we calculate, using a specific algorithm, the optimal state-dependent tax in three different functional forms: a tax scheme with a fixed rate, a tax scheme with a tax rate proportional to the state of the system, and a tax scheme which is a quadratic function of the state. We study this problem in detail for the non-linear state dynamic specification corresponding to the lake problem. Benchekroun and Long (1998) studied the optimal taxation of a polluting oligopoly using a state-dependent tax rate, in the context of a differential game with linear dynamics and concave objective. Their specific results for the case of feedback Nash equilibria focus on linear-quadratic problems and linear feedback strategies. In this paper we study not only a more general problem but also a problem which represents the non-linear dynamics characterizing common-pool resources.

2 Regulation of Non-Linear Differential Games

We consider the class of non-linear differential games studied by Mäler et al. (2003) and Kossioris et al. (2008). Consider a situation where n economic agents take actions a_i, $i = 1, 2, ..., n$, at each point in time t, with which they affect the state of a natural system that is shared by all the agents. The natural system is characterized by thresholds, hysteresis, and irreversibilities which could lead to the type of regime shifts which have been extensively studied in lake ecosystems.[2] In the context of the analysis of the lake ecosystems, these actions would be phosphorus loadings into the lake due to agricultural activities, while the economic agents are communities concerned about the eutrophication of the lake that they share. The action a_i generates benefits according to a strictly increasing and concave utility function $U(a_i)$, which is assumed to be the same for all agents. The evolution of the pollutant in the natural system is described by the non-linear transition equation

$$\dot{x}(t) = \sum_{i=1}^{n} a_i(t) - bx(t) + f(x(t)), x(0) = x_0. \tag{1}$$

In the lake context the state variable x is interpreted as accumulated phosphorus in a lake. Besides the standard linear degradation term $-bx$, non-linear feedbacks occur that are represented by the function $f(x)$, which is an increasing non-linear function of the state variable x. Following the standard literature (e.g. Mäler et al. 2003) the function $f(x)$ is a convex-concave function with a switching point in between, where $f'(x)$ is maximal. The stock of pollutants x causes environmental damage (or equivalently, reduces the flow of useful services generated by the natural system) according to a strictly increasing and convex damage function $D(x)$, which is also assumed to be the same for all agents. It follows that the flow of net benefits accruing to each agent at each point in time is given by $U(a_i(t)) - D(x(t))$. Each agent chooses a strategy a_i in order to maximize the present value of net benefits over an infinite time horizon, or

$$\max_{a_i(.)} \int_0^\infty e^{-\rho t}[U(a_i(t)) - D(x(t))]dt, i = 1, 2, ..., n, \tag{2}$$

[2] See, for example, Brock and Starrett (2003), Mäler et al. (2003), Wagener (2003), Dechert and O'Donnell (2006), Hein (2006), Kossioris et al. (2008).

subject to (1), where $\rho > 0$ is a discount rate, common for all agents.

The game aspect is standard: all actions add to the public bad, so that each agent generates a negative externality for the other agents. Three types of solutions are regarded as important for this game. A solution corresponding to the socially optimal management (SOM), where strategies $\{a_1, a_2, ..., a_n\}$ are chosen to maximize the sum of agents' net benefits, and two non-cooperative solutions which correspond to the open-loop Nash equilibrium (OLNE) and the feedback Nash equilibrium (FBNE). If SOM is regarded as the socially desirable solution, deviations of the paths for the state and the control variables for OLNE or FBNE from the corresponding paths for SOM call for regulation. Regulation should induce the OLNE or the FBNE of the regulated agents to converge in some well-defined way to the SOM solution.

3 Optimal Taxation in Non-Linear Differential Games

We consider the attainment of the SOM solution by a decentralized scheme which consists of a *state-dependent tax* $\tau(x)$ on individual phosphorous loadings. To characterize this tax we need a description of the three solutions described above.

The SOM solution requires choosing the set of strategies $\{a_1, a_2, ..., a_n\}$ in order to maximize the sum of individual net benefits, or

$$\max_{\{a_1(.),...a_n(.)\}} \int_0^\infty e^{-\rho t}[\sum_{i=1}^n U(a_i(t)) - nD(x(t))]dt, \tag{3}$$

subject to (1). The current-value Hamiltonian H for this problem is given by

$$H = \sum_{i=1}^n U(a_i) - nD(x) + \lambda[a - bx + f(x)], a = \sum_{i=1}^n a_i, \tag{4}$$

and Pontryagin's maximum principle yields the necessary conditions. In

(4), the costate variable λ should be interpreted as the social shadow cost of accumulated phosphorous. Following Mäler et al. (2003), the Modified Hamiltonian Dynamic System (MHDS), in the state-control space (x, a), for the optimal control problem associated with SOM is

$$\begin{aligned} \dot{x}(t) &= a - bx(t) + f(x(t)),\ x(0) = x_0, & (5) \\ \dot{a}(t) &= -\left[\rho + b - f'(x(t))\right] a(t) + a^2(t) 2cx(t), & (6) \end{aligned}$$

where a denotes the total loadings of all the agents together. As it is shown in Brock and Starrett (2003), under the assumptions made on the $U(a_i)$, $D(x)$ and $f(x)$ functions, this MHDS has an odd number of steady states. The first and the last steady states are locally stable. The locally stable steady states have the saddle-point property with a one-dimensional globally stable manifold, and the locally unstable steady states, with possibly complex eigenvalues, lie between two locally stable steady states. A local socially-optimal steady state (OSS) (x^*, a^*) is defined as a solution of the system

$$\begin{aligned} a &= bx + f(x), & (7) \\ a &= \frac{\rho + b - f'(x)}{2cx}. & (8) \end{aligned}$$

The solution corresponding to the OLNE of this game is obtained in a straightforward way by applying Pontryagin's maximum principle to the individual optimal control problems (2). The MHDS under symmetry in the state-control space (x, a) is

$$\dot{x}(t) = a(t) - bx(t) + f(x(t)),\ x(0) = x_0, \quad (9)$$

$$\dot{a}(t) = -\left[\rho + b - f'(x(t))\right] a(t) + \frac{1}{n} a^2(t) 2cx(t), \quad (10)$$

while an open-loop Nash equilibrium steady state (OLNE-SS) $\left(x^{OL}, a^{OL}\right)$ is defined as a solution of the system

$$a = bx + f(x), \tag{11}$$

$$a = \frac{[\rho + b - f'(x)]\, n}{2cx}. \tag{12}$$

OLNE-SSs have the same properties as OSSs. There are an odd number of steady states, the first and the last steady states are locally stable with the saddle-point property, and the locally unstable steady states lie between two locally stable steady states. This similarity between the SOM solution and the OLNE can easily be verified by comparing (8) and (12), which show that the OSS is a special case of the OLNE-SS for $n = 1$.

Comparing the MHDSs and the steady-state conditions for SOM and OLNE, it is clear that the two solutions differ. Thus regulation is required if the OLNE is inferior to the SOM solution and the aim is to approach the SOM solution. As shown by Mäler et al. (2003), the steady-state concentration of phosphorus is higher at the OLNE-SS relative to the OSS. This calls for a regulatory scheme which would induce the agents to choose loadings so that the SOM solution is attained. Mäler et al. (2003) show that in order to obtain the loading that corresponds to SOM, the tax on loading should be chosen such that $\tau(t) = -\lambda(t) + \lambda^{OL}(t)$. This implies that the tax bridges the gap between the social shadow cost of the accumulated phosphorus $\lambda(t)$ and the private shadow cost of the accumulated phosphorus $\lambda^{OL}(t)$ at the OLNE, which causes the steady-state phosphorus levels in the OLNE to exceed the (unique) steady-state phosphorus level under optimal management. The tax rate, however, is time-dependent, since it has to equalize cooperative and non-cooperative loading at every point in time. Mäler et al. (2003) studied in detail a simpler tax scheme, consisting of a fixed tax rate on loading. This fixed time-independent tax is called optimal steady-state tax (OSST), and when set as $\tau^*_{OL} = (n-1)/a^*$, the regulated OLNE reaches an OLNE-SS which is the same as the OSS, provided that the number of agents n is sufficiently low.[3]

[3]For the parameter constellation used by Mäler et al. (2003), if $n > 7$ then multiple steady states occur under regulation. The attainment of the steady state which corre-

The FBNE for the class of non-linear differential games, described by (2), with symmetric and stationary feedback Nash equilibrium strategies $a_i = h(x)$, $i = 1, 2, ..., n$, can be described using Pontryagin's maximum principle and a MHDS representation or by using dynamic programming and the resulting Hamilton-Jacobi-Bellman equation in the value function V. The current value Hamiltonian characterizing the FBNE is given by

$$\begin{aligned} H_i &= U(a_i) - D(x) + \lambda_i^{FB}[a_i + (n-1)\,h\,(x) - bx + f(x)], && (13) \\ & \left(a_i, \lambda_i^{FB}\right) \text{ the same for all } i. && (14) \end{aligned}$$

The MHDS in the state-control space (x, a) is

$$\dot{x}(t) = a(t) - bx(t) + f(x(t)),\ x(0) = x_0, \quad (15)$$

$$\dot{a}(t) = -\left[\rho + b - f'(x(t)) - (n-1)\,h'\,(x(t))\right] a(t) + \frac{1}{n} a^2(t) 2cx(t), \quad (16)$$

while a feedback Nash equilibrium steady state (FBNE-SS) $\left(x^{FB}, a^{FB}\right)$ is defined as the solution of the system

$$\begin{aligned} a &= bx + f(x), && (17) \\ a &= \frac{\left[\rho + b - f'(x) - (n-1)\,h'\,(x)\right] n}{2cx}. && (18) \end{aligned}$$

Comparing (16),(18) with (10),(12) suggests that the OLNE is a special case of the FBNE for $h'\,(x) = 0$. Comparison of the conditions characterizing SOM with the corresponding FBNE conditions suggests that regulation is required. The Hamiltonian formulation reveals the deviations between SOM and FBNE and the need for regulation if the FBNE is 'worse' than the SOM solution. The Hamiltonian formulation is, however, not as useful for determining the optimal tax $\tau(t) = -\lambda(t) + \lambda^{FB}(t)$, as in the case of regulating the OLNE. This is because it is difficult to determine $\lambda^{FB}(t)$ due to the presence of the unknown feedback Nash equilibrium strategies $a_i = h(x)$, which should

sponds to the OSS depends on initial conditions.

emerge as part of the solution of the problem. In order to overcome this difficulty we choose to use the dynamic programming approach for determining the optimal tax.

3.1 Non-linear feedback strategies and the optimal steady-state tax

The feedback Nash equilibrium strategies for the unregulated non-linear differential game described by (2) have been recently obtained by Kossioris at al. (2008), using the dynamic programming approach. It is shown in that paper that for the non-linear differential game described by (2), the steady state of the best feedback Nash equilibrium is not necessarily close to the OSS. Moreover, the paper shows that even if these steady states are close, the value of the corresponding feedback Nash equilibrium is generally much worse than the value of optimal management. Thus regulation is required to approach the SOM outcome in some well-defined way. In our case we require that the regulation is implemented by a state-dependent tax on loadings which is designed in a way such that the regulated system attains in the long run the desired optimal steady state of the SOM solution.

In order to study this type of regulation in the non-linear differential game, we start with the HJB equation of each agent i for the problem without regulation. The HJB equation for the unregulated problem is

$$\rho V(x) = \max_{a_i}\{U(a_i) - D(x) + V'(x)[a_i + (n-1)h(x) - bx + f(x)]\}. \quad (19)$$

Regulation is introduced in the form of a time-stationary tax rate per unit loading a_i which depends on the state of the system. The tax rate is defined as $\tau(x)$. Under the state-dependent tax the HJB equation becomes

$$\rho V(x) = \max_{a_i}\{U(a_i) - \tau(x)\, a_i - D(x) + V'(x)[a_i + (n-1)h(x) - bx + f(x)]\}. \quad (20)$$

The optimality condition is

$$U'(a_i) - \tau(x) + V'(x) = 0. \quad (21)$$

In equilibrium $a_i = h(x)$, so that

$$V'(x) = -U'(h(x)) + \tau(x) \ , \text{ and} \tag{22}$$
$$V''(x) = -U''(h)\, h'(x)) + \tau'(x) \, . \tag{23}$$

By differentiating (20) with respect to x, using the optimality conditions (22) and (23) and rearranging terms, a non-linear ordinary differential equation in $h(x)$, which depends on $\tau(x)$, is obtained:

$$[(nh(x) - bx + f(x))U''(h(x)) - (n-1)\tau(x) + (n-1)U'(h(x))]h'(x) = \tag{24}$$
$$(\rho + b - f'(x))\,[U'(h(x)) - \tau(x)] + [(n-1)\,h(x) - bx + f(x)]\,\tau'(x) - D'(x).$$

Using the specifications $U(a) = \ln a$, $D(x) = cx^2$, $f(x) = \frac{x^2}{x^2+1}$, which have been used in the lake analyses (Mäler et al. 2003, Kossioris et al. 2008) we obtain:

$$\begin{aligned}
&[-h(x) - (n-1)\,\tau(x)\,h^2(x) + bx - \frac{x^2}{x^2+1}]h'(x) = \\
&\left[\left(\rho + b - \frac{2x}{(x^2+1)^2}\right)(1 - \tau(x)\,h(x)) - 2cxh(x)\right] h(x) + \\
&\left[(n-1)\,h(x) - bx + \frac{x^2}{x^2+1}\right] \tau'(x)\,h^2(x) \, .
\end{aligned} \tag{25}$$

Equation (25) is a non-linear differential equation with two unknown functions $h(x)$ and $\tau(x)$. If the tax function was known then (25) could be solved for $h(x)$, although (25) does not have an analytic solution.[4] Our approach is to specify a functional form for the state-dependent tax and then to solve for the unknown equilibrium strategy $h(x)$. The parameters of the tax function are chosen such that the steady state of the SOM solution is attained as a feedback Nash equilibrium steady state of the regulated system.

[4]It should be noted that the Abel differential equation of the second kind, without an analytic solution, derived by Kossioris et al. (2008) for the unregulated problem, is a special case of (25) for $\tau = 0$.

We consider polynomial tax functions of the general form

$$\tau(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + ... + \beta_p x^p \ , \ \beta_j \gtreqless 0, \ , \ j = 0, 1, ..., p. \tag{26}$$

For example, if $\beta_j = 0, j \geq 1$, then $\tau(x) = \beta_0$ and we have the case of a fixed tax rate. If $\beta_0 = 0$, $\beta_j = 0, j \geq 2$, then $\tau(x) = \beta_1 x$ and we have the case of a tax function with a tax rate that is proportional to the current state. More complex tax structures can be defined in a straightforward way. Let $\boldsymbol{\beta} = \left(\beta_0, \beta_1, ..., \beta_p\right)$ be the vector of the parameters of the tax function. Depending on the specific structure of the tax function some of the elements of this vector will be zero. Since $\boldsymbol{\beta}$ is a parameter vector in (25), general theorems on the solutions of differential equations suggest that the solution of (25) will depend on this vector. Let $h(x, \boldsymbol{\beta})$ be such a solution. For an initial value $x = x_0$, the values for the feedback Nash equilibria, for each agent, are given by

$$V_f(x_0, x_f, \boldsymbol{\beta}) = \int_0^\infty e^{-\rho t}[\ln h(x(t), \boldsymbol{\beta}) - cx^2(t) - \tau(x(t))\, h(x(t), \boldsymbol{\beta})]dt, \tag{27}$$

where x_f is a steady state and $h(x, \boldsymbol{\beta})$ is the solution of the differential equation (25), with boundary condition

$$h(x, \boldsymbol{\beta}) = \frac{1}{n}(bx - \frac{x^2}{x^2+1}), \tag{28}$$

and $x(t)$ is the solution of the differential equation

$$\dot{x}(t) = nh(x(t), \boldsymbol{\beta}) - bx(t) + \frac{x^2(t)}{x^2(t)+1}, x(0) = x_0. \tag{29}$$

It will be clear that in general not every steady state x_f can be reached from any initial state x_0 for a given $\tau(x)$. It will also be clear that in general not every steady state x_f will be stable. If, however, a number of stable steady states can be reached from some initial state x_0 for a given tax $\tau(x)$, it is assumed that the agents will be able to coordinate on the best *regulated* feedback Nash equilibrium, in the sense of attaining the maximum value (27), if it exists. Thus the regulator sets the tax scheme and the agents, by taking the scheme as given, coordinate on a non-linear feedback Nash equilibrium profile $h(x, \boldsymbol{\beta})$ which converges to a steady state x_f and maximizes the value (27). This means that in this case the value will be only a function of the initial state and the tax parameters $\boldsymbol{\beta}$:

$$V_f(x_0, \boldsymbol{\beta}) = \max_{x_f} V_f(x_0, x_f, \boldsymbol{\beta}) \tag{30}$$

where x_f must be stable and reachable from x_0, and where it is assumed that the maximum exists.

Then an optimal steady-state tax function (OSSTF) can be defined as follows.

Definition 1 (OSSTF) *Let x_0 be a given initial state and let x^* be the optimal steady state which corresponds to the SOM solution. The state-dependent tax function $\tau^*(x)$ for which the corresponding best regulated feedback Nash equilibrium $h^*(x, \boldsymbol{\beta}^*)$ generates a path that converges starting from x_0 to the FBNE-SS x_f, which is equal to x^*, will be the optimal steady-state tax function. In this sense the specific tax function $\tau^*(x)$ and the regulated feedback strategies $h^*(x, \boldsymbol{\beta}^*)$ reproduce the OSS of the SOM solution.*

The numerical algorithm to determine the best regulated feedback Nash equilibrium and the OSSTF for the lake problem consists of the following steps:

Step 1. For each candidate x_f and tax parameters $\boldsymbol{\beta} \in \mathcal{B} = [\mathbf{0}, \boldsymbol{\beta}^{\max}]$, the non-linear ordinary differential equation (25) with boundary condition (28) is solved, with the ode solver *ode15s* of Matlab, in the intervals $[p, x_f]$ and $[x_f, q]$, where p and q are chosen appropriately, and the $h(x, \boldsymbol{\beta})$ profile is determined.

Step 2. The numerical solution for $h(x, \boldsymbol{\beta})$ is used to solve the transition equation (29) in the interval $[0, T]$, where T is chosen appropriately.

Step 3. The value (27) is computed, using a Matlab *quad* function.

Step 4. The value is maximized over the set of admissible x_f, according to (30).

Step5a. If the tax function corresponds to a fixed tax $\tau_0(x) = \beta_0$ or to a tax that is proportional to the current state $\tau_1(x) = \beta_1 x$ (i.e. the state-dependent tax is determined by a single parameter β_j, $j = 0$ or 1), then we proceed as follows. We construct the relationship $x_f = \phi\left(\beta_j\right)$, $j = 0$ or 1, that determines the FBNE-SS which can be reached from a given initial state x_0 with the best regulated feedback Nash equilibrium when the tax function is $\tau_0(x) = \beta_0$ or $\tau_1(x) = \beta_1 x$. Since we search for a tax function that will steer the regulated system to the SOM steady state x^*, the parameter of the OSSTF should satisfy $x^* = \phi\left(\beta_j^*\right)$, $j = 0$ or 1. If x^* is in the domain of ϕ, the OSSTF will be $\tau_0^*(x) = \beta_0^*$ or $\tau_1^*(x) = \beta_1^* x$.

Step5b. If the tax function corresponds to the quadratic function $\tau_2(x) = \beta_1 x + \beta_2 x^2$ (i.e. the state-dependent tax is determined by the parameter vector (β_1, β_2)), then we proceed as follows. Provided that the set of x_f in step 4 contains x^*, we construct the contour $\psi(\beta_1, \beta_2) = x^*$ which describes combinations of the tax-function parameters (β_1, β_2) that attain the SOM steady state x^*. We choose from this contour as the parameters of the OSSTF, the pair (β_1^*, β_2^*) that maximizes the social welfare given by

$$W(\boldsymbol{\beta}) = \int_0^\infty e^{-\rho t} \sum_{i=1}^{n} [\ln h(x(t), \boldsymbol{\beta}) - cx^2(t)] dt. \tag{31}$$

The OSSTF will be $\tau_2^*(x) = \beta_1^* x + \beta_2^* x^2$.

It is straightforward, although computationally very demanding, to define the algorithm for tax functions with more than three parameters. If a set of values for the parameter vector leads to the desired steady state, the regulator chooses the element of this set that maximizes social welfare.

3.2 Numerical results

In order to be able to compare our results with the earlier results of Mäler et al. (2003) and Kossioris et al. (2008), the basic parameters are fixed at the same values as in these studies: $b = 0.6$, $\rho = 0.03$, $c = 1$ and $n = 2$. For these parameter values the saddle-point stable optimal steady state for SOM is $x^* = 0.353$, while the socially optimal steady-state phosphorous loading is

$a^* = 0.101$. Thus the open-loop OSST is $\tau^*_{OL} = (n-1)/0.101$ which means, for example, that for $n = 2$ a regulated OLNE with $\tau^*_{OL} = 1/0.101 \simeq 9.9$ will converge in the long run, along the stable manifold, to the OSS $x^* = 0.353$ as shown in Mäler et al. (2003). In order to study the structure of the optimal tax function, for the feedback Nash equilibrium, we set the initial state at $x_0 = 0.6$ which is well above the OSS $x^* = 0.353$, so that the OSSTF indeed has to steer the system towards the OSS. We examine three cases for the OSSTF, the fixed tax $\tau_0(x) = \beta_0$, the proportional tax $\tau_1(x) = \beta_1 x$, and the quadratic tax $\tau_2(x) = \beta_1 x + \beta_2 x^2$.

3.2.1 The fixed tax

In figure 1, the inverse of the $x_f = \phi(\beta_0)$ relationship described in step 5a of the algorithm is presented.

[Figure 1]

The relationship has the expected negative slope indicating that the higher the tax parameter, the lower the FBNE-SS. The cross in figure 1 indicates the OSSTF $\beta_0 = 5.9$ which attains the desired socially optimal steady state $x^* = 0.353$. Figure 2 presents feedback profiles $h(x, \beta_0)$ for the data of figure 1. The dashed line corresponds to the regulated profile with fixed tax $\tau^* = 5.9$, and the value for $x^* = 0.353$ is indicated by the square. The solid line corresponds to $\beta_0 = 0$, which means that this profile corresponds to the unregulated FBNE with steady state $x_f = 0.3825$ (the value of this profile at the steady state is indicated by the circle). Both the regulated and the unregulated steady states are stable.

[Figure 2]

Figure 3 shows the time path of the phosphorous stock under the fixed tax (dashed line) towards $x^* = 0.353$, along with the path corresponding to the unregulated equilibrium (solid line) converging to $x_f = 0.3825$.

[Figure 3]

The regulated social welfare for each player under the fixed tax, as defined by (31), is $SW|_{\beta_0=5.9} = -108.018644$.

3.2.2 The proportional tax

Following again step 5a of the algorithm, the proportional tax that attains $x^* = 0.353$ as a regulated steady state is $\beta_1 = 4.4$. The $x_f = \phi(\beta_1)$ relationship, the regulated, and unregulated feedback profiles and state paths are similar to the paths corresponding to the fixed tax (figures 1,2,3 above). The comparison will follow below in section 3.2.4. The regulated steady state is stable, while the regulated social welfare for each player under the proportional tax, as defined by (31), is $SW|_{\beta_1=4.4} = 107.993973$.

3.2.3 The quadratic tax

Following step 5b of the algorithm, figure 4 shows the steady-state contours for different combinations of (β_1, β_2). The contour of interest is the one corresponding to $x^* = 0.353$. From the different combinations of (β_1, β_2) on this contour the combination $(\beta_1^*, \beta_2^*) = (0.9, 6)$ is the one that maximizes welfare with $SW(\beta_1^*, \beta_2^*) = -107.893107$. This combination can therefore be considered as the optimal quadratic tax scheme that attains $x^* = 0.353$ as the regulated steady state. The quadratic tax is an improvement in terms of social welfare relative to the fixed tax and the proportional tax.

[Figure 4]

The regulated feedback profile $h(x, \beta_1, \beta_2)$ and the regulated time path of the phosphorous stock are similar to those of the fixed tax while again the regulated FBNE-SS is stable. The comparison will follow below in section 3.2.4.

3.2.4 Summary of numerical results

Our numerical results are summarized in table 1. Table 1 presents social welfare per individual for the SOM solution, the unregulated feedback Nash equilibrium and the regulated feedback Nash equilibrium for the fixed, the proportional and the quadratic tax.

Table 1: Social Welfare

Socially Optimal Management	-107.227 , $x^* = 0.353$
Unregulated	-108.709334 , $x_f = 0.3825$
Fixed tax	-108.018644 , $x^* = 0.353$
Proportional tax	-107.993973 , $x^* = 0.353$
Quadratic tax	-107.891307 , $x^* = 0.353$

Note that all the three tax schemes, by construction, induce the socially optimal steady state but they differ in terms of social welfare. The reason is that the proportional tax moves the trajectory towards the steady state closer to the socially optimal one than the fixed tax, and the quadratic tax moves it closer than the proportional tax. The differences are small but it can be expected that higher-order tax schemes would move the trajectory further and decrease the gap in social welfare between the SOM solution and the regulated feedback Nash equilibrium.

Figure 5 puts together the $h(x)$ profiles for the unregulated feedback Nash equilibrium (solid line) and for the fixed, linear, and quadratic tax cases (other lines, at the right from top to bottom, respectively).

[Figure 5]

The comparison shows that for states higher than the socially optimal steady state $x^* = 0.353$, loadings decrease as we move from the fixed tax to the quadratic tax. As is to be expected, unregulated loadings are the highest. Finally, figure 6 puts together the time paths of the phosphorus stock for the cases of socially optimal management (lower solid line), unregulated feedback Nash equilibrium (upper solid line) and regulated feedback Nash equilibria with fixed, linear and quadratic taxes, respectively.

[Figure 6]

The upper path in figure 6 corresponds to the unregulated feedback equilibrium and converges to the steady state $x_f = 0.3825$. The next three paths (close together) correspond, starting from the top, to the fixed, linear, and quadratic tax cases. All these paths converge to the socially optimal steady state $x^* = 0.353$. The lower path is the one corresponding to socially optimal management. We observe that the optimal management path converges very fast to the socially optimal steady state $x^* = 0.353$: convergence is almost complete for $t > 25$. The regulated paths also converge to the socially optimal steady state, but they converge in large time, that is convergence is almost complete for $t > 2000$.[5] Thus the regulated paths converge to the socially optimal steady state much slower relative to the optimal path. The deviation between the socially optimal path and the regulated paths can be regarded as a measure of inefficiency of the specific regulatory scheme to attain the social optimum. This inefficiency is reduced as we move from fixed to quadratics taxes. It is to be expected that higher order tax schemes will reduce this inefficiency further.

4 Concluding Remarks

This paper considers regulation of common-pool resources characterized by complex dynamics. This requires solving for the feedback Nash equilibrium of a non-linear differential game under an appropriate tax scheme. The Hamilton-Jacobi-Bellman equation for this problem leads to a complicated ordinary differential equation that has to be solved. A numerical algorithm is developed for a fixed tax, a proportional tax and a quadratic tax, and the algorithm is applied to the model of the lake as a metaphor for this type of models. It is shown that these tax schemes induce convergence to the socially optimal steady state and that higher-order tax schemes lower the gap in social welfare between the regulated feedback Nash equilibrium and the socially optimal outcome. Further study of this problem could address more complicated tax schemes as well as the impact on the solution from increasing the number of agents n.

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[5] We calculate paths up to $t = 2500$. The maximum error of convergence of all three regulated paths, relative to the point of convergence $x^* = 0.353$, is of the order of $0.5\text{x}10^{-5}$.

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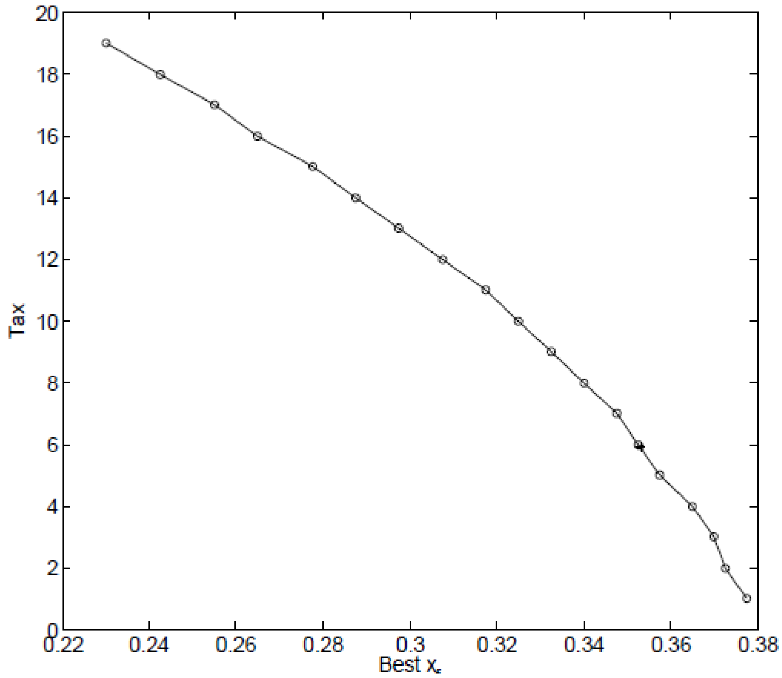

Figure 1: The inverse of the $x_f = \phi(\beta_0)$ relationship

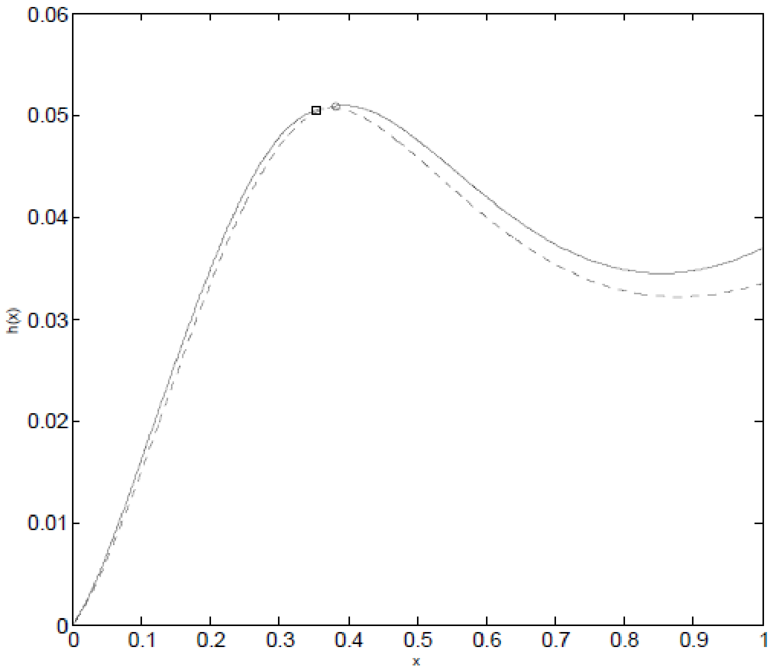

Figure 2: Feedback profiles, fixed tax

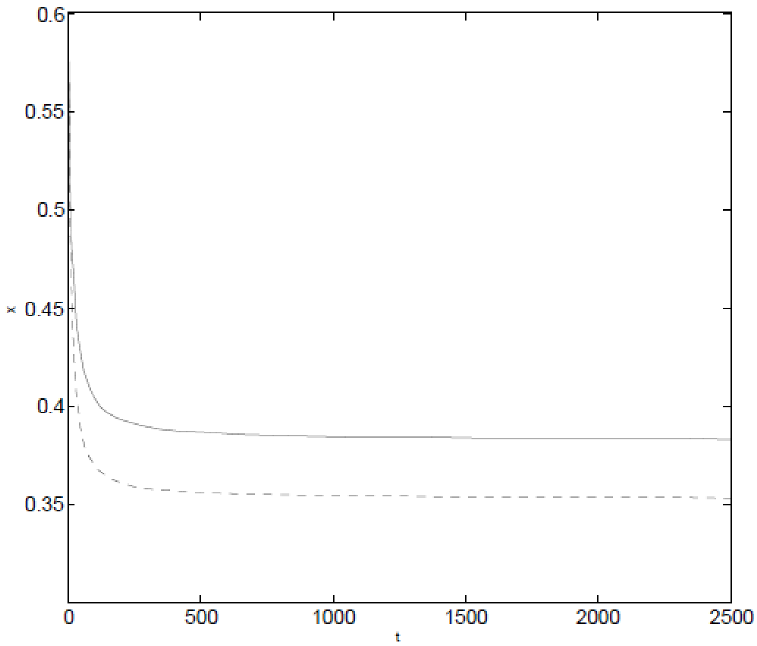

Figure 3: Phosphorous stock, fixed tax

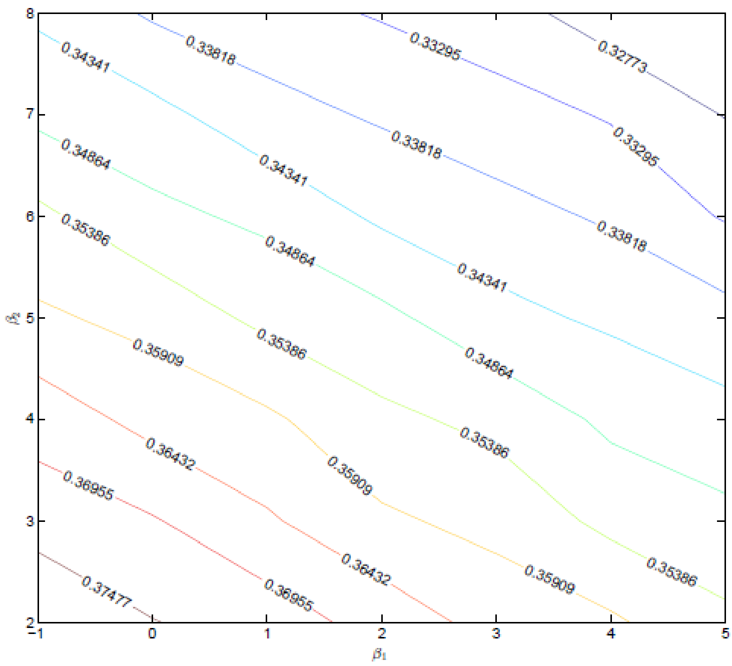

Figure 4: Steady-state contours (β_1, β_2)

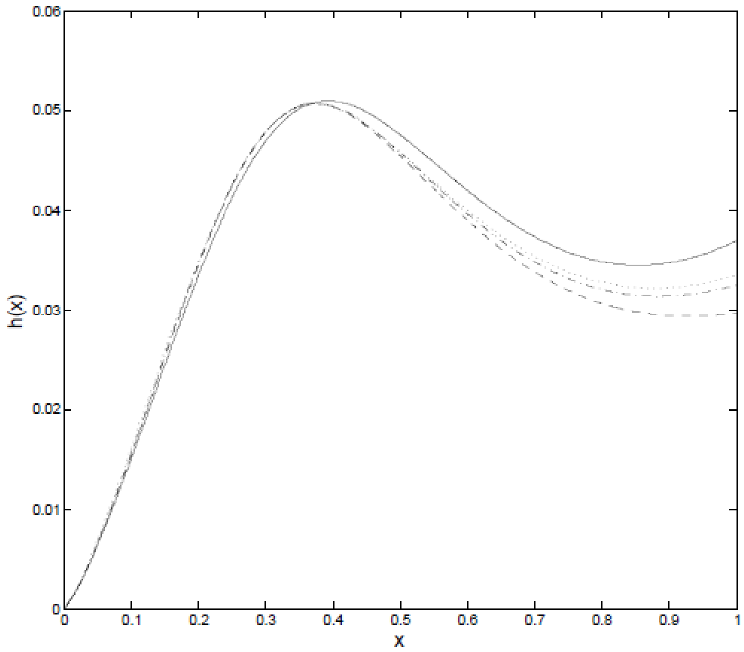

Figure 5: Feedback profiles

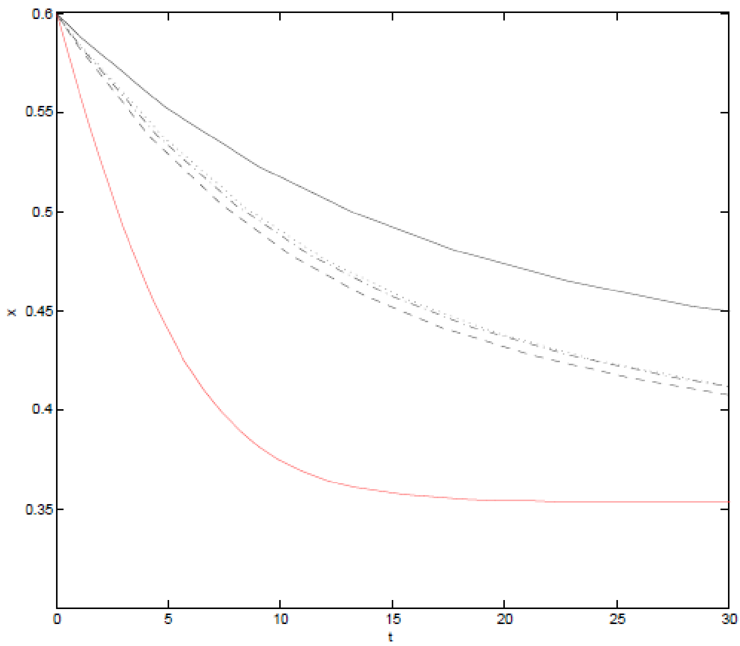

Figure 6: Phosphorous stock time paths