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# Managing Interacting Populations under Time Scale Separation

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**Abstract**

Renewable resource modelling is usually characterized by different time scales where some state variables such as biomass may evolve relatively faster than other state variables such as carrying capacity. Ignoring this time scale separation means that a slowly changing variable is treated as constant over time. Management rules that ignore time scale separation do not account for a time scale externality and this may induce inefficiencies in resource management. In the current work, we study multispecies resource management under time scale separation by adopting the framework of singular perturbation reduction methods. By extending recent work by Vardas and Xepapadeas (2015) to interacting populations, we study regulation with full internalization of the time scale externality. We further study regulation and noncooperative outcomes when the time scale separation is ignored, and identify deviations in harvesting and biomass paths

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among these cases. Deviations indicate the inefficiencies associated with ignoring time scale separation.

*Keywords*: interacting populations, resource harvesting, fast slow dynamics, singular perturbation, regulation, open loop Nash equilibrium.

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# 1 Introduction

The interaction of fast and slow processes and the separation of time scales is an integral part of ecosystem analysis (Gunderson and Pritchard, 2002; Walker et al., 2012; Levin et al., 2013). Time scale separation is taken into account in ecosystem modeling by including state variables evolving in slow and fast time scales, and appears in models of antagonistic species coevolution in which population (or biomass) dynamics interact with mutation (or trait) dynamics leading to the so-called Red Queen cycles. Modeling economic/ecological systems as fast/slow systems has also been associated with issues like biological resource management, water management and pest control (e.g. Milik et al., 1996; Brock and Xepapadeas, 2004; Grimsrud and Huffaker, 2006; Huffaker and Hotchkiss, 2006; Crepin et al., 2011).

In terms of the mathematical approach, dynamical systems evolving in a fast/slow time framework can be analyzed using singular perturbation methods (e.g., Wasow, 1965; Fenichel, 1979; Berglund and Gentz, 2003). In environmental and resource economics there have been a few attempts to study ecosystems in separate time scales. In particular Huffaker and Hotchkiss (2006) apply singular equations of motion to accommodate the disparate time scale and analyze the economic dynamics of reservoir sedimentation management using the hydrosuction-dredging sediment-removal system. Grimsrud and Huffaker (2006) apply singular perturbation methods in a bio-economic model to investigate the optimal management of pest resistance to pesticide crops and Rinaldi and Scheffer (2000) use a range of examples from natural and terrestrial ecosystems to study the effects of slow and fast variables to ecosystems. Crepin (2007) presents a general framework to handle systems with fast and slow variables, and illustrates the approach by using a model of coral reefs subject to fishing pressure. Crepin et al. (2011) explore how non-convexities and slow-fast dynamics affect coupled human-nature systems, while Milik et al., (1996), considering a simple model of demographic, economic and environmental interactions, illustrate the use of geometric singular perturbation theory in environmental economics.

When time scale separation exists with state variables evolving in different time scales, ignoring this separation and treating everything in the same time scale - the fast one - introduces an externality: the time scale externality. This is because when agents consider a slowly varying state variable (e.g., carrying capacity) as fixed, they ignore the impact of their actions on this state variable as well as on other interacting variables. However, the agents' actions affect, this state variable as well as the agents' utility or profits slowly, without been internalized. This is a source of externality which we will call time scale externality. It should be noticed that even if the agents' actions generate a well defined externality - such as emissions - which is regulated by conventional policy instruments (e.g., emissions taxes or tradable emission permits) but time-scale separation is ignored, then regulation is inefficient because it does not internalize all the external effects.

In the present paper we contribute to the discussion of optimal resource management under time scale separation by analyzing externalities emerging because of time-scale separation and potential inefficiencies of regulation related to harvesting rules, in the context of interacting populations. This extends earlier results of Vardas and Xepapadeas (2015) to a multispecies renewable resource harvesting model. In particular we study, by applying the singular perturbation reduction methods (Fenichel, 1979), optimal regulation when emissions cause environmental damages and at the same time cause a slowly varying carrying capacity of the interacting populations. We compare optimal regulation that accounts for time scale separation to: (i) regulation emerging under conventional modeling where the carrying capacity is regarded as fixed, and (ii) noncooperative outcomes associated with open loop Nash equilibrium where competing agents treat carrying capacity as fixed. By identifying deviations between optimal regulation and the other two cases, we obtain insights of the implications of ignoring the time scale externality. Furthermore we point out that ignoring time scale separation leads to time inconsistencies.

# 2 Optimal regulation when emissions cause a slowly varying carrying capacity

We consider the case of two renewable resources growing according to:

$$\begin{aligned} dx_1 &= \left[\rho_1 x_1 \left(1 - \frac{x_1}{K_1(S)} - a_{12}\frac{x_2}{K_1(S)}\right) - h_1\right] dt = f_1(x_1, x_2)dt \qquad (1) \\ dx_2 &= \left[\rho_2 x_2 \left(1 - \frac{x_2}{K_2(S)} - a_{21}\frac{x_1}{K_2(S)}\right) - h_2\right] dt = f_2(x_1, x_2)dt, \\ & x_1(0) = x_{10}, x_2(0) = x_{20}, \end{aligned}$$

where $x_i, i = 1, 2$ are the biomasses of resources and $K_i$ refers to carrying capacity which can be either constant or a function of some other state variable such as the stock of pollution $S$ that affects the carrying capacity, that is $K_i(S)$. In (1) $\rho_m, m = 1, 2$ denote the intrinsic growth rates of the two biomasses and $h_m = \sum_{j=1}^{J} h_{mj}$ denotes total harvesting of the two resources undertaken by a finite number of agents. We assume that harvesting can be expressed in terms of a generalized production function as a function of biomass and effort, or

$$\begin{aligned} h_{mj} &= q_m x_m^{\alpha} E_{mj}^{\beta} \ , \alpha > 0, 0 < \beta < 1. \\ \mathbf{E}_m &= (E_{m1}, ..., E_{mJ}) \, , m = 1, 2, , j = 1, ..J \\ E &= (\mathbf{E}_1, \mathbf{E}_2)^T, \end{aligned}$$

where $q_m$ is the catchability coefficient for the $m^{\text{th}}$ biomass and $E_{mj}$ is the fishing effort of the $j^{\text{th}}$ agent, in harvesting the two biomasses. Finally with $a_{12}, a_{21}$ we denote the interaction coefficients between the two populations. When both of them are positive, then there is competition between the two resources. When one is positive and the other is negative we are in a prey-predator context.

The model can extended to $M$ interacting populations with biomasses $x_m$, $m = 1, 2, .., M$. Denoting the interaction coefficients between the $\mu^{\text{th}}$, $\nu^{\text{th}}$ populations with $a_{\mu\nu}$ and the intrinsic growth rates $m = 1, .., \mu, \nu, ...M$ with $\rho_m$, population dynamics (1) can be written in the general case as:

$$
\begin{aligned}
dx &= (\Xi - H)dt \qquad (2)\\
dx &= \begin{bmatrix} dx_1 \\ .. \\ dx_M \end{bmatrix}, H = \begin{bmatrix} h_1 \\ ... \\ h_M \end{bmatrix}, h_m = \sum_{j=1}^{J} h_{mj}, m = 1, .., M \\
\Xi &= \begin{bmatrix} \xi_1 \\ ..\xi_m.. \\ \xi_M \end{bmatrix}, \xi_m = \rho_m x_m \left( 1 - \sum_{\mu=1}^{M} a_{m\mu} \frac{x_\mu}{K_m(S)} \right), \\
a_{mm} &= 1, \sum_{\mu=1, m\neq\mu}^{M} a_{m\mu} < 1.
\end{aligned}
$$

At this point we introduce a link between emissions and the evolution of carrying capacity following Vardas and Xepapadeas (2015) by assuming that the non-fishing sector of the economy generates emissions through production processes. Emissions are generated by a finite number of homogeneous agents $i = 1, ..I$, and generate benefits according to a strictly concave benefit function

$$B_i\left(s_i\right), B_i' \geq 0, B_i'' < 0,$$

with aggregate emissions defined by

$$s = \sum_{i=1}^{I} s_i.$$

Emissions accumulate into the ambient environment to form a stock, according to:

$$S' = \varphi s - lS, \ \varphi > 0, l > 0, \ S\left(0\right) = S_0 > 0, S\left(t\right) \geq 0.$$

Overall carrying capacity $K_o$, as well as individual carrying capacities, depend on the emissions' stock. We assume that the stock of emissions affects the carrying capacity according to:

$$
\begin{aligned}
K_o(t) &= \omega(S(t)) = A - \theta S(t)\,,\ \theta > 0, K_o(t) \geq 0 && (3)\\
K_o' &= -\varepsilon S' = -\varepsilon(\varphi s - lS) && (4)
\end{aligned}
$$

Assuming that $\varepsilon$ is small, we consider a situation where the evolution of the pollutant's stock in the ambient environment induces a slow evolution of the carrying capacity $K$. In this case we have time-scale separation between the fast resource and pollution dynamics, and the slow dynamics of carrying capacity. If $\varepsilon = 0$, then carrying capacity is fixed and does not respond to changes in pollution stock. If $\varepsilon$ is small but is ignored, that is we take $\varepsilon \to 0$, the carrying capacity is treated as fixed, while in reality it is slowly changing in response to changes in the pollution stock. This is the source of the time-scale externality.

Using (3) to solve for $S$ and replacing in (4), we obtain

$$K_o' = -\varepsilon\left(\varphi s - l\left(\frac{A - K_o}{\theta}\right)\right).$$

Defining $\gamma$ so that $l = \gamma\theta$ we obtain

$$K_o' = \varepsilon\left(\gamma(A - K_0) - \varphi\sum_{i=1}^{I} s_i\right).$$

In this case the dynamical system can written in slow time as:

$$
\begin{aligned}
\varepsilon x'(\tau) &= \Xi - H && (5)\\
K_o'(\tau) &= \gamma(A - K_o(\tau)) - \varphi\sum_{i=1}^{I} s_i(\tau)\,,\ \gamma = \frac{l}{\theta} > 0, && (6)\\
K_o(0) &= A - \varepsilon S(0) = K_0 > 0,
\end{aligned}
$$

or in fast time as:

$$
\begin{aligned}
x^{'}(t) &= \Xi - H \\
K_o'(t) &= \varepsilon\left(\gamma\left(A - K_o(t)\right) - \varphi\sum_{i=1}^{I} s_i(t)\right).
\end{aligned} \tag{7}
$$

## 2.1 The problem of the regulator

Given the dynamics (5)-(6) we can define the regulator's problem, in slow time,[1] as the problem of choosing harvesting effort and emission paths to maximize discounted aggregate benefits from harvesting and emissions net of environmental damages associated with the ambient pollutant stock, or

$$
\begin{gathered}
\max_{\mathbf{E},\mathbf{s}} \int_0^{\infty} \mathrm{e}^{-\delta\tau}\pi d\tau \\
\varepsilon x^{'}(\tau) = \Xi - H \\
K_o'(\tau) = \gamma\left(A - K_o(\tau)\right) - \varphi\sum_{i=1}^{I} s_i(\tau),\ \gamma = \frac{l}{\varepsilon} > 0, \\
\pi = \sum_{j=1}^{J}\pi_j(x, E_j) + \sum_{i=1}^{I} B_i(s_i) - D\left(\sum_{i=1}^{I} s_i\right) \\
\pi_j(x, E_j) = \sum_{m=1}^{M}\pi_m(x, E_{mj}) \\
\pi_m(x, E_{mj}) = p_m q_m x_m^{\alpha} E_{mj}^{\beta} - w_m E_{mj},
\end{gathered} \tag{8}
$$

where $\tau = t/\varepsilon$, with $\varepsilon$ being a small positive parameter. Then, denoting by $\lambda = \begin{bmatrix}\lambda_1 & ... & \lambda_M\end{bmatrix}$ the vector of the costate variables associated with each one of the $M$ biomasses, the Hamiltonian $\mathbf{H}$ takes the form:

$$
\begin{aligned}
\mathbf{H} &= \lambda(\Xi - H) + \mu G \\
G &= \gamma\left(A - K_0(\tau)\right) - \varphi\sum_{i=1}^{I} s_i(\tau),
\end{aligned}
$$

[1] We denote with $\delta$ the discount rate in slow time, i.e., the ten year discount rate.

and results in the following optimality conditions:

$$\pi_{E_{mJ}} + \lambda_1 f_{1_{E_{mJ}}} + \lambda_2 f_{2_{E_{mJ}}} + \mu G_{E_{mJ}} = 0, j = 1.., J, m = 1, 2 \tag{9}$$

$$B_i'(s_i) - D'\left(\sum_{i=1}^{I} s_i\right) - \mu\varphi = 0, i = 1.., I$$

$$\varepsilon \begin{bmatrix} \lambda_1' - \delta\lambda_1 \\ \ldots \\ \lambda_M' - \delta\lambda_M \end{bmatrix} + \begin{bmatrix} (\pi + \lambda(\Xi - H) + \mu G)_{x_1} \\ \ldots. \\ (\pi + \lambda(\Xi - H) + \mu G)_{x_M} \end{bmatrix} = 0$$

$$\mu' - \delta\mu + \pi_K + (\lambda(\Xi - H))_K + \mu G_K = 0$$

$$\varepsilon x' = \Xi - H$$

$$K_0' = \gamma (A - K_0) - \varphi \sum_{i=1}^{I} s_i$$

$$\pi = \sum_{j=1}^{J} \pi_j (x, E_j) + \sum_{i=1}^{I} B_i (s_i) - D\left(\sum_{i=1}^{I} s_i\right). \tag{10}$$

## 2.2 Optimal regulation of two competing resources

In order to obtain more tractable results we study the case of harvesting two competing resources. More specifically, without loss of generality we assume that the two carrying capacities $K_1(S), K_2(S)$ are equal and denote them by $K(S)$ or - to simplify notation - $K$, which implies that the overall carrying capacity is $K_o = 2K$. Then the evolution of the two biomasses is given by:

$$\begin{aligned} dx_1 &= \left[\rho_1 x_1 \left(1 - \frac{x_1}{K} - a_{12}\frac{x_2}{K}\right) - h_1\right] dt,\ x_1(0) = x_{10} \\ dx_2 &= \left[\rho_2 x_2 \left(1 - \frac{x_2}{K} - a_{21}\frac{x_1}{K}\right) - h_2\right] dt,\ x_2(0) = x_{20}. \end{aligned} \tag{11}$$

The application of (9) provides the following optimality conditions:

$$p_m \beta q_m x_m^\alpha {E_{mj}}^{\beta-1} - w - \lambda_m \beta q_m x_m^\alpha {E_{mj}}^{\beta-1} = 0, j = 1.., J, m = 1, 2 \qquad (12)$$

$$B_i^{'}(s_i) - D'\left(\sum_{i=1}^{I} s_i\right) - \mu\varphi = 0, i = 1.., I$$

$$\varepsilon(\lambda_1' - \delta\lambda_1) + (p_1 - \lambda_1)\alpha q_1 x_1^{\alpha-1} \textstyle\sum_{j=1}^{J} {E_{1j}}^{\beta} +$$
$$\lambda_1\rho_1(1 - (2x_1 + a_{12}x_2)/K) - \lambda_2\rho_2 x_2 a_{21}/K = 0$$

$$\varepsilon(\lambda_2' - \delta\lambda_2) + (p_2 - \lambda_2)\alpha q_2 x_2^{\alpha-1} \textstyle\sum_{j=1}^{J} {E_{2j}}^{\beta} +$$
$$\lambda_2\rho_2(1 - (2x_2 + a_{21}x_1)/K) - \lambda_1\rho_1 x_1 a_{12}/K = 0$$

$$\mu' - \delta\mu + \lambda_1\rho_1 x_1 (x_1 + a_{12}x_2)/K^2$$
$$+\lambda_2\rho_2 x_2 (x_2 + a_{21}x_1)/K^2 - \mu\gamma = 0$$

$$\varepsilon x_1' = \rho_1 x_1 \left(1 - \tfrac{x_1}{K} - a_{12}\tfrac{x_2}{K}\right) - h_1$$
$$\varepsilon x_2' = \rho_2 x_2 \left(1 - \tfrac{x_2}{K} - a_{21}\tfrac{x_1}{K}\right) - h_2$$

$$K_0' = \gamma(A - K_0) - \varphi\sum_{i=1}^{I} s_i.$$

In order to improve tractability we assume without loss of generality that $B_i(s_i) = \sqrt{s_i}$, $i = 1, 2$ and that damages can be modeled by a quadratic damage function. Furthermore we assume that $\alpha = \beta = 1/2$ and $\varphi = 2$. System (12) consists of nine equations. The first three of them are algebraic equations from which we can solve for the control variables of our problem. Thus we obtain:[2]

$$E_{mj} = \left(\frac{(p_m - \lambda_m)\beta_m q x_m^\alpha}{w_m}\right)^{\frac{1}{1-\beta}}, E_{mj}^\beta = \frac{(p_m - \lambda_m)\beta q_m x_m^\alpha}{w_m}, m = 1, 2 \qquad (13)$$

$$s_i : 1/2\sqrt{s_i} - 2\sum_{i=1}^{I} s_i - \mu\varphi = 0.$$

The system of the remaining six equations is a system with fast and slowly evolving variables. In particular we obtain the following set of equations

[2] We can obtain similar results by assuming a linear damage function of the form $D(\cdot) = (\cdot)$. Then $\mathrm{s}_i : 1/2\sqrt{s_i} - 1 - \mu\varphi = 0, , i = 1.., I$. Here we present the results corresponding to a quadratic damage function.

which characterizes the evolution along an optimal path of biomasses, carrying capacities, and their corresponding shadow values:

$$\begin{aligned} &\varepsilon\left(\lambda_1' - \delta\lambda_1\right) + (p_1 - \lambda_1)\alpha q_1 x_1^{\alpha-1} \textstyle\sum_{j=1}^J E_{1j}{}^{\beta} + \\ &\lambda_1\rho_1\left(1 - (2x_1 + a_{12}x_2)/K\right) - \lambda_2\rho_2 x_2 a_{21}/K = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} &\varepsilon\left(\lambda_2' - \delta\lambda_2\right) + (p_2 - \lambda_2)\alpha q_2 x_2^{\alpha-1} \textstyle\sum_{j=1}^J E_{2j}{}^{\beta} + \\ &\lambda_2\rho_2\left(1 - (2x_2 + a_{21}x_1)/K\right) - \lambda_1\rho_1 x_1 a_{12}/K = 0 \\ &\mu' - \delta\mu + \lambda_1\rho_1 x_1\left(x_1 + a_{12}x_2\right)/K^2 \\ &+\lambda_2\rho_2 x_2\left(x_2 + a_{21}x_1\right)/K^2 - \mu\gamma = 0 \\ &\varepsilon x_1' = \rho_1 x_1\left(1 - \tfrac{x_1}{K} - a_{12}\tfrac{x_2}{K}\right) - h_1 \\ &\varepsilon x_2' = \rho_2 x_2\left(1 - \tfrac{x_2}{K} - a_{21}\tfrac{x_1}{K}\right) - h_2 \\ &K_o' = \gamma\left(A - K_o\right) - \varphi\sum_{i=1}^{I} s_i, \end{aligned} \tag{15}$$

in slow time $\tau$. The above system is called "the slow system." Rescaling with $\tau = \varepsilon t$, we obtain the so-called "fast system"

$$\begin{aligned} &\left(\lambda_1' - \varepsilon\delta\lambda_1\right) + (p_1 - \lambda_1)\alpha q_1 x_1^{\alpha-1} \textstyle\sum_{j=1}^J E_{1j}{}^{\beta} + \\ &\lambda_1\rho_1\left(1 - (2x_1 + a_{12}x_2)/K\right) - \lambda_2\rho_2 x_2 a_{21}/K = 0 \end{aligned} \tag{16}$$

$$\begin{aligned} &\left(\lambda_2' - \varepsilon\delta\lambda_2\right) + (p_2 - \lambda_2)\alpha q_2 x_2^{\alpha-1} \textstyle\sum_{j=1}^J E_{2j}{}^{\beta} + \\ &\lambda_2\rho_2\left(1 - (2x_2 + a_{21}x_1)/K\right) - \lambda_1\rho_1 x_1 a_{12}/K = 0 \\ &\mu' + \varepsilon\{-\delta\mu + \lambda_1\rho_1 x_1\left(x_1 + a_{12}x_2\right)/K^2 \\ &+\lambda_2\rho_2 x_2\left(x_2 + a_{21}x_1\right)/K^2 - \mu\gamma\} = 0 \\ &x_1' = \rho_1 x_1\left(1 - \tfrac{x_1}{K} - a_{12}\tfrac{x_2}{K}\right) - h_1 \\ &x_2' = \rho_2 x_2\left(1 - \tfrac{x_2}{K} - a_{21}\tfrac{x_1}{K}\right) - h_2 \\ &K_o' = \varepsilon\left[\gamma\left(A - K_o\right) - \varphi\sum_{i=1}^{I} s_i\right], \end{aligned} \tag{17}$$

where in the second case, derivatives are evaluated with respect to fast evolving time $t$.

If we consider (17) we can obtain the dynamics of individual carrying capacities as:

$$K' = \varepsilon \left[ \gamma \left( A/2 - K \right) - \left( \varphi/2 \right) \sum_{i=1}^{I} s_i \right].$$

Systems ((14)-(15), (16 )-(17)), can be rewritten in a matrix notation as:

$$Fast \quad \begin{array}{c} dX/dt = \mathbf{F}(X,\mathbf{K},\varepsilon) \\ d\mathbf{K}/dt = \varepsilon\mathbf{G}(X,\mathbf{K},\varepsilon) \end{array}, \quad Slow \quad \begin{array}{c} \varepsilon dX/d\tau = \mathbf{F}(X,\mathbf{K},\varepsilon) \\ d\mathbf{K}/d\tau = \mathbf{G}(X,\mathbf{K},\varepsilon) \end{array}, \tag{18}$$

with $X = (\lambda_1, \lambda_2, x_1, x_2)^T, \mathbf{K} = (\mu, K_o)^T$ the vectors of fast and slow variables respectively. Furthermore $\mathbf{F} = (F_1, F_2, F_3, F_4)^T$ and $\mathbf{G} = (G_1, G_2)^T$, with

$$\begin{aligned} F_1 &= \begin{array}{c} \varepsilon\delta\lambda_1 - (p_1 - \lambda_1)\alpha q_1 x_1^{\alpha-1} \sum_{j=1}^{J} E_{1j}{}^{\beta} - \\ \lambda_1\rho_1 \left(1 - (2x_1 + a_{12}x_2)/K\right) + \lambda_2\rho_2 x_2 a_{21}/K \end{array} \qquad (19) \\ F_2 &= \begin{array}{c} \varepsilon\delta\lambda_2 - (p_2 - \lambda_2)\alpha q_2 x_2^{\alpha-1} \sum_{j=1}^{J} E_{2j}{}^{\beta} - \\ \lambda_2\rho_2 \left(1 - (2x_2 + a_{21}x_1)/K\right) + \lambda_1\rho_1 x_1 a_{12}/K \end{array} \\ F_3 &= \rho_1 x_1 \left(1 - \frac{x_1}{K} - a_{12}\frac{x_2}{K}\right) - h_1 \\ F_4 &= \rho_2 x_2 \left(1 - \frac{x_2}{K} - a_{21}\frac{x_1}{K}\right) - h_2 \\ G_1 &= \delta\mu - \lambda_1\rho_1 x_1 \left(x_1 + a_{12}x_2\right)/K^2 - \lambda_2\rho_2 x_2 \left(x_2 + a_{21}x_1\right)/K^2 + \mu\gamma \\ G_2 &= \gamma \left(A_2 - K\right) - \varphi_2 \sum_{i=1}^{I} s_i, with, A_2 = A/2, \varphi_2 = \varphi/2, \end{aligned}$$

where $E_{mj}^{\beta}$ and $s_i$ are given in (13). By setting $\varepsilon = 0$ in the fast system we define the *layer problem,* while by setting $\varepsilon = 0$ in the slow system we define the *reduced problem.*

### 2.2.1 Slow Manifolds and Optimal Regulation

To approximate the "slow manifolds" which characterize the solution of our management problem, we apply Fenichel's invariant manifold theorem (Fenichel 1979). The application of this theorem requires three conditions. The first is related to the requirement that the functions $\mathbf{F}, \mathbf{G}$ be continuous. This requirement is satisfied. The second one is related to the reduced problem and requires the existence of functions of the form:

$$\mathbf{X} = \mathbf{H}^o(\mathbf{K}) = [H_1^o(\mathbf{K}), H_2^o(\mathbf{K}), H_3^o(\mathbf{K}), H_4^o(\mathbf{K})]$$

such that $\mathbf{F}(\mathbf{H}^o(\mathbf{K}), \mathbf{K}, \varepsilon = 0) = 0$, that is the fast evolving variables can be solved as functions of the slow variables. In particular, taking into account relationship (13) which gives the effort rate and manipulating, we obtain:

$$\begin{gathered}(p_1 - \lambda_1)^2 q_1^2 \alpha\beta J w_1^{-1} + \\ \lambda_1\rho_1 \left(1 - (2x_1 + a_{12}x_2)/K\right) - \lambda_2\rho_2 x_2 a_{21}/K = 0\end{gathered} \tag{20}$$

$$\begin{gathered}(p_2 - \lambda_2)^2 \alpha\beta q_2^2 J w_2^{-1} + \\ \lambda_2\rho_2 \left(1 - (2x_2 + a_{21}x_1)/K\right) - \lambda_1\rho_1 x_1 a_{12}/K = 0\end{gathered} \tag{21}$$

$$1 - x_1/K - a_{12}x_2/K - J\frac{(p_1 - \lambda_1)\beta q_1^2}{\rho_1 w_1} = 0 \tag{22}$$

$$1 - x_2/K - a_{21}x_1/K - J\frac{(p_2 - \lambda_2)\beta q_2^2}{\rho_2 w_2} = 0. \tag{23}$$

Equations (22), (23), can be considered as a system on $x_1/K, x_2/K$ , which can be solved as function of the two costate variables. Then replacing $x_1/K, x_2/K$ into (20), (21), we obtain a system of two equations with the $\lambda_1, \lambda_2$ being unknown variables. Solving and replacing back into (22), (23), we obtain $x_1, x_2$, as functions of $K$. In particular we obtain, by solving the linear system of (22)-(23),

$$\begin{aligned} x_1/K + a_{12}x_2/K &= 1 - J\frac{(p_1 - \lambda_1)\beta q_1^2}{\rho_1 w_1} \\ a_{21}x_1/K + x_2/K &= 1 - J\frac{(p_2 - \lambda_2)\beta q_2^2}{\rho_2 w_2} \end{aligned}$$

$$\begin{aligned} x_1/K &= \frac{1 - a_{12} - J\frac{(p_1-\lambda_1)\beta q_1^2}{\rho_1 w_1} + a_{12}J\frac{(p_2-\lambda_2)\beta q_2^2}{\rho_2 w_2}}{1 - a_{12}a_{21}} \\ x_2/K &= \frac{1 - J\frac{(p_2-\lambda_2)\beta q_2^2}{\rho_2 w_2} - a_{21} + a_{21}J\frac{(p_1-\lambda_1)\beta q_1^2}{\rho_1 w_1}}{1 - a_{12}a_{21}}. \end{aligned} \tag{24}$$

To obtain a clear picture of the results, given the complexity of the problem, we resort to numerical simulations. Replacing into (20) and (21) the following parameter setting (see Da Rocha et al., 2014; Vardas and Xepapadeas, 2015):

$$\begin{aligned}\beta &= \alpha = 1/2, J = 2, p_1 = p_2 = 10, w_1 = w_2 = 5, q_1 = 0,048, q_2 = 0,042, \\ \rho_1 &= 0.45, \rho_2 = 0.35, r = 0,05, a_{12} = a_{12} = 0.3,\end{aligned}$$

we obtain, using (24), the following solutions:[3]

$$\begin{aligned}sol_1 &= (x_1, x_2, \lambda_1, \lambda_2)_1 = (0.315786K, 0.707977K, -367.664, 94.785) \quad (25)\\ sol_2 &= (x_1, x_2, \lambda_1, \lambda_2)_2 = (0.33198K, 0.309629K, -249.870, 40.6759)\\ sol_3 &= (x_1, x_2, \lambda_1, \lambda_2)_3 = (0.75375K, 0.755806K, 0.05816, 0.0447838)\\ sol_4 &= (x_1, x_2, \lambda_1, \lambda_2)_4 = (0.697341K, 0.278107K, 75.49248, -522.37705).\end{aligned}$$

Finally accordingly to the third condition, we want the real parts of the eigenvalues of the Jacobian matrix $J = \frac{\partial F}{\partial \mathbf{X}}(\mathbf{H}^o(\mathbf{K}), \mathbf{K}, \varepsilon = 0)$ to be nonzero. Negative real parts induce an attracting manifold while if there is at least one positive real part the manifold is repelling. In our case the matrix $J$ is given by[4]

$$J = \frac{\partial F}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial \mathbf{x}_1} & \frac{\partial F_1}{\partial \mathbf{x}_2} & \frac{\partial F_1}{\partial \lambda_1} & \frac{\partial F_1}{\partial \lambda_2} \\ \frac{\partial F_2}{\partial \mathbf{x}_1} & \frac{\partial F_2}{\partial \mathbf{x}_2} & \frac{\partial F_2}{\partial \lambda_1} & \frac{\partial F_2}{\partial \lambda_2} \\ \frac{\partial F_3}{\partial \mathbf{x}_1} & \frac{\partial F_3}{\partial \mathbf{x}_2} & \frac{\partial F_3}{\partial \lambda_1} & \frac{\partial F_3}{\partial \lambda_2} \\ \frac{\partial F_4}{\partial \mathbf{x}_1} & \frac{\partial F_4}{\partial \mathbf{x}_2} & \frac{\partial F_4}{\partial \lambda_1} & \frac{\partial F_4}{\partial \lambda_2} \end{bmatrix}.$$

Using our parametrization we obtain the eigenvalues associated with each of the $sol_i$ $i = 1, 2, 3, 4$. It turns out that all eigenvalues have no zero real parts.

To select the slow manifold that the regulator is seeking to direct the controlled system, we use Dockner's (1985) Theorem 2 (iii) to identify the point with the property of conditional stability (saddle point property, for

[3] This approach can be extended in a straightforward way to the case of $M$ resources. In this case we can solve for $x_m/K, m = 1, .., M$.

[4] Seethe Appendix for an analytic description of the matrix $J$.

$\varepsilon = 0$). Following this criterion only $sol_4$ is acceptable with all other solutions being completely unstable.[5]

Then Fenichel's theorem extends the analysis for an arbitrary small parameter $\boldsymbol{\varepsilon}$ and provides a slow manifold $M_\varepsilon = \{(X, \mathbf{K}) \in R^6 : X = (\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}), K, \boldsymbol{\varepsilon})\}$ such that:

$$d\mathbf{K}/d\tau = \mathbf{G}(\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}), K, \boldsymbol{\varepsilon}),$$

where the vector $\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}) = \mathbf{H}^{\mathbf{0}}(\mathbf{K}) + \boldsymbol{\varepsilon}\mathbf{H}^{(\mathbf{1})}(\mathbf{K}) + ...$ as $\boldsymbol{\varepsilon} \to 0$, with

$$\begin{aligned} \mathbf{H}^0(\mathbf{K}) &= \mathbf{H}^0(\mathbf{K}) \\ \mathbf{H}^{(1)}(\mathbf{K}) &= [\frac{\partial \mathbf{F}}{\partial \mathbf{X}}]^{-1}[\frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} G^* - \frac{\partial \mathbf{F}}{\partial \boldsymbol{\varepsilon}}] \\ \frac{\partial \mathbf{F}}{\partial \boldsymbol{\varepsilon}} &= [\frac{\partial F_1}{\partial \boldsymbol{\varepsilon}}, \frac{\partial F_2}{\partial \boldsymbol{\varepsilon}}, \frac{\partial F_3}{\partial \boldsymbol{\varepsilon}}, \frac{\partial F_4}{\partial \boldsymbol{\varepsilon}}]^T = [\delta\lambda_1, \delta\lambda_2, 0, 0]^T \\ \frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} &= \begin{bmatrix} \frac{\partial H_1^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_1^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_2^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_2^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_3^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_3^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_4^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_4^o(\mathbf{K})}{\partial K} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial H_1^o(\mathbf{K})}{\partial K} \\ 0 & \frac{\partial H_2^o(\mathbf{K})}{\partial K} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{26}$$

Manipulating $sol_4$ and using the following setting for the parameters[6]

$$\begin{aligned} \beta &= \alpha = 1/2, J = 2, p_1 = p_2 = 10, w_1 = w_2 = 5, q_1 = 0,048, q_2 = 0,042, \\ \rho_1 &= 0.45, \rho_2 = 0.35, r = 0,05, a_{12} = a_{12} = 0.3, I = 2 \\ D(\cdot) &= (\cdot)^2, \delta = 0.05, \varphi = 0.1, A = 50, \varepsilon = 0.04, l = 0.4, \gamma = l/\varepsilon = 10, \end{aligned}$$

we obtain that the dynamics of the slow variables on $M_\varepsilon$, for $\varepsilon = 0.04$, are

[5] If we calculate the corresponding eigenvalues of the Jacobian matrix $J$, for all $K$ in the range between 22 and 26, $sol_4$ is the only one with two eigenvalues with positive and two eigenvalues with negative real parts. Furthermore all four eigenvalues are real numbers. For example for $K = 24$ and $K = 25$ we obtain

$$\begin{gathered} \{6.30289, -5.32471, -0.0079588, 0.00100309\} \\ \{6.05072, -5.11102, -0.00829154, 0.0010449\}. \end{gathered}$$

We use the above range of values for $K$ because it includes the optimal solutions for $K$.

[6] See Appendix for an analytic derivation of the the dynamics of the slow variables on $M$ and the associated steady state.

given by:

$$\begin{bmatrix} d\mu/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\mu - \lambda_1^\varepsilon \rho_1 x_1^\varepsilon \left(x_1^\varepsilon + a_{12} x_2^\varepsilon\right)/K^2 - \\ -\lambda_2^\varepsilon \rho_2 x_2^\varepsilon \left(x_2^\varepsilon + a_{21} x_{1,}^\varepsilon\right)/K^2 + \mu\gamma \\ \gamma\left(A_2 - K\right) - \varphi_2 \sum_{i=1}^I s_i \end{bmatrix}. \tag{27}$$

The steady state $(K, \mu)$ can be obtained as:

$$\begin{aligned} \begin{bmatrix} 0 \\ K \end{bmatrix} &= \begin{bmatrix} \delta\mu - \lambda_1^\varepsilon \rho_1 x_1^\varepsilon \left(x_1^\varepsilon + a_{12} x_2^\varepsilon\right)/K^2 - \lambda_2^\varepsilon \rho_2 x_2^\varepsilon \left(x_2^\varepsilon + a_{21} x_{1,}^\varepsilon\right)/K^2 + \mu\gamma \\ A_2 - \frac{\varphi_2}{\gamma} \sum_{i=1}^I s_i \end{bmatrix} \\ \begin{bmatrix} \mu \\ K \end{bmatrix} &= \begin{bmatrix} -0.64470 \\ 24.9001 \end{bmatrix}, K_o = 2K = 49.8002 \\ s_i &: \quad 1/2\sqrt{s_i} - 2\sum_{i=1}^I s_i - \mu\varphi = 0, \longrightarrow s_i = 0.49926. \end{aligned} \tag{28}$$

The Jacobian matrix $J$ associated with system (27) is the following $2 \times 2$ matrix[7]

$$J = \begin{bmatrix} (\delta + \gamma) & 0 \\ (\gamma\left(A_2 - K\right) - \varphi_2 \sum_{i=1}^I s_i)\mu & -\gamma \end{bmatrix}$$

with determinant equal to $-(\gamma + \delta)\gamma < 0$, and trace $\delta > 0$. Thus the steady state is a saddle point.

Assume that the initial carrying capacity of each resource is $K(0) = 20$. Then applying a shooting method from the initial state $(-1.2933, 20)$, we obtain a good approximation for convergence at the steady state $(\mu, K) = (-0.64470, 24.8448)$ and $s_i = 0.49926$, along the stable manifold of system (27). Convergence is obtained at $\tau = 0.5$. The corresponding initial value for $s_i$ is $s_i(0) = 0.787$ and the solution paths are shown in Figure 1. This result means that if the regulator sets initial emissions at $s_i(0) = 0.787$ then the optimal paths for emissions and harvesting are the paths shown in Figure 1. The optimal paths for harvesting and emissions can then be implemented using taxes or quotas on harvesting and emissions. This would be the optimal regulatory scheme that internalizes the time scale externality by taking into

[7] Taking into account (45) and (27) we can see that $K$ disappears after some manipulations.

account time scale separation.[8] Figure (1) also depicts the optimal paths of the two biomasses $x_1, x_2$, for which the optimal steady-state biomass values are $(x_1, x_2) = (17.3253, 6.9095)$. These are the paths that will be attained if the optimal regulatory scheme is applied.

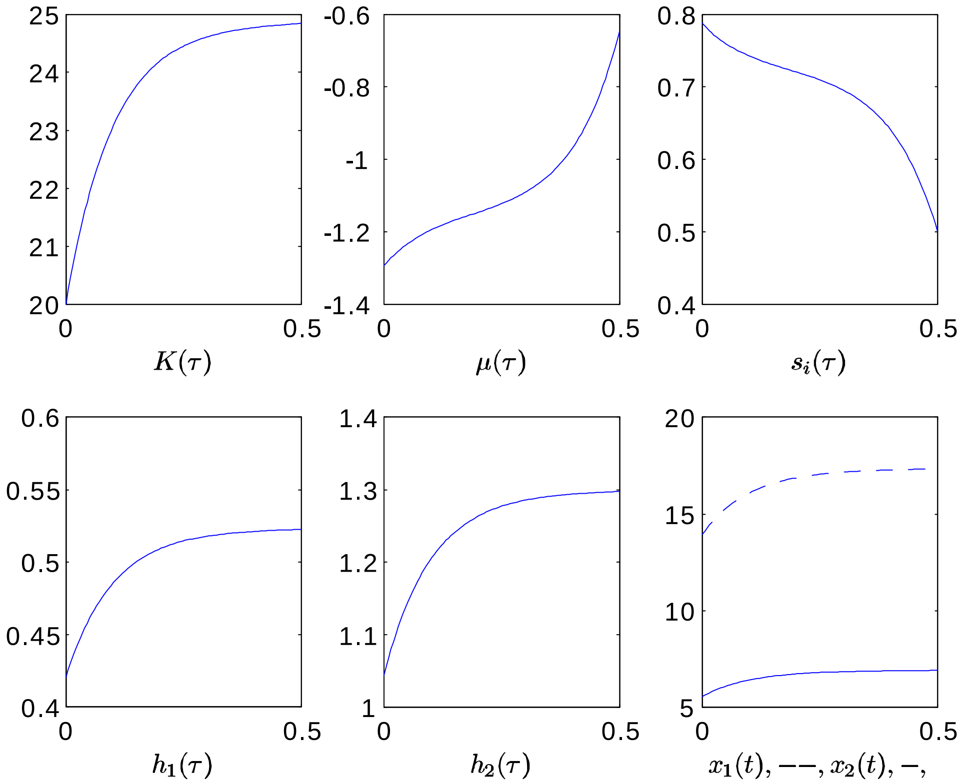


Figure 1: Optimal regulation

# 3 Ignoring the Time Scale Externality

The previous section characterized optimal regulation of resource harvesting and emissions when the regulator takes into account the time scale externality. However the most commonly followed approach when time scale

[8]In the Appendix we present the optimal paths for the case of a linear damage function.

separation exists, is for the regulator or the agents competing for resource harvesting to ignore it, and treat slowly evolving variables as fixed parameters. In this section we explore the implications of ignoring the time scale externality on the actions of the regulator or the competing agents.

## 3.1 Regulation when the time scale externality is ignored

In order to study the impact on regulation when the time scale externality is ignored, we assume a constant carrying capacity $\bar{K} = K_1(S) = K_2(S)$ which is not affected by the pollution stock. In this case the dynamics of the two competing resources become

$$\begin{aligned} dx_1 &= \left[\rho_1 x_1 \left(1 - \frac{x_1}{\bar{K}} - a_{12}\frac{x_2}{\bar{K}}\right) - h_1\right] dt = f_1(x_1, x_2)dt, x_1(0) = x_{10} \quad (29) \\ dx_2 &= \left[\rho_2 x_2 \left(1 - \frac{x_2}{\bar{K}} - a_{21}\frac{x_1}{\bar{K}}\right) - h_2\right] dt = f_2(x_1, x_2)dt, x_2(0) = x_{20}. \end{aligned}$$

Thus we can define in fast time the following resource management problem:

$$\max_E \int_0^\infty \mathrm{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j(x, E_j) = \sum_{m=1}^{2}\sum_{j=1}^{J} \pi(x, E_{mj}) = \pi(x, E)\right] dt \qquad (30)$$

$$\text{subject to , (29)}$$

$$\pi(x, E_{m_J}) = p_m q_m x_m^\alpha E_{mj}^\beta - w_m E_{mj}, \text{and}, \pi_j(x, E_j) = \sum_{m=1}^{2} \pi(x, E_{mj}),$$

with $w_m$ the cost per unit effort, $p_m$ the exogenous price of the harvested resources and $\pi_j(x, E_j)$ the individual profits.[9]

The extension of the above model to $M$ interactive populations with biomass $x_m$ for $m = 1, 2, .., M$, is straightforward. Denoting with $a_{\mu\nu}$ the interaction coefficients between the $\mu^{th}, \nu^{th}$ populations, with $\rho_\mu$, $\mu = 1, ...M$,

[9] The extension of the above model to $M$ interactive populations with biomass $x_m$ for $m = 1, 2, .., M$, is straightforward. Denoting with $a_{\mu\nu}$ the interaction coefficients between the $\mu^{th}, \nu^{th}$ populations, $\rho_\mu$, $\mu = 1, ...M$, the corresponding intrinsic growth rates and keeping the other variables and functions as in the previous sections, equation (29) can be written in the general case as:

the corresponding intrinsic growth rates and keeping the other variables and functions as in the previous sections, equation (29) can be written in the general case as:

$$dx = (\Xi - H)dt, \text{ where} \tag{31}$$
$$dx = \begin{bmatrix} dx_1 \\ .. \\ dx_M \end{bmatrix}, H = \begin{bmatrix} h_1 \\ ... \\ h_M \end{bmatrix}, h_m = \sum_{j=1}^{J} h_{mj}, m = 1, .., M$$
$$\Xi = \begin{bmatrix} \xi_1 \\ ..\xi_m.. \\ \xi_M \end{bmatrix}, \xi_m = \rho_m x_m \left(1 - \sum_{\mu=1}^{M} a_{m\mu} \frac{x_\mu}{\bar{K}}\right),$$
$$a_{mm} = 1, \sum_{\mu=1, m\neq\mu}^{M} a_{m\mu} < 1, h_{mj} = q_m x_m^\alpha E_{mj}^\beta$$

Then, problem (30) becomes:

$$\max_E \int_0^\infty \mathsf{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j (x, E_j) = \sum_{m=1}^{M} \sum_{j=1}^{J} \pi (x, E_{mj}) = \pi(x, E)\right] dt \tag{32}$$
$$s.t., (31)$$

Manipulating we obtain the optimality conditions as:

$$\lambda' = \Upsilon$$
$$x' = \Xi - H$$
$$\Upsilon = \begin{bmatrix} r\lambda_1 - J\alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}((p_1 - \lambda_1)q_1)^{\frac{1}{1-\beta}} x_1^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_1\rho_1 \left(1 - \sum_{\mu=1}^{M} A_{1\mu}\frac{x_\mu}{\bar{K}}\right) \\ + \sum_{\mu=2}^{M} \lambda_\mu \rho_\mu x_\mu a_{\mu 1}/\bar{K} \\ ..r\lambda_m - J\alpha(\frac{\beta}{w_m})^{\frac{\beta}{1-\beta}}((p_m - \lambda_m)q_1)^{\frac{1}{1-\beta}} x_m^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_m\rho_m \left(1 - \sum_{\mu=1}^{M} A_{m\mu}\frac{x_\mu}{\bar{K}}\right) \\ + \sum_{\mu=1,\mu\neq m}^{M} \lambda_\mu \rho_\mu x_\mu a_{\mu m}/\bar{K}.. \\ r\lambda_M - J\alpha(\frac{\beta}{w_M})^{\frac{\beta}{1-\beta}}((p_M - m_M)q_M)^{\frac{1}{1-\beta}} x_M^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_M\rho_M \left(1 - \sum_{\mu=1}^{M} A_{M\mu}\frac{x_\mu}{\bar{K}}\right) \\ + \sum_{\mu=1}^{M-1} \lambda_\mu \rho_\mu x_\mu a_{\mu M}/\bar{K} \end{bmatrix}$$

$$
\begin{aligned}
&dx = (\Xi - H)dt, \text{ where} \qquad (33)\\
&dx = \begin{bmatrix} dx_1 \\ .. \\ dx_M \end{bmatrix}, H = \begin{bmatrix} h_1 \\ ... \\ h_M \end{bmatrix}, h_m = \sum_{j=1}^{J} h_{mj}, m = 1, .., M \\
&\Xi = \begin{bmatrix} \xi_1 \\ ..\xi_m.. \\ \xi_M \end{bmatrix}, \xi_m = \rho_m x_m \left(1 - \sum_{\mu=1}^{M} a_{m\mu} \frac{x_\mu}{\bar{K}}\right), \\
&a_{mm} = 1, \sum_{\mu=1, m\neq\mu}^{M} a_{m\mu} < 1, h_{mj} = q_m x_m^{\alpha} E_{mj}^{\beta}.
\end{aligned}
$$

Then, problem (30) becomes:

$$
\max_E \int_0^\infty \mathrm{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j \left(x, E_j\right) = \sum_{m=1}^{M} \sum_{j=1}^{J} \pi\left(x, E_{mj}\right) = \pi(x, E)\right] dt \qquad (34)
$$

$$
s.t.\ (31).
$$

Manipulating we obtain the optimality conditions as:

$$
\lambda' = \Upsilon
$$

$$
x' = \Xi - H
$$

$$
\Upsilon = \begin{bmatrix}
r\lambda_1 - J\alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}((p_1 - \lambda_1)q_1)^{\frac{1}{1-\beta}} x_1^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_1\rho_1 \left(1 - \sum_{\mu=1}^{M} A_{1\mu} \frac{x_\mu}{\bar{K}}\right) \\
+ \sum_{\mu=2}^{M} \lambda_\mu \rho_\mu x_\mu a_{\mu 1}/\bar{K} \\
..r\lambda_m - J\alpha(\frac{\beta}{w_m})^{\frac{\beta}{1-\beta}}((p_m - \lambda_m)q_1)^{\frac{1}{1-\beta}} x_m^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_m\rho_m \left(1 - \sum_{\mu=1}^{M} A_{m\mu} \frac{x_\mu}{\bar{K}}\right) \\
+ \sum_{\mu=1,\mu\neq m}^{M} \lambda_\mu \rho_\mu x_\mu a_{\mu m}/\bar{K}.. \\
r\lambda_M - J\alpha(\frac{\beta}{w_M})^{\frac{\beta}{1-\beta}}((p_M - m_M)q_M)^{\frac{1}{1-\beta}} x_M^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_M\rho_M \left(1 - \sum_{\mu=1}^{M} A_{M\mu} \frac{x_\mu}{\bar{K}}\right) \\
+ \sum_{\mu=1}^{M-1} \lambda_\mu \rho_\mu x_\mu a_{\mu M}/\bar{K}
\end{bmatrix}.
$$

Returning to the two-interacting-populations model, defining the current value Hamiltonian $H$ as:

$$H(x, E, \lambda) = \pi(x, E) + \lambda F, m = \lambda(t) = \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix}, \tag{35}$$

and manipulating we finally obtain the following Hamiltonian system for biomasses and their costate variables:[10]

$$\begin{aligned} \lambda' &= \begin{bmatrix} r\lambda_1 - J\alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}((p_1-\lambda_1)q_1)^{\frac{1}{1-\beta}} x_1^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_1\rho_1\left(1 - \frac{2x_1+a_{12}x_2}{\bar{K}}\right) \\ +\lambda_2\rho_2 x_2 a_{21}/\bar{K} \\ r\lambda_2 - J\alpha(\frac{\beta}{w_2})^{\frac{\beta}{1-\beta}}((p_2-\lambda_2)q_2)^{\frac{1}{1-\beta}} x_2^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_2\rho_2\left(1 - \frac{2x_2+a_{21}x_1}{\bar{K}}\right) \\ +\lambda_1\rho_1 x_1 a_{12}/\bar{K} \end{bmatrix} \\ x' &= \begin{bmatrix} \rho_1 x_1\left(1 - \frac{x_1}{\bar{K}} - a_{12}\frac{x_2}{\bar{K}}\right) - h_1 \\ \rho_2 x_2\left(1 - \frac{x_2}{\bar{K}} - a_{21}\frac{x_1}{\bar{K}}\right) - h_2 \end{bmatrix}, h_m = \sum_{j=1}^{J} h_{mj}. \end{aligned} \tag{36}$$

### 3.1.1 Numerical simulations

In order to obtain a clear picture of the solution, that will also allow us to provide comparisons with the case where carrying capacity is slowly changing, we resort again to numerical simulations using our usual parameter setting

$$\begin{aligned} \beta &= \alpha = 1/2, J = 2, p_1 = p_2 = 10, w_1 = w_2 = 5, q_1 = 0,048, q_2 = 0,042, \\ \rho_1 &= 0.45, \rho_2 = 0.35, r = 0,05, a_{12} = a_{12} = 0.3, \bar{K} = 25. \end{aligned}$$

Then (36) becomes:

$$\begin{aligned} \lambda' &= \begin{bmatrix} r\lambda_1 - J(\frac{1}{4w_1})((p_1-\lambda_1)q_1)^2 - \lambda_1\rho_1\left(1 - \frac{2x_1+a_{12}x_2}{\bar{K}}\right) + \lambda_2\rho_2 x_2 a_{21}/\bar{K} \\ r\lambda_2 - J(\frac{1}{4w_2})((p_2-\lambda_2)q_2)^2 - \lambda_2\rho_2\left(1 - \frac{2x_2+a_{21}x_1}{\bar{K}}\right) + \lambda_1\rho_1 x_1 a_{12}/\bar{K} \end{bmatrix} \\ x' &= \begin{bmatrix} \rho_1 x_1\left(1 - \frac{x_1}{\bar{K}} - a_{12}\frac{x_2}{\bar{K}}\right) - J\frac{1}{2w_1}(p_1-\lambda_1)q_1^2 x_1 \\ \rho_2 x_2\left(1 - \frac{x_2}{\bar{K}} - a_{21}\frac{x_1}{\bar{K}}\right) - J\frac{1}{2w_2}(p_2-\lambda_2)q_2^2 x_2 \end{bmatrix}. \end{aligned} \tag{37}$$

[10] See Appendix for details.

The equations in system (37) are similar to the third and fourth equations of (19), with $\bar{K} = 25$. Thus the steady states $S_i = (x_1, x_2, \lambda_1, \lambda_2)$, for $i = 1, .., 4$ which lead to nonextinction of both resources, are determined by setting $\bar{K} = 25$ in (25)and are given by:

$$\begin{aligned}
sol_1 &= (x_1,x_2,\lambda_1,\lambda_2)_1 = (7.89465, 17.6994, -367.664, 94.785) \\
sol_2 &= (x_1,x_2,\lambda_1,\lambda_2)_2 = (8.2995, 7.74072, -249.870, 40.6759) \\
sol_3 &= (x_1,x_2,\lambda_1,\lambda_2)_3 = (18.8438, 18.8952, 0.05816, 0.0447838) \\
sol_4 &= (x_1,x_2,\lambda_1,\lambda_2)_4 = (17.4335, 6.95268, 75.49248, -522.37705)
\end{aligned}$$

Stability properties are the same as in Section 2.2.1 and thus the acceptable solution is the fourth one.

Assuming initial biomass values $x_1(0) = 1$ and $x_2(0) = 5$, Figure 2 depicts the results of the multiple shooting method. With initial values $(481, 9020, -519, 0229)$ for the costates, we obtain a satisfactory approximation of $sol_4$ for $t = 8.25$.

# 4 Non-cooperative harvesting when the time scale externality is ignored

This section studies the harvesting behavior of competing agents. Each agent in the renewable resource sector ignores the pollution stock $S$ and its link with carrying capacity. The agents take the carrying capacity $\bar{K}$ as parametric and solve in fast time the problem:

$$\max_E \int_0^\infty \mathrm{e}^{-rt} \left[ \sum_{j=1}^J \pi_j (x, E_j) = \sum_{m=1}^2 \sum_{j=1}^J \pi (x, E_{mj}) = \pi(x, E) \right] dt \qquad (38)$$

$$s.t., (29),$$

$$\pi (x, E_{m_J}) = p_m q_m x_m^\alpha E_{mj}^\beta - w_m E_{mj}, and, \pi_j (x, E_j) = \sum_{m=1}^2 \pi (x, E_{mj}).$$

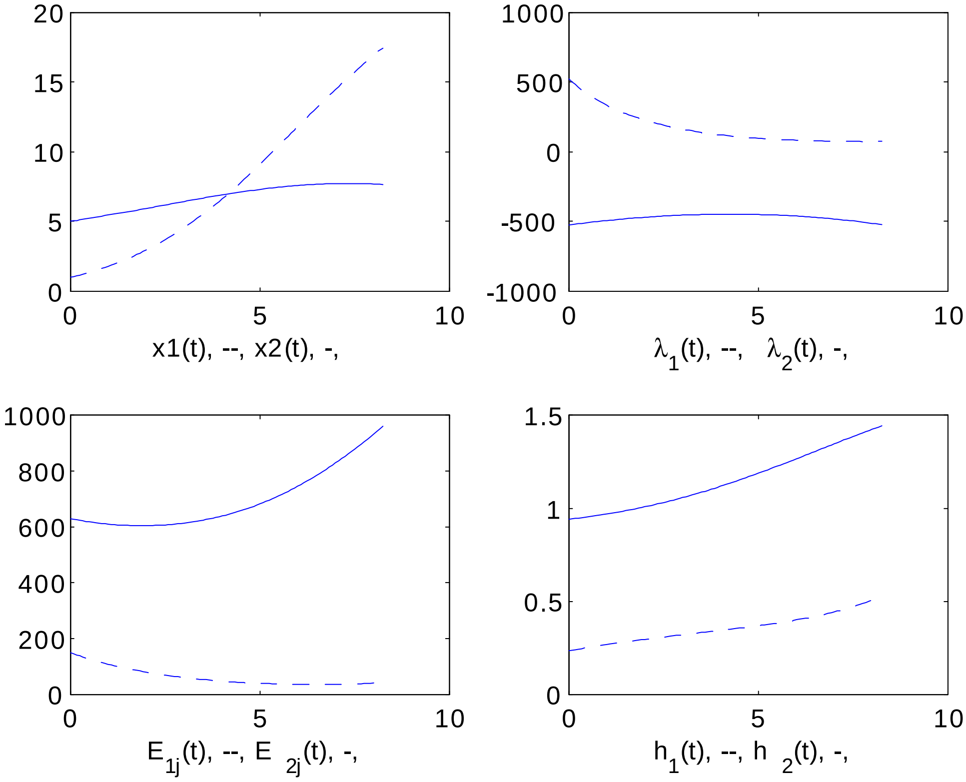


Figure 2: Regulation when time separation is ignored

Assuming open loop information structure, the open loop Nash equilibrium (OLNE) is obtained by defining the current value Hamiltonian as:

$$H(x,E,\lambda)=\pi(x,E)+\lambda F, m=\lambda(t)=\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\end{bmatrix}$$
$$F=\begin{bmatrix}\rho_1 x_1\left(1-\frac{x_1}{\bar{K}}-a_{12}\frac{x_2}{\bar{K}}\right)-h_{1j}-\sum_{l\neq j}\bar{h}_{1l}\\ \rho_2 x_2\left(1-\frac{x_2}{\bar{K}}-a_{21}\frac{x_1}{\bar{K}}\right)-h_{2j}-\sum_{l\neq j}\bar{h}_{2l}\end{bmatrix}.$$

In particular we assume that for agent $j$, harvesting is given by $h_{mj}=q_m x_m^{\alpha} E_{mj}^{\beta}$, $m=1,2$. Then similarly to the previous analysis, we can derive open loop solutions for harvesting and effort, as functions of the costate variables $\lambda_m$ $m=1,2$:[11]

$$E_{mj}=(\frac{p_m-\lambda_m}{w_m}q_m x_m^{\alpha}\beta)^{\frac{1}{1-\beta}},, j=1,..J, m=1,2.$$

The evolution of the costate variables is given by:

$$\begin{aligned}\lambda_m^{'}&=r\lambda_m-\lambda_m\rho_m\left(1-\frac{2x_1+a_{12}x_2}{\bar{K}}\right)-\alpha(\frac{\beta}{w_0})^{\frac{\beta}{1-\beta}}((p_m-\lambda_m)q_m)^{\frac{1}{1-\beta}}x_m^{\frac{\alpha+\beta-1}{1-\beta}}\\ &+\lambda_m(J-1)\alpha(\frac{\beta}{w_m})^{\frac{\beta}{1-\beta}}(p_m-\lambda_0)^{\frac{\beta}{1-\beta}}q_m^{\frac{1}{1-\beta}}x_m^{\frac{\alpha+\beta-1}{1-\beta}}, m=1,2. \end{aligned} \tag{39}$$

[11] See the previous section and Vardas and Xepapadeas (2014) for more details regarding the derivation of this formula.

Adopting the same setting for the parameters, we obtain the following Hamiltonian system: [12][13]

$$
\begin{aligned}
\lambda' &= \begin{bmatrix} r\lambda_1 - \frac{1}{4w_1}((p_1-\lambda_1)q_1)^2 - \lambda_1\rho_1\left(1-\frac{2x_1+a_{12}x_2}{\bar{K}}\right) \\ +\lambda_1(J-1)\frac{1}{4w_1}\ (p_1-\lambda_1)q_1^2 + \lambda_2\rho_2x_2a_{21}/\bar{K} \\ r\lambda_2 - \frac{1}{4w_2}((p_2-\lambda_2)q_2)^2 - \lambda_2\rho_2\left(1-\frac{2x_2+a_{21}x_1}{\bar{K}}\right) + \\ \lambda_2(J-1)\frac{1}{4w_2}\ (p_2-\lambda_2)q_2^2 + \lambda_1\rho_1x_1a_{12}/\bar{K} \end{bmatrix} \qquad (40) \\
x' &= \begin{bmatrix} \rho_1x_1\left(1-\frac{x_1}{\bar{K}}-a_{12}\frac{x_2}{\bar{K}}\right) - J\frac{1}{2w_1}(p_1-\lambda_1)q_1^2x_1 \\ \rho_2x_2\left(1-\frac{x_2}{\bar{K}}-a_{21}\frac{x_1}{\bar{K}}\right) - J\frac{1}{2w_2}(p_2-\lambda_2)q_2^2x_2 \end{bmatrix}.
\end{aligned}
$$

## 4.1 Numerical simulations

To obtain meaningful comparisons with our previous results we resort again to numerical simulations and study the nature of the steady states of our system for $\bar{K} = 25$, with the other parameters being the same as in the previous sections. The steady states $S_i = (x_1, x_2, \lambda_1, \lambda_2)$ for $i = 1, .., t$ are the following:

$$
\begin{aligned}
sol_1 &= (x_1, x_2, \lambda_1, \lambda_2)_1 = (7.69264652849, 17.1831869, -369.3, 74.12) \\
sol_2 &= (x_1, x_2, \lambda_1, \lambda_2)_2 = (7.95728377, 6.82868, -252.252, -399.0722) \\
sol_3 &= (x_1, x_2, \lambda_1, \lambda_2)_4 = (18.8416, 18.8937, 0.0214518319, 0.02724053584) \\
sol_4 &= (x_1, x_2, \lambda_1, \lambda_2)_3 = (16.80487, 6.2272628, 61.2206, -546.723).
\end{aligned}
$$

[12] See Section 3.3.1 for more details regarding the derivation of these equations.

[13] In a noncooperative setup in the general case with $M$ harvesters, the system of equations which characterizes the optimality conditions becomes:

$$
\lambda' = \begin{bmatrix} r\lambda_1 - \alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}((p_1-\lambda_1)q_1)^{\frac{1}{1-\beta}}x_1^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_1\rho_1\left(1-\sum_{\mu=1}^{M} A_{1\mu}\frac{x_\mu}{K}\right) \\ +\lambda_1(J-1)\alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}(p_1-\lambda_1)^{\frac{\beta}{1-\beta}} q_1^{\frac{1}{1-\beta}} x_1^{\frac{\alpha+\beta-1}{1-\beta}} + \sum_{\mu=2}^{M} \lambda_\mu\rho_\mu x_\mu a_{\mu 1}/\bar{K} \\ ..r\lambda_m - \alpha(\frac{\beta}{w_m})^{\frac{\beta}{1-\beta}}((p_m-\lambda_m)q_1)^{\frac{1}{1-\beta}}x_m^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_m\rho_m\left(1-\sum_{\mu=1}^{M} A_{m\mu}\frac{x_\mu}{K}\right) \\ +\lambda_m(J-1)\alpha(\frac{\beta}{w_m})^{\frac{\beta}{1-\beta}}(p_m-\lambda_m)^{\frac{\beta}{1-\beta}} q_m^{\frac{1}{1-\beta}} x_m^{\frac{\alpha+\beta-1}{1-\beta}} + \sum_{\mu=1,\mu\neq m}^{M} \lambda_\mu\rho_\mu x_\mu a_{\mu m}/\bar{K}.... \\ r\lambda_M - \alpha(\frac{\beta}{w_M})^{\frac{\beta}{1-\beta}}((p_M-m_M)q_M)^{\frac{1}{1-\beta}}x_M^{\frac{\alpha+\beta-1}{1-\beta}} - \lambda_M\rho_M\left(1-\sum_{\mu=1}^{M} A_{M\mu}\frac{x_\mu}{K}\right) \\ +\lambda_M(J-1)\alpha(\frac{\beta}{w_M})^{\frac{\beta}{1-\beta}}(p_M-\lambda_M)^{\frac{\beta}{1-\beta}} q_M^{\frac{1}{1-\beta}} x_M^{\frac{\alpha+\beta-1}{1-\beta}} + \sum_{\mu=1}^{M-1} \lambda_\mu\rho_\mu x_\mu a_{\mu M}/\bar{K} \end{bmatrix}
$$

$$
x' = \Xi - H, m = 1, ..M, A_{ii} = 2a_{ii}, A_{ij} = a_{ij}.
$$

Note that these are OLNE steady states. We select $sol_4$ which has the saddle point property, the rest being completely unstable (see Dockner, 1985).

Assuming as before initial biomass values $x_1(0) = 1$ and $x_2(0) = 5$, Figure 3 depicts the results of the multiple shooting method. With initial values $(499.421, -562.963)$ for the costate variables we obtain a satisfactory approximation of $sol_4$ for $t = 8.05$.

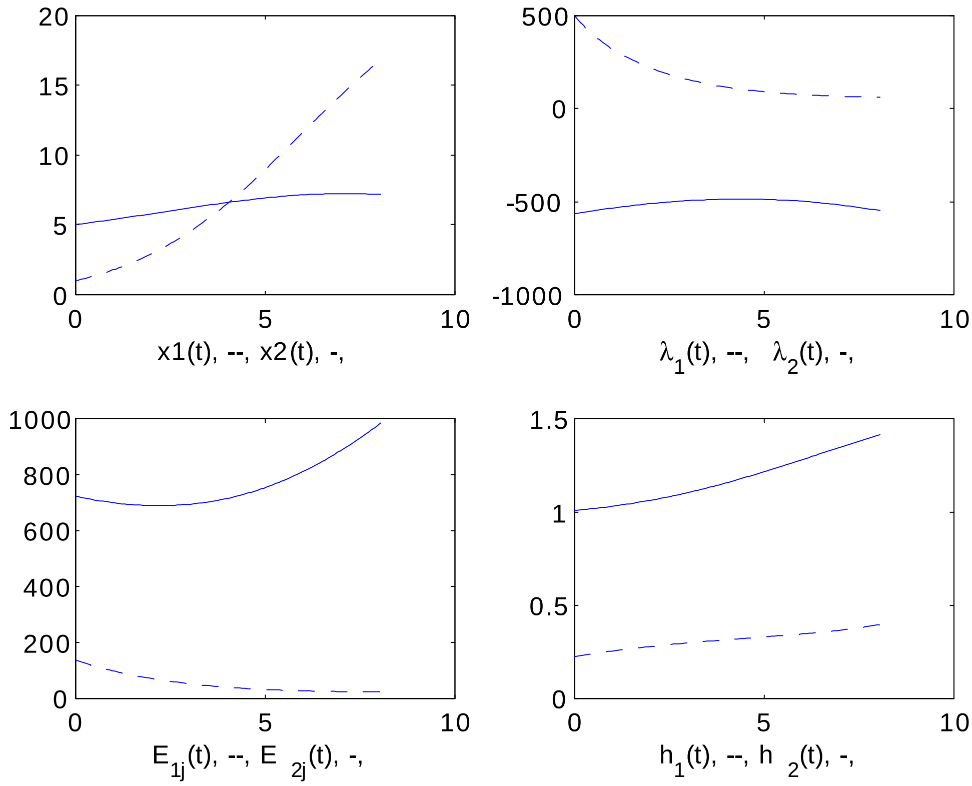


Figure 3: Open loop Nash equilibrium for competing agents

## 4.2 Implications from Ignoring the Time Scale Externality

To obtain a picture of the implications from ignoring the time scale externality, we present in Table 1 the steady-state levels of the biomasses of the two resources and the corresponding harvesting and emissions levels in the cases of (i) optimal regulation (OR), (ii) regulation when the time scale externality is ignored (R), and (iii) OLNE when the competing agents ignore time scale separation. In case (i) the optimal steady state carrying capacity is $K^* = 24.85$, while in cases (ii) and (iii) where carrying capacity is taken as fixed, we consider two values $\bar{K} = 23$ and $\bar{K} = 27$ and use the formulas, $h_m = \sum_{j=1}^{J} h_{mj}$, $h_{mj} = q_m x_m^\alpha E_{mj}^\beta, m, j = 1, 2, E_{mj}^\beta = \frac{p_m - \lambda_m}{w_m} q_m x_m^\alpha \beta$.

Table 1. Biomass comparison

| **Steady States** | **OR** | **R** | | **OLNE** | |
|---|---|---|---|---|---|
| | $K^* = 24.85$ | $\bar{K} = 23$ | $\bar{K} = 27$ | $\bar{K} = 23$ | $\bar{K} = 27$ |
| $x_1$ | 17.32 | 16.04 | 18.83 | 15.46 | 18.15 |
| $x_2$ | 6.91 | 6.40 | 7.51 | 5.73 | 6.72 |
| $h_1$ | 0,53 | 0.48 | 0.57 | 0.37 | 0.43 |
| $h_2$ | 1,30 | 1.20 | 1.41 | 1.13 | 1.32 |
| $s_1 = s_2$ | 0.499 | — | — | — | — |

It should be made clear that the steady states for cases (ii) and (iii), that is R and OLNE respectively, can be understood only as *ex ante* steady states which will never be attained because in the actual natural system carrying capacity will change slowly and thus the *ex ante* biomasses paths under R or OLNE will deviate from the actual paths. This also implies that when time separation is ignored, the solutions of the corresponding optimization problems associated with R or OLNE are not time consistent. This is because actual carrying capacity at $t = s > 0$ will be different from the actual carrying capacity at $t = 0$, and therefore the optimal path starting at $t = s$ will be different from the optimal path starting at $t = 0$. When time separation is taken into account, the resulting optimal path under OR is time consistent.

The fixed-carrying-capacity paths deviate therefore from the optimal paths. This deviation can be clearly seen if we consider the impact from the industrial sector on the carrying capacity, which exists but is not taken into account when the regulator assumes a fixed $K$. Assume that the regulator chooses and implements optimal emissions $(\hat{s}_1, \hat{s}_2)$ by solving the static problem

$$\max_{s_i} \sum_{i=1}^{I} B_i(s_i) - D\left(\sum_{i=1}^{I} s_i\right). \tag{41}$$

Then pollution, taking into account the link between carrying capacity and pollution stock given by

$$\begin{aligned} K_o(t) &= \omega(S(t)) = A - \theta S(t), \ \theta > 0, K(t) \geq 0 \\ K_o' &= -\varepsilon S' = -\varepsilon(\varphi s - lS), K_o = 2K, \end{aligned} \tag{42}$$

will accumulate according to $S'(t) = \varphi(\hat{s}_1 + \hat{s}_2) - lS(t), S(0) = S_0$. Let $\hat{S}(t)$ be the resulting path of accumulated pollution. Then the carrying capacity will evolve slowly according to:

$$K(t) = (A - \theta\hat{S}(t))/2, K(0) = (A - \theta S_0)/2. \tag{43}$$

If, for example, we use our parametrization with $\varphi = 2, l = 0.4, \varepsilon = \theta = 0.04$, $S_0 = 1$, and $A = 50$, and $2\hat{s} = 2$ for uncontrolled emissions with respect to time scale separation, then $\hat{S}(t) = 5 - 4e^{-0.4t}$ and $K(t) = 25 - 0.02\hat{S}(t)$. Thus carrying capacity is slowly reducing and at a steady state $\lim_{t\to\infty} \hat{S}(t) = 5$ and $\lim_{t\to\infty} K(t) = 24$. Since our numerical results indicate that the optimal steady-state biomasses are declining along with carrying capacity, treating the carrying capacity as fixed by ignoring the time scale externality, implies that the time inconsistent regulatory scheme R will induce excess harvesting.

# 5 Conclusions

In the present paper we study the impact of time scale separation and the implications of the resulting time scale externality in a model of renewable resource harvesting with interacting populations. In our setup the carrying capacity of each population is evolving slowly in response to pollution accumulating because of emissions generated in the industrial sector of the economy. Using singular perturbation methods, we analyze the problem of a regulator seeking to internalize the time scale externality and we derive the optimal paths for the population biomasses and pollution along with the optimal regulatory scheme that includes the adjustments necessary to internalize the time scale separation. By solving the problem of a regulator and the problem of competing harvesters when time scale separation is ignored and carrying capacity is regarded as fixed, we identify the deviations relative to the case where the time scale externality is internalized. We also show that ignoring the time scale externality leads to time inconsistent regulation. Our results are supported by numerical simulations.

Areas for further research include the study of prey-predator models with time scale separation and the introduction of nonconvexities in ecosystems dynamics. In particular if pollution dynamics, which induce a slow variation of the carrying capacity, are characterized by nonconvexities then the slow manifold might contain more than one feasible branch. In this case an additional task of optimal regulation would be to identify the optimal slow branch and steer the fast system towards this branch.

Uncertainty is also another open issue, in particular the case where the evolution of the interacting populations might be characterized by risk, or measurable uncertainty, and the slow evolution of the carrying capacity might be characterized by ambiguity. The application of robust control methods when the structure of uncertainty differs according to the time scale could be a promising area for further research.

# 6 Appendix

## 6.1 Appendix 1: Matrix $J$

The matrix is defined as: $J=\frac{\partial F}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial \mathbf{x}_1} & \frac{\partial F_1}{\partial \mathbf{x}_2} & \frac{\partial F_1}{\partial \lambda_1} & \frac{\partial F_1}{\partial \lambda_2} \\ \frac{\partial F_2}{\partial \mathbf{x}_1} & \frac{\partial F_2}{\partial \mathbf{x}_2} & \frac{\partial F_2}{\partial \lambda_1} & \frac{\partial F_2}{\partial \lambda_2} \\ \frac{\partial F_3}{\partial \mathbf{x}_1} & \frac{\partial F_3}{\partial \mathbf{x}_2} & \frac{\partial F_3}{\partial \lambda_1} & \frac{\partial F_3}{\partial \lambda_2} \\ \frac{\partial F_4}{\partial \mathbf{x}_1} & \frac{\partial F_4}{\partial \mathbf{x}_2} & \frac{\partial F_4}{\partial \lambda_1} & \frac{\partial F_4}{\partial \lambda_2} \end{bmatrix}$, where

$$\frac{\partial F_1}{\partial \mathbf{x}_1} = 2\lambda_1\rho_1/K$$
$$\frac{\partial F_1}{\partial \mathbf{x}_2} = (a_{12}\lambda_1\rho_1 + \lambda_2\rho_2 a_{21})/K$$
$$\frac{\partial F_1}{\partial \lambda_1} = -\rho_1\left(1-(2x_1+a_{12}x_2)/K\right) + 2(p_1-\lambda_1)\alpha\beta q_1^2 J w_1^{-1}$$
$$\frac{\partial F_1}{\partial \lambda_2} = \rho_2 x_2 a_{21} S$$
$$\frac{\partial F_2}{\partial \mathbf{x}_1} = 2\lambda_2\rho_2/K$$
$$\frac{\partial F_2}{\partial \mathbf{x}_2} = (a_{21}\lambda_2\rho_2 + \lambda_1\rho_1 a_{12})/K$$
$$\frac{\partial F_2}{\partial \lambda_1} = \rho_1 x_1 a_{12}/K$$
$$\frac{\partial F_2}{\partial \lambda_2} = -\rho_2\left(1-(2x_2+a_{21}x_1)/K\right) + 2(p_2-\lambda_2)\alpha\beta q_2^2 J w_2^{-1}$$
$$\frac{\partial F_3}{\partial \mathbf{x}_1} = \rho_1\left(1-2x_1/K - a_{12}x_2/K\right) - J\frac{(p_1-\lambda_1)\beta q_1^2}{\rho_1 w_1}$$
$$\frac{\partial F_3}{\partial \mathbf{x}_2} = -\rho_1 x_1 a_{12}/K$$
$$\frac{\partial F_3}{\partial \lambda_1} = J\frac{\beta q_1^2}{\rho_1 w_1}x_1$$
$$\frac{\partial F_3}{\partial \lambda_2} = 0$$
$$\frac{\partial F_4}{\partial \mathbf{x}_1} = \rho_2\left(1-2x_2/K - a_{21}x_1/K\right) - J\frac{(p_2-\lambda_2)\beta q_2^2}{\rho_2 w_2}$$
$$\frac{\partial F_4}{\partial \mathbf{x}_2} = -\rho_2 x_2 a_{21}/K$$
$$\frac{\partial F_4}{\partial \lambda_1} = 0$$
$$\frac{\partial F_4}{\partial \lambda_2} = J\frac{\beta q_2^2}{\rho_2 w_2}x_2.$$

## 6.2 Appendix 2: The Slow Manifold

Fenichel's theorem extends the analysis to an arbitrary small parameter $\boldsymbol{\varepsilon}$ and provides a slow manifold $M_\varepsilon$ defined as $M_\varepsilon = \{(X, \mathbf{K}) \in R^6 : X = (\mathbf{H}^{\varepsilon}(\mathbf{K}), K, \boldsymbol{\varepsilon})\}$ such that:

$$d\mathbf{K}/d\tau = \mathbf{G}(\mathbf{H}^{\varepsilon}(\mathbf{K}), K, \boldsymbol{\varepsilon}),$$

where the vector $\mathbf{H}^{\varepsilon}(\mathbf{K}) = \mathbf{H^0}(\mathbf{K}) + \boldsymbol{\varepsilon}\mathbf{H^{(1)}}(\mathbf{K}) + ...$ as $\boldsymbol{\varepsilon} \to 0$, with

$$\mathbf{H}^0(\mathbf{K}) = \mathbf{H}^0(\mathbf{K}) \tag{44}$$

$$\mathbf{H}^{(1)}(\mathbf{K}) = [\frac{\partial \mathbf{F}}{\partial \mathbf{X}}]^{-1}[\frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} G^* - \frac{\partial \mathbf{F}}{\partial \varepsilon}]$$

$$\frac{\partial \mathbf{F}}{\partial \varepsilon} = [\frac{\partial F_1}{\partial \varepsilon}, \frac{\partial F_2}{\partial \varepsilon}, \frac{\partial F_3}{\partial \varepsilon}, \frac{\partial F_4}{\partial \varepsilon}]^T = [\delta\lambda_1, \delta\lambda_2, 0, 0]^T$$

$$\frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} = \begin{bmatrix} \frac{\partial H_1^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_1^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_2^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_2^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_3^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_3^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_4^o(\mathbf{K})}{\partial \mu} & \frac{\partial H_4^o(\mathbf{K})}{\partial K} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial H_1^o(\mathbf{K})}{\partial K} \\ 0 & \frac{\partial H_2^o(\mathbf{K})}{\partial K} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Manipulating $sol_4$ we obtain that:

$$[\frac{\partial F}{\partial \mathbf{X}}]^{-1} = \begin{bmatrix} 0.00064223K & -0.000687398K & -1.68311 & 5.49517 \\ 0.000501827K & 0.00129511K & -6.68298 & -15.4481 \\ 5.71002 & -0.743681 & \frac{429.902}{K} & -\frac{8981.07}{K} \\ -2.65132 & 8.2908 & -\frac{9027.47}{K} & \frac{20478.8}{K} \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} = \begin{bmatrix} 0 & 0.69734 \\ 0 & 0.27810 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \frac{\partial \mathbf{H^0}}{\partial \mathbf{K}}\mathbf{G} - \frac{\partial \mathbf{F}}{\partial \varepsilon} = \begin{bmatrix} 0.69734G_2 - \delta\lambda_1 \\ 0.27810G_2 - \delta\lambda_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{H}^{(1)}(\mathbf{K}) = \begin{bmatrix} 0.00064223(0.69734G_2 - \delta\lambda_1^{\varepsilon}) - 0.000687398K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0.000501827K(0.69734G_2 - \delta\lambda_1^{\varepsilon}) + 0.00129511K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{H}^{\varepsilon}(\mathbf{K}) = \mathbf{H^0}(\mathbf{K}) + \varepsilon\mathbf{H^{(1)}}(\mathbf{K}) + ...$$

Then with $\varepsilon = 0.04$, the optimal trajectories of the fast variables $X^{\varepsilon}$ as functions of the slow variables are given by:

$$X^{\varepsilon} = (x_{1,}^{\varepsilon} x_{2}^{\varepsilon}, \lambda_{1}^{\varepsilon}, \lambda_{2}^{\varepsilon})^{T} = \mathbf{H}^{\varepsilon}(\mathbf{K}) = \begin{bmatrix} 0.69734K \\ 0.27810K \\ 75.4924 \\ -522.3770 \end{bmatrix} + 0.04 \tag{45}$$

$$\begin{bmatrix} 0.00064223(0.69734G_2 - \delta\lambda_1^{\varepsilon}) - 0.000687398K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0.000501827K(0.69734G_2 - \delta\lambda_1^{\varepsilon}) + 0.00129511K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0 \\ 0 \end{bmatrix}$$

which provides $d\mathbf{K}/d\tau = \mathbf{G}(X^{\varepsilon}, \mathbf{K},\varepsilon)$. The dynamics of the slow variables on $\mathrm{M}_{\varepsilon}$ are given by:

$$\begin{bmatrix} d\mu/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\mu - \lambda_1^{\varepsilon}\rho_1 x_1^{\varepsilon}\left(x_1^{\varepsilon} + a_{12}x_2^{\varepsilon}\right)/K^2 \\ -\lambda_2^{\varepsilon}\rho_2 x_2^{\varepsilon}\left(x_2^{\varepsilon} + a_{21}x_1^{\varepsilon}\right)/K^2 + \mu\gamma \\ \gamma\left(A_2 - K\right) - \varphi_2 \sum_{i=1}^{I} s_i \end{bmatrix}. \tag{46}$$

Taking into account the relationship (13) and adopting the following parameterization

$$\begin{aligned} \beta &= \alpha = 1/2, J = 2, p_1 = p_2 = 10, w_1 = w_2 = 5, q_1 = 0,048, q_2 = 0,042, \\ \rho_1 &= 0.45, \rho_2 = 0.35, r = 0,05, a_{12} = a_{12} = 0.3, \\ D(\cdot) &= (\cdot)^2, \delta = 0.05, \varphi = 0.1, A = 50, \varepsilon = 0.04, l = 0.4, \gamma = l/\varepsilon = 10, I = 2, \end{aligned}$$

we obtain the dynamical system characterizing the slow variables.

By further manipulation we obtain the steady states of the above system which give the full system equilibria. Their derivatives with respect to $\mathbf{K}$ evaluated at a specific steady state, determine the stability properties of the specific steady state. We obtain for the steady state $(K,\mu)$:

$$\begin{aligned}
\begin{bmatrix} 0 \\ K \end{bmatrix} &= \begin{bmatrix} \delta\mu - \lambda_1^{\varepsilon}\rho_1 x_1^{\varepsilon}\left(x_1^{\varepsilon}+a_{12}x_2^{\varepsilon}\right)/K^2 - \lambda_2^{\varepsilon}\rho_2 x_2^{\varepsilon}\left(x_2^{\varepsilon}+a_{21}x_{1,}^{\varepsilon}\right)/K^2+\mu\gamma \\ A_2 - \frac{\varphi_2}{\gamma}\sum_{i=1}^{I} s_i \end{bmatrix} \\
\begin{bmatrix} \mu \\ K \end{bmatrix} &= \begin{bmatrix} -0.64470 \\ 24.9001 \end{bmatrix}, K_o = 2*K = 49.8003 \qquad (47) \\
s_i &: \quad 1/2\sqrt{s_i} - 2\sum_{i=1}^{I} s_i - \mu\varphi = 0, \longrightarrow s_i = 0.49926
\end{aligned}$$

This steady state has the saddle property stability.

## 6.3 Appendix 3: Linear damage function

In this part we derive our results assuming that the damage function associated with emissions is linear. Similarly to the quadratic damage function we obtain regarding $sol_4$:

$$X^{\varepsilon} = (x_{1,}^{\varepsilon} x_2^{\varepsilon}, \lambda_1^{\varepsilon}, \lambda_2^{\varepsilon})^T = \mathbf{H}^{\varepsilon}(\mathbf{K}) = \begin{bmatrix} 0.69734K \\ 0.27810K \\ 75.4924 \\ -522.3770 \end{bmatrix} + 0.04 \qquad (48)$$

$$\begin{bmatrix} 0.00064223K(0.69734G_2 - \delta\lambda_1^{\varepsilon}) - 0.000687398K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0.000501827K(0.69734G_2 - \delta\lambda_1^{\varepsilon}) + 0.00129511K(0.27810G_2 - \delta\lambda_2^{\varepsilon}) \\ 0 \\ 0 \end{bmatrix},$$

which provides $d\mathbf{K}/d\tau = \mathbf{G}(X^{\varepsilon}, \mathbf{K}, \varepsilon)$. The dynamics of the slow variables on $\mathrm{M}_{\varepsilon}$ are given by:

$$\begin{bmatrix} d\mu/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\mu - \lambda_1^{\varepsilon}\rho_1 x_1^{\varepsilon}\left(x_1^{\varepsilon}+a_{12}x_2^{\varepsilon}\right)/K^2 \\ -\lambda_2^{\varepsilon}\rho_2 x_2^{\varepsilon}\left(x_2^{\varepsilon}+a_{21}x_1^{\varepsilon}\right)/K^2+\mu\gamma \\ \gamma\left(A-K\right) - \varphi\sum_{i=1}^{I} s_i \end{bmatrix}. \qquad (49)$$

Adopting the usual parameterization,

$$\begin{aligned}
\beta &= \alpha = 1/2, J = 2, p_1 = p_2 = 10, w_1 = w_2 = 5, q_1 = 0,048, q_2 = 0,042, \\
\rho_1 &= 0.45, \rho_2 = 0.35, r = 0,05, a_{12} = a_{12} = 0.3, \\
D(\cdot) &= (\cdot), \delta = 0.05, \varphi = 0.1, A_2 = 25, \varepsilon = 0.04, l = 0.4, \gamma = l/\varepsilon = 10, I = 2,
\end{aligned}$$

we obtain the dynamical system characterizing the slow variables.

Manipulating we obtain the steady states of the above system which give the full system equilibria. For $s_i$ and the steady state $(K, \mu)$ we obtain:

$$\begin{bmatrix} 0 \\ K \end{bmatrix} = \begin{bmatrix} \delta\mu - \lambda_1^{\varepsilon}\rho_1 x_{1,}^{\varepsilon}\left(x_{1,}^{\varepsilon} + a_{12}x_2^{\varepsilon}\right)/K^2 \\ -\lambda_2^{\varepsilon}\rho_2 x_2^{\varepsilon}\left(x_2^{\varepsilon} + a_{21}x_{1,}^{\varepsilon}\right)/K^2 + \mu\gamma \\ \gamma\left(A_2 - K\right) - \varphi_2 \sum_{i=1}^{I} s_i \end{bmatrix}$$

$$\begin{bmatrix} \mu \\ K \end{bmatrix} = \begin{bmatrix} -0.64470 \\ 24.9971 \end{bmatrix},$$

$$s_i : 1/2\sqrt{s_i} - 1 - \mu\varphi = 0, \longrightarrow s_i = 0.285644. \tag{50}$$

The steady state $(\mu, K) = (-0.64470, 24.9971)$ is a saddle point as in the case with quadratic damages.

Assuming $K(0) = 20$, we present the main findings in Figure 4 where we obtain, by using shooting methods, a good approximation of the final steady state $(-0.64470, 24, 9636)$ using as initial state $(-1.2878, 20)$. The steady state is attained at $\tau = 0.5$. The corresponding steady state values for the biomasses are $(x_1, x_2) = (17, 4081, 6, 9425)$.

Comparing to the quadratic damage function case, it can be noticed that linear damages result in relatively higher carrying capacity and resource biomasses.

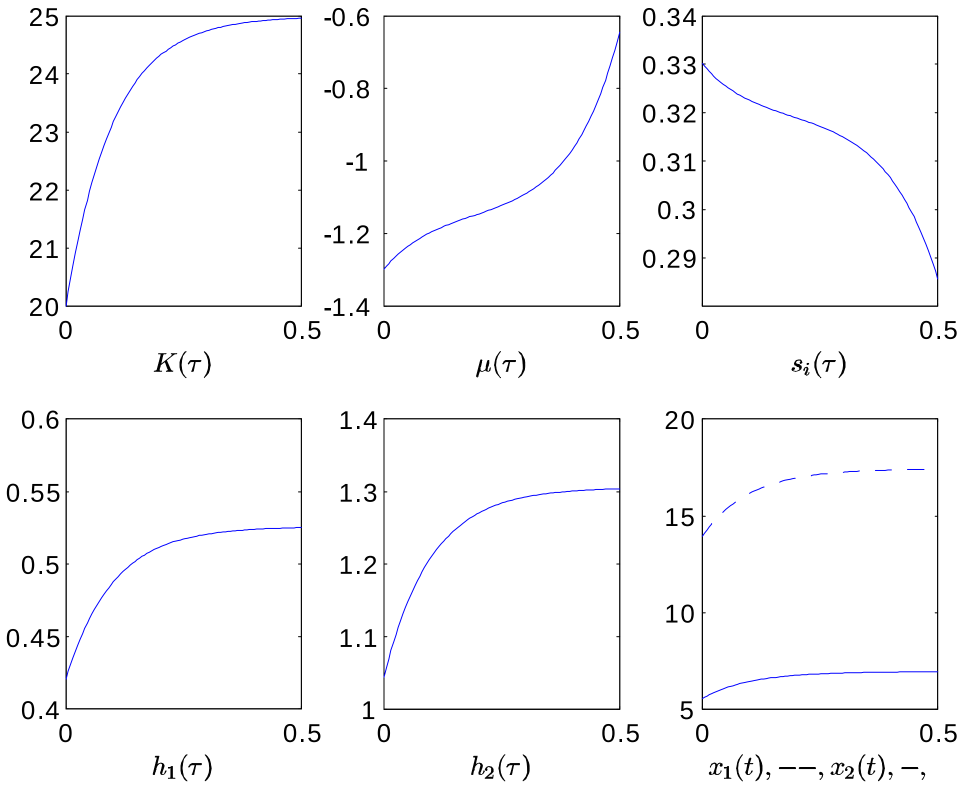


Figure 4: Regulation with linear damage function

## 6.4 Appendix 4: Regulation with fixed K

Using the Hamiltonian

$$H(x,E,\lambda)=\pi(x,E)+\lambda F, m=\lambda(t)=\begin{bmatrix}\lambda_1(t)\\ \lambda_2(t)\end{bmatrix}, \tag{51}$$

we derive the following optimality conditions:

$$H_{E_{m_J}}(x,E,\lambda)=\beta p_m q_m x_m^\alpha E_{mj}^{\beta-1}-w_m-\beta\lambda_m q_m x_m^\alpha E_{mj}^{\beta-1}=0, j=1,..J. \tag{52}$$

$$H_{E_{m_J}E_{m_i}}=(p-\lambda_m)(\beta-1)q_m x_m^\alpha E_{mj}^{\beta-2}<0, m=1,2.$$

$$\lambda^{'}=r\lambda-H_x(x,E,\lambda)=$$

$$\begin{bmatrix} r\lambda_1-\alpha p_1 q_1 x_1^{\alpha-1}\sum_{j=1}^J E_{1j}^\beta-\lambda_1\{\rho_1\left(1-\frac{2x_1+a_{12}x_2}{\bar{K}}\right)-\alpha q_1 x_1^{\alpha-1}\sum_{j=1}^J E_{1j}^\beta\} \\ +\lambda_2\rho_2 x_2 a_{21}/\bar{K} \\ r\lambda_2-\alpha p_2 q_2 x_2^{\alpha-1}\sum_{j=1}^J E_{2j}^\beta-\lambda_2\{\rho_2\left(1-\frac{2x_2+a_{21}x_1}{\bar{K}}\right)-\alpha q_2 x_2^{\alpha-1}\sum_{j=1}^J E_{2j}^\beta\} \\ +\lambda_1\rho_1 x_1 a_{12}/\bar{K} \end{bmatrix}$$

$$x'(t)=F(x_1,x_2)=\begin{bmatrix}\rho_1 x_1\left(1-\frac{x_1}{\bar{K}}-a_{12}\frac{x_2}{\bar{K}}\right)-h_1\\ \rho_2 x_2\left(1-\frac{x_2}{\bar{K}}-a_{21}\frac{x_1}{\bar{K}}\right)-h_2\end{bmatrix}, h_m=\sum_{j=1}^J h_{mj}, m=1,2.$$

Solving for $E_{mj}$, we obtain

$$E_{mj}=(\frac{p_m-\lambda_m}{w_m}q_m x_m^\alpha\beta)^{\frac{1}{1-\beta}}, , j=1,..J, m=1,2,$$

and thus (52) becomes:

$$\begin{aligned}\lambda^{'} &= \begin{bmatrix} r\lambda_1-J\alpha(\frac{\beta}{w_1})^{\frac{\beta}{1-\beta}}((p_1-\lambda_1)q_1)^{\frac{1}{1-\beta}}x_1^{\frac{\alpha+\beta-1}{1-\beta}}-\lambda_1\rho_1\left(1-\frac{2x_1+a_{12}x_2}{\bar{K}}\right)\\ +\lambda_2\rho_2 x_2 a_{21}/\bar{K}\\ r\lambda_2-J\alpha(\frac{\beta}{w_2})^{\frac{\beta}{1-\beta}}((p_2-\lambda_2)q_2)^{\frac{1}{1-\beta}}x_2^{\frac{\alpha+\beta-1}{1-\beta}}-\lambda_2\rho_2\left(1-\frac{2x_2+a_{21}x_1}{\bar{K}}\right)\\ +\lambda_1\rho_1 x_1 a_{12}/\bar{K}\end{bmatrix}\\ x' &= \begin{bmatrix}\rho_1 x_1\left(1-\frac{x_1}{\bar{K}}-a_{12}\frac{x_2}{\bar{K}}\right)-h_1\\ \rho_2 x_2\left(1-\frac{x_2}{\bar{K}}-a_{21}\frac{x_1}{\bar{K}}\right)-h_2\end{bmatrix}, h_m=\sum_{j=1}^J h_{mj}.\end{aligned}$$

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