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Optimal Agglomerations in Dynamic Economics

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Abstract

We study rational expectations equilibrium problems and social optimum problems in infinite horizon spatial economies in the context of a Ramsey type capital accumulation problem with geographical spillovers. We identify sufficient local and global conditions for the emergence (or not) of optimal agglomeration, using techniques from monotone operator theory and spectral theory in infinite dimensional Hilbert spaces. We show that agglomerations may emerge, with any type of returns to scale (increasing or decreasing) and with the marginal productivity of private capital increasing or decreasing with

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respect to the spatial externality. This is a fairly general result indicating the importance of the network structure of the spatial externality relative to the properties of the aggregate production function. Our analytical methods can be used to systematically study optimal potential agglomeration and clustering in dynamic economics.

Keywords: Agglomeration, spatial spillovers, spillover induced instability, rational expectations equilibrium, social optimum, monotone operators.

1 Introduction

This paper shows how monotone operator theory can be used to study rational expectations equilibrium (REE) problems and social optimum (SO) problems in infinite horizon, infinite dimensional spatial economies. Our analysis is applied to an illustrative infinite horizon, infinite dimensional spatial Ramsey type capital accumulation problem where borrowing and lending on world capital markets at a rate of interest equal to the rate of discount on subjective utility are the same; quadratic adjustment costs penalize rapid movements of capital; and geographical spillovers stemming from capital accumulation across location generate a positive externality. We locate sufficient conditions on primitives that may cause potential agglomerations and spatial clustering to form or not form for both problems. Furthermore, we show how the spectral theory of compact operators allows decomposition of the infinite dimensional problem into a countable collection of tractable finite dimensional problems. Using this technique we provide explicit local stability criteria for the linearized system.

Related literature in terms of new economic geography includes work by Krugman (1996), Fujita et al. (2001), Lucas (2001), Quah (2002), Desmet and Rossi-Hansberg (2007), Ioannides and Overman (2007), Lucas and Rossi-Hansberg (2007), and others. However, to our knowledge, no one has yet provided a concise framework in which the combination of monotone operator theory, the theory of compact operators, and the decomposition techniques we develop here can be applied to infinite horizon, infinite dimensional spatial economies to study endogenous agglomeration (or non-agglomeration) for rational expectations equilibrium and the social optimum in terms of local and global analysis for Ramsey type growth models as we do here.

There is a small but growing literature which studies optimal dynamic social welfare, e.g., an analog of our SO problem in terms of Solow, Ramsey and AK type models with a trade balance where capital is mobile across space and growth occurs. In these models capital movement is modelled by local diffusion and spatial dynamics are governed by a parabolic partial differential equation (e.g.,Camacho and Zou (2004), Camacho et al. (2008), Boucekkine et al. (2009), Brito (2011), Boucekkine et al. (2013a), Boucekkine et al. (2013b)). These models also study the long-run structure of the spatial distribution of the stock of capital and the results suggest convergence to a spatially homogeneous steady state.

In contrast and complementary to this literature, the present paper studies both the SO and REE problems under non-local spatial effects which can be regarded as a spatial Romer-Lucas type of externality and derives conditions, based on the network structure of the spatial externality, for potential agglomeration emergence.

It is known that agglomerations and spatial clusters may appear with localized positive spatial spillovers when there are increasing returns. In this case the increasing returns activity concentrates to one location (e.g. Grossman and Helpman (1991)). Actually increasing returns underlie the generation of centripetal forces that favor cumulative causation and thus spatial clustering (e.g. Nocco (2005)). In our model the production technology exhibits diminishing marginal productivity with respect to private capital for any fixed value of the spatial externality and diminishing returns with respect to the spatial externality for fixed levels of private capital, although increasing social returns, in the sense of Romer (1986), are possible.

In the context of a Ramsey type growth model with the above production technology, our contribution consists of providing conditions for the emergence or not of agglomerations which does not depend on the structure of the aggregate production technology, that is increasing versus decreasing returns, but rather on the network structure of the spatial externality. We show that this structure is important since it may induce endogenous agglomeration with decreasing returns to scale and without exogenous agglomeration drivers such as boundary conditions or location advantages. Thus, combining global analysis based on monotone operator theory with local analysis based on spectral theory, we obtain valuable insights regarding the endogenous emergence (or not) of optimal agglomerations at an REE and the SO of dynamic economic systems modelled by a Ramsey type growth model.

In terms of global analysis we show that long run agglomerations do not emerge at the SO, where full internalization of the spatial externality by the social planner occurs, and the system converges to a unique spatially homogenous (or flat) steady state. This result suggests that we have identified a class of problems for which the usual value loss methods of proving turnpike theorems, which depend upon a small enough discount rate on future payoffs (e.g., McKenzie (1976), Araujo and Scheinkman (1977), Bewley (1982)), can be replaced by monotone operator methods of getting asymptotic convergence results in settings where the state space is infinite dimensional and where the results are not so dependent upon a small discount rate on future payoffs. Furthermore we treat cases in which there are externalities and solutions are not necessarily Pareto optimal. For an REE, where incomplete internalization of the spatial externality by optimizing agents occurs, we show that if the network structure of the spatial interactions is such that the operator characterizing spatial spillovers is monotonic then the system also converges to a unique spatially homogenous steady state, in general not the same as the SO steady state. On the other hand, if monotonicity does not hold, then multiple steady states are possible in an REE. In this case only one steady state will be spatially homogeneous and therefore the long-run REE could be characterized by agglomerations.

In terms of local analysis we use spectral theory to derive conditions for instability to spatial perturbations of a spatial homogeneous steady state which may lead to agglomerations. We show that having potential agglomerations at an REE does not require increasing returns to scale or the marginal productivity of private capital to be positively related to the spatial externality. On the contrary, given the structure of the spatial externality, spatial agglomeration may emerge with decreasing returns to scale and diminishing marginal productivity of private capital with respect to the spatial externality. This result indicates the importance of the network structure relative to the properties of the aggregate production function, suggesting that spatial agglomeration may emerge as the outcome of an REE when returns to scale are decreasing and nonlocal spatial spillovers occur. It is also important to note that long-run agglomerations do not emerge at the SO where the spatial externality is fully internalized and returns to scale are decreasing. Thus our result establishes a potential divergence, in terms of the spatial distribution of the stock of capital, between equilibrium outcomes and socially optimal outcomes. This divergence may appear even under decreasing returns to scale.[1] Numerical simulations confirm the results of our local analysis.

We would also like to note that there is a large literature in mathematical biology (e.g., Murray, 2003) that studies spatial agglomeration problems in infinite dimensional spaces. However, as far as we know, none of this literature deals with optimization problems as we do here. There are many differences between the "backward-looking" dynamics in mathematical biology problems and other natural science problems, and the "forward-looking" dynamics of economic problems. It is not just a simple adaptation of dynamical systems techniques to two-point boundary value problems similar to the familiar phase diagrams in textbook analysis of Ramsey type optimal growth problems and Ramsey type rational expectations problems in finite dimensional spaces. For example, our development of techniques from operator theory mentioned above allows us to locate sufficient conditions on primitives for all potential agglomerations to be removed in infinite horizon optimization problems. Furthermore, and contrary to the spirit of the Turing instability, which provides local results for non optimizing linearized dynamical systems, we obtain global results valid for the fully nonlinear optimized dynamical system.

The paper is organized as follows: Section 2 introduces the model and Section 3 characterizes equilibria with spatial spillovers. Sections 4 and 5 provide global and local analysis for the emergence (or not) of optimal potential agglomerations while Section 6 presents a detailed analytic and numerical example. Section 7 discusses intuition, shows how our methods can be used to study generalizations to spatial domains of similarly structured economic problems - in this case the well known investment problem of the firm with adjustment costs - and outlines other ways in which the present paper can be extended. So as not to disrupt the flow of the presentation, all proofs are contained in Section 8 which serves as an Appendix.

[1] It would be interesting to model the effect of introducing spillover externalities like ours as well as introducing growth and local effects like Boucekkine et al. (2013a,b) and comparing the solutions for SO and RE as we do in our paper. This is, however, beyond the scope of the current paper.

2 Geographical Spillovers in Forward-Looking Optimizing Economies

Consider a spatial economy occupying a bounded domain $\mathcal{O} \subset \mathbb{R}^d$. It is worth noting that space may be considered as either geographical (physical) space or as economic space (space of attributes related to economic quantities of interest). Without loss of generality we may assume $d = 1$.

Capital stock is assumed to be a scalar quantity that evolves in time and depends on the particular point z of the domain $\mathcal{O}$ under consideration. Thus capital is described as a function of time t and space z, i.e. $x : I \times \mathcal{O} \to \mathbb{R}$ where $I = (0, T)$ is the time interval over which the temporal evolution of the phenomenon takes place. We assume an infinite horizon model, i.e. $I = \mathbb{R}_+$, and denote the capital stock at point $z \in \mathcal{O}$ at time t by $x\,(t, z)$. The spatial behavior of x is modelled by assuming that the functions $x(t, \cdot)$ belong for all t to an appropriately chosen function space $\mathbb{H}$. Therefore, as is common in the abstract theory of evolution equations, we assume that x is described by a vector-valued function $\breve{x} : I \to \mathbb{H}$, where $I = (0, T)$, and $\mathbb{H}$ is the function space that describes the spatial properties of the function x.[2] Different choices for $\mathbb{H}$ are possible. A convenient choice is to let $\mathbb{H}$ be a Hilbert space,[3] e.g., $\mathbb{H} = L^2(\mathcal{O})$, the space of square integrable functions on $\mathcal{O}$, or an appropriately chosen subspace, e.g. $L^2_{per}(\mathcal{O})$, the space of square integrable functions on $\mathcal{O} = [-L, L]$ satisfying periodic boundary conditions (this would model a circular economy).

Consumption is assumed to be a local procedure and modeled by a vector valued function $c : I \to \mathbb{H}$ interpreted in a similar fashion to the capital stock function x discussed above. By the scalar quantity $c(t, z)$ we denote consumption at time $t \in I$ at the spatial point $z \in \mathcal{O}$. Consumption is associated with a utility function $U : I \times \mathbb{H} \to \mathbb{R}$. The utility of consumption at time $t \in I$ and at point $z \in \mathcal{O}$ is given by $U(c(t, z))$.

Production at each location is determined by local inputs and by nonlocal

[2] This function is defined such that $(\breve{x}(t))(z) := x(t, z)$; to avoid cumbersome notation in the sequel we denote the $\mathbb{H}$-valued function $\breve{x}$ using the same notation x and therefore by $x(t)$ we denote an element of $\mathbb{H}$, which is in fact a function $x(t) : \mathcal{O} \to \mathbb{R}$ which describes the spatial structure of the capital stock at time t.

[3] Other choices are possible, where $\mathbb{H}$ is a Banach space, e.g., $\mathbb{H} = C(\mathcal{O})$ the set of continuous functions on $\mathcal{O}$ or $\mathbb{H} = L^p(\mathcal{O})$, $p \neq 2$, $1 \leq p \leq \infty$, the set of p-integrable functions on $\mathcal{O}$. In the present paper, we restrict our attention to Hilbert spaces, though many of our results may be extended to Banach space.

procedures. At time t, output production at each location z is described by the production function f with inputs being capital $x(t,z)$ and labour $\ell(t,z)$, at this location, and also spatial effects describing the effect that capital stocks on locations $s \in \mathcal{O}$ at time t have on production at location z. Without loss of generality and to concentrate on the impact of geographical spillovers, we assume that labour input is normalized to unity $\ell(t,z) = 1$.

Spillover effects play a very important role in this study. We adopt the notation $\check{X}(t,z)$ for the spillover effects at time t on site z. Production at time t and site z is given by the production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ in terms of $f(x(t,z), \check{X}(t,z))$.

Clearly the modelling of spillover effects is crucial. We will adopt two alternative ways:

(a) Exogenously given spillovers $\check{X}(t,z) = X^e(t,z)$ where $X^e : I \to \mathbb{H}$ is a known function or

(b) Endogenously determined spillovers by the state of the system, i.e. $\check{X}(t,z) = (\mathsf{T}x)(t,z)$ where $\mathsf{T} : \mathbb{H} \to \mathbb{H}$ is a mapping (operator) taking the state of the system at time t, $x(t,.) \in \mathbb{H}$ and providing the spillovers $\check{X}(t,\cdot) \in \mathbb{H}$.

If we regard spillovers as the spatial externality case, (b) indicates internalization of the externality. When adopting modelling strategy (b), spillover effects at time t and site z are given by the intermediate quantity:

$$X(t,z) = \int_{\mathcal{O}} w(z,s)x(t,s)ds \tag{1}$$

where $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is an integrable kernel function modeling the effect that position s has on position z. This introduces nonlocal (spatial) effects, and may be understood as defining a mapping which takes an element $x(t,\cdot) \in \mathbb{H}$ and maps it to a new element $X(t,\cdot) \in \mathbb{H}$ such that (1) holds for every $z \in \mathcal{O}$. This mapping is understood as an operator $\mathsf{T} : \mathbb{H} \to \mathbb{H}$.

Some comments are due on the interpretation of the intermediate variable X. The quantity $X(t,z)$ will have different interpretations in different contexts. If $X(t,z)$ represents a type of knowledge which is produced proportionately to capital usage, it is natural to assume that the kernel $w(\zeta), \zeta = z - s$ is single peaked bell-shaped, with a maximum at $\zeta = 0$, and with $w(\zeta) \to 0$ for sufficiently large ζ. If $X(t,z)$ reflects aggregate benefits of knowledge produced at (t, s) for producers at (t, z) and damages to production at (t, z) from usage of capital at (t, s), then non-monotonic shapes of w, with for example a single peak at $\zeta = 0$ and two local minima located symmetrically around $\zeta = 0$, with negative values indicating damages to production at z from usage of capital at s, are plausible. This production function could be considered as a spatial version of a neoclassical production function with Romer (1986) and Lucas (1988) externalities modelled by geographical spillovers given by a Krugman (1996), Chincarini and Asherie (2008) specification.

Let us now fix a time t and consider a site $z \in \mathcal{O}$. Let $x(t, z)$ be the capital stock at this site and $\check{X}(t, z)$ the spillover effects at site z from all the other sites in $\mathcal{O}$. We treat the site as analytically equivalent to an agent located on the site who has access to valuable technology $f\left(x\left(t, z\right), \check{X}(t, z)\right)$ that generates rents. The individual (or the site) has access to the world capital market and can borrow x at an exogenous interest rate $r\left(t\right)$ against the present value of future rents from operating $f\left(x\left(t, z\right), \check{X}(t, z)\right)$. The agent faces quadratic adjustment costs, $\frac{\alpha}{2}\left[\frac{\partial x(t,z)}{\partial t}\right]^2$, to adjusting the capital stock and experiences geographical spillovers $\check{X}$, while the capital stock depreciates at a fixed rate η. The instantaneous budget constraint facing the individual or the site, z, at t can therefore be written as:

$$c\left(t, z\right) + \frac{\partial x\left(t, z\right)}{\partial t} = f\left(x\left(t, z\right), \check{X}\left(t, z\right)\right) - \eta x\left(t, z\right) - \frac{\alpha}{2}\left[\frac{\partial x\left(t, z\right)}{\partial t}\right]^2. \quad (2)$$

Assuming for simplicity constant r, the lifetime budget constraint expressed in present discounted value form for the agent is obtained if (2) is multiplied by e^{-rt} , integrated over t from $t = 0$ to ∞ (assuming momentarily the existence of exponentially bounded solutions of (2)) and all debts are required to be paid off.[4] Defining[5]

$$\lambda = r + \eta, \quad u(t,z) = \frac{\partial x}{\partial t}(t,z) = x'(t,z)$$

leads to a reformulation of the instantaneous budget constraint in a static form as:

$$0 = \mathcal{C}(z) := \tag{3}$$
$$\int_0^\infty e^{-rt}[x_0 + f\left(x(t,z), \check{X}(t,z)\right) - \lambda x(t,z) - c(t,z) - \frac{\alpha}{2}u^2(t,z)]dt$$

which holds a.e. in $\mathcal{O}$. The same constraint over the whole domain $\mathcal{O}$ takes the form:

$$0 = \mathcal{C}^\Diamond := \tag{4}$$
$$\int_{\mathcal{O}}\int_0^\infty e^{-rt}[x_0 + f\left(x(t,z), \check{X}(t,z)\right) - \lambda x(t,z) - c(t,z) - \frac{\alpha}{2}u^2(t,z)]dtdz.$$

To summarize, in this model the use or production of capital on a site affects other sites through the geographical spillovers, while a site can borrow or lend capital using the world capital markets. Depending on the type of the agent we can specify $\check{X}$ accordingly. An individual located at z treats geographical spillovers as parametric and exogenously given, $\check{X}(t,z) = X^e(t,z)$, while a social planner fully internalizes geographical spillovers so that $\check{X}(t,z) = X(t,z) = \int_{\mathcal{O}} w(z-s)x(t,s)ds$.

[4]We assume that the agent (or the site) has discounted future income greater than any desired borrowing at any point in time. Thus the capitalized, at the rate r, sum of the site's future income is large enough to pay off the debt incurred by borrowing. To put it differently, we assume that each site z has enough capital so that it is "solvent" in the present value sense at each point in time. If for example initial capital is zero and initial bonds are zero, then the solvency condition is obtained by multiplying both sides of (2) by e^{-rt} and using $x(0,z) = b(0,z) = 0$, as:

$$\int_0^\infty e^{-rt}\left[f\left(x(t,z), \check{X}(t,z)\right) - (r+\eta)\,x(t,z) - \frac{a}{2}\frac{d^2x}{dt^2}(t,z)\right]dt \geq 0$$

where $b(t,z)$ is "bonds" held by z at time t, $b(t,z) < 0$ is debt, and $b(t,z) > 0$ is assets.

[5]By $'$ we denote the derivative with respect to time of the Hilbert space valued function $x: I \to \mathbb{H}$.

3 Equilibria with Geographical Spillovers

3.1 Rational expectations and social optimum equilibria

The objective is to maximize the utility of consumption either locally or globally. Both cases are considered in this work: the maximization of local consumption when spillovers are exogenous will be called a rational expectations (RE) problem, while the maximization of global utility with endogenous spillovers will be called a social optimum (SO) problem.

Given the (local) utility function U we now define the functionals $J_{RE} : \mathbb{H} \to \mathbb{R}$ and $J_{SO} : \mathbb{H} \to \mathbb{R}$ whose action on the consumption function c is as follows:

$$(J_{RE}(c))(z) := \int_0^\infty e^{-\rho t} U(c(t,z))dt, \tag{5}$$

$$J_{SO}(c) := \int_{\mathcal{O}} \psi(z)(J_s c)(z)dz = \int_{\mathcal{O}} \int_0^\infty e^{-\rho t}\psi(z) U(c(t,z))dtdz. \tag{6}$$

The functional J_{RE} provides the discounted - by a subjective utility discount rate $\rho > 0$ - utility of consumption $c(t,z)$ in the infinite horizon at location z. On the other hand, the functional J_{SO} provides the discounted utility of consumption averaged over the whole domain $\mathcal{O}$, with a weight function ψ which will be set to one without loss of generality.

We are now in a position to define the two optimization problems faced by either an arbitrary representative agent at location z (RE problem) or a social planner (SO problem).

Definition 1 (RE and SO problems)**.**

$$\text{RE problem: } \max_{c\in\mathcal{A}} J_{RE} \quad \text{subject to (3) with } \check{X}(t,z) = X^e\,(t,z) \tag{7}$$

$$\text{SO problem: } \max_{c\in\mathcal{A}} J_{SO} \quad \text{subject to (4) with } \check{X}(t,z) = X\,(t,z) \tag{8}$$

where J_{RE} and J_{SO} are the functionals defined in (5) and (6) respectively and $\mathcal{A}$ is the acceptable consumption set (typically $c(t,z) \geq 0$ a.e. in $\mathcal{O}$ would suffice).

3.2 Standing assumptions

In developing our model we make the three assumptions below, which will be assumed to hold through the paper unless explicitly stated otherwise.

Note that some of these assumptions can be relaxed considerably for some of our results. To simplify the exposition, we assume the stronger conditions that guarantee that all of our results hold uniformly, and we make specific remarks concerning the possibility of relaxing them in the particular cases where this is feasible.

When RE equilibria are concerned, we need to make an assumption on X^e.

Assumption 1. We assume that the exogenously given spillover function $X^e \in \mathbb{H}$.

We make the following assumptions on the primitives of the economy.

Assumption 2. We assume that

(a) The influence kernel function $w : \mathcal{O} \times \mathcal{O} \to \mathbb{R}$ is continuous and symmetric, i.e. $w(z,s) = w(s,z) = w(z-s)$.

(b) The production function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly increasing, strictly concave function of the (real) variables (x, X).

(c) The utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is an increasing and strictly concave C^2 function in consumption c and satisfies the Inada conditions[6]

$$\lim_{c \to 0} \partial_c U(c) = +\infty, \quad \lim_{c \to +\infty} \partial_c U(c) = 0.$$

Under Assumption 2(a), equation (1) defines an integral operator $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ whose action on a function x is defined as:

$$(\mathsf{K}x)(t,z) := \int_{\mathcal{O}} w(z-s)x(t,s)ds. \tag{9}$$

Since $\mathcal{O}$ is a bounded domain, the continuity assumption leads us to the result that $w \in L^2(\mathcal{O})$ so that by standard results in the theory of integral operators, K is a compact bounded operator which, furthermore, by the symmetry of the kernel function w, is a self-adjoint operator.

We impose the following smoothness assumptions on the production function.

[6] $\partial_u \phi$, $\partial_{uv} \phi$ denote first and second order partial derivatives of a function ϕ, with respect to variables u, v.

Assumption 3. We assume that $f \in C^2(\mathbb{R}^2)$, with uniformly bounded second derivatives and furthermore, it holds that

$$\lim_{(x,X)\to(0,0)} f_x(x,X) > C, \text{ and } \lim_{(x,X)\to(0,0)} f_X(x,X) > C,$$

for a positive constant C.

The positive constant C in Assumption 3 will be chosen typically larger than λ (in the REE case, or a multiple of that depending on the choice of the kernel w in the SO case). This will be needed to guarantee the existence of steady-state solutions (see Theorem 2 and its proof). Assumption 3 holds for typical production functions, e.g., for the Cobb-Douglas production function.[7]

3.3 The rational expectations and social optimum equilibria

The rational expectations (RE) and SO equilibrium problems, (7) and (8) respectively, can be reformulated into a form which is more convenient to handle, using a generalization of the Fisher separation principle, for this infinite dimensional economy.

The optimization problem (7) can be broken down, by expressing the associated Lagrangian in a separable form, into two distinct but interrelated sub-problems: A problem corresponding to the choice of the agent's consumption, $c(t,z)$, to maximize discounted lifetime utility subject to a lifetime budget constraint; and a problem corresponding to the choice of the agent's investment, $u(s,z) = x^{'}(t,z)$, to maximize the agent's interests in the economy by maximizing the location's present value. This is essentially a generalization of the Fisher separation principle for a single-owner firm which implies that if the optimization problem (7) admits a solution (c^*, x^*), then there exists a $\Lambda : \mathcal{O} \to \mathbb{R}_+$ such that the solution of this problem can be split into two separate problems:
(a) A consumption optimization problem which, upon choice of Λ, assumes the form

$$\max_{c(\cdot,z)} \int_0^\infty e^{-\rho t}(U(c(t,z) - \Lambda(z)e^{-rt}c(t,z))dt. \tag{10}$$

(b) An investment optimization problem, independent of the choice of Λ,

[7]In fact for the Cobb-Douglas, these limits are infinite.

according to which $x(t,z) = x_0(z) + \int_0^t u(s,z)dz$ is chosen so as to solve

$$\max_{x'(\cdot,z)} \int_0^\infty e^{-rt}\{f(x(t,z), X^e(t,x)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt, \tag{11}$$

with $\lambda = r + \eta$. As the following remark shows, it is reasonable to assume that $r = \rho$.

Remark 1. The first order necessary condition for problem (10) is $U'(c(t,z)) = e^{(\rho-r)t}\Lambda(z)$. Since $\Lambda(z)$ is positive and independent of time, we see that marginal utility goes to infinity, i.e. $c(t,z)$ goes to zero, if the individual discounts the future higher than r and vice versa if the consumer discounts the future less than r. Thus, if we want to study a steady state for $c(t,z)$, we can assume that the consumer discounts at the same rate as r.

Similarly an application of the Fisher separation to the social planner problem (8) shows that the solution of this problem may be obtained by the solution of two separate problems:
(a) A consumption optimization problem which upon choice of $\Lambda^\lozenge \in \mathbb{R}_+$ independent of z assumes the form

$$\max_c \int_0^\infty \int_{\mathcal{O}} e^{-r\,t}\left(\psi(z)U(c(t,z)) - \Lambda^\lozenge c(t,z)\right) dz\,dt. \tag{12}$$

(b) An investment optimization problem, independent of the choice of $\Lambda^\lozenge$, where u is chosen so that $x(t,z) = x_0(z) + \int_0^t u(s,z)ds$ solves

$$\max_{x'} \int_0^\infty \int_{\mathcal{O}} e^{-r\,t}\left(f(x(t,z), (\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\,dt. \tag{13}$$

The above results can be easily shown, e.g., for the REE problem, by using the Lagrangian $\mathcal{L}(z) = \int_0^\infty e^{-\rho t}U(c)\,dt + \Lambda(z)\,\mathcal{C}(z)$.

We note that in both cases only the solution of the second problems (11) and (13) respectively, which are independent of the choice of the Lagrange multiplier Λ, is required to characterize the spatial structure of the capital stock. This problem is essentially equivalent to a calculus of variations problem. At this point the following definition is required:

Definition 2 (The RE and SO problems)**.**

(i) RE problem:

$$\max_{x(\cdot,z)} \int_0^{\infty} e^{-rt}\{f(x(t,z), X^e(t,x) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt, \ \ \forall z \in \mathcal{O}. \tag{14}$$

(ii) SO problem:

$$\max_{x} \int_0^{\infty}\int_{\mathcal{O}} e^{-r\,t}\left(f(x(t,z),(\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\,dt. \tag{15}$$

Note that the RE problem is a calculus of variations problem where for each fixed $z \in \mathcal{O}$ we find a function $x(\cdot,z) : \mathbb{R}_+ \to \mathbb{R}$ that maximizes the functional

$$J_{RE}(x(\cdot,z);z) := \int_0^{\infty} e^{-rt}\{f(x(t,z), X^e(t,x)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\}dt.$$

On the other hand the SO problem is a calculus of variations problem where we find a vector valued function $x : \mathbb{R}_+ \to \mathbb{H}$ that maximizes the functional

$$J_{SO}(x(\cdot)) := \int_0^{\infty}\int_{\mathcal{O}} e^{-r\,t}\left(f(x(t,z),(\mathsf{K}x)(t,z)) - \lambda x(t,z) - \frac{\alpha}{2}(x^{'}(t,z))^2\right) dz\,dt.$$

3.4 Existence of equilibria and first order conditions

We now discuss the existence of RE and SO equilibria. The following operators will be needed.

Definition 3. Define the nonlinear operators $A_\nu : \mathbb{H} \to \mathbb{H}$, $\mathsf{A}_\nu : \mathbb{H} \to \mathbb{H}$, $\nu = RE, SO$, by

$$\begin{aligned} A_{RE}x &:= -\alpha^{-1}(f_x(x,\check{X}) - \lambda), \ \ \check{X} = X^e, \\ A_{SO}x &:= -\alpha^{-1}(f_x(x,\check{X}) + \mathsf{K}f_X(x,\check{X}) - \lambda), \ \ \check{X} = \mathsf{K}x \end{aligned}$$

and

$$\begin{aligned} \mathsf{A}_{RE}x &:= -\alpha^{-1}(f_x(x,\check{X}) - \lambda), \ \ \check{X} = \mathsf{K}x, \\ \mathsf{A}_{SO}x &:= -\alpha^{-1}(f_x(x,\check{X}) + \mathsf{K}f_X(x,\check{X}) - \lambda), \ \ \check{X} = \mathsf{K}x. \end{aligned}$$

Note that the operators A_{SO} and A_{SO} coincide, but we include both for notational consistency.

Theorem 1.

(a) *The optimization problems (14) and* (15) *admit a solution.*

(b) *The first order necessary condition for problems (14) and* (15) *is of the form*[8]

$$x'' - rx' - A_\nu x = 0, \quad \nu = RE, SO \tag{16}$$

where A_ν are the nonlinear operators of Definition 3.

Remark 2. An alternative would be to use the maximum principle, in terms of the current value Hamiltonian $\mathcal{H}_\nu$, $\nu = RE, SO$ where

$$\mathcal{H}_{RE} := f(x, \check{X}) - \lambda x - \frac{\alpha}{2}u^2 + p\,u, \;\; \check{X} = X^e, \tag{17}$$

$$\mathcal{H}_{SO} := \int_{\mathcal{O}} (f(x, \check{X}) - \lambda x - \frac{\alpha}{2}u^2 + p\,u) dz, \;\; \check{X} = \mathsf{K}x. \tag{18}$$

Applying the Pontryagin maximum principle formally and maximizing over u, the Hamiltonian equations are easily seen to be equivalent to (16).

The form of the first order conditions (16) motivates the following definition:

Definition 4 (RE and SO equilibrium)**.** A solution $x : \mathbb{R}_+ \to \mathbb{H}$ of the nonlinear integro-differential equation

$$x'' - rx' - \mathsf{A}_\nu x = 0 \tag{19}$$

is called an RE equilibrium if $\nu = RE$ and an SO equilibrium if $\nu = SO$.

Remark 3. Note that in the RE equilibrium we use the operator A_{RE} rather than the operator A_{RE}. This means that the agent makes her decision locally using $\check{X} = X^e$ but her decision changes the background spillovers to $\check{X} = \mathsf{K}x$.

[8] If the maximization is performed on a closed convex subspace of $\mathbb{H}$ then the first order condition (16) must be replaced by a variational inequality. The first order necessary conditions have to be complemented with the transversality condition

$$\lim_{t\to\infty} e^{-rt} x x' = \lim_{t\to\infty} \frac{1}{2} e^{-rt} (x^2)' = 0.$$

where the limits are interpreted in the weak sense.

4 Optimal Agglomerations in the Long Run: Global Analysis

Having defined the RE equilibrium and the SO equilibrium in the context of geographical spillovers, we turn to the study of the long-run characteristics of these equilibria. These characteristics will provide information about the potential emergence of optimal agglomerations as long-run equilibria, as well as information about potential differences in the long run between the RE equilibrium and the SO. Analyzing these issues requires global analysis to study existence, uniqueness and stability regarding both types of equilibria.

The long-term behavior of the system will be provided by the steady-state solutions of the equilibrium equations given in Definition 4. These are solutions without any temporal variability, and are given as solutions of the nonlinear operator equations $\mathsf{A}_\nu x = 0$, $\nu = RE, SO$. In general, the solution of this operator equation presents spatial variability, i.e., $x = x(z)$, but if operator K has the following property (which we will henceforth refer to as Property $\mathcal{P}$),

$$K\bar{x} \text{ is independent of } z, \text{ if } \bar{x} \text{ is independent of } z, \qquad (\mathcal{P})$$

then the solution of $\mathsf{A}_\nu x = 0$ may be uniform in space. We will call such an equilibrium a flat equilibrium or flat steady state. Property $\mathcal{P}$ holds in the case of periodic boundary conditions (see Proposition 3) so that a flat steady state always exists when periodic boundary conditions are taken into account. The flat steady state is the solution of an algebraic nonlinear equation.

If a flat steady state exists, it is globally stable and if, furthermore, the steady states of the system are unique, then the dynamics of the system preclude the emergence of spatially varying solutions in the long run; therefore, they preclude the emergence of spatial pattern formation in the economy. We will refer to such patterns as potential "agglomeration patterns" to be in accordance with the terminology of economic geography. Furthermore, since such spatial patterns may occur as a result of optimizing behavior, we will henceforth refer to them as "optimal agglomeration".

The following theorem provides important information on the long-run dynamics.

Theorem 2. *Consider the case $\nu = SO$.*

(a) The operator equation $\mathsf{A}_\nu x = 0$*, has a unique bounded positive solution.*

(b) All bounded solutions of $x'' - rx' - \mathsf{A}_\nu x = 0$ *have as weak limit the solution of* $\mathsf{A}_\nu x = 0$*.*

(c) If A_ν *has the additional property of having* $(I + \mathsf{A}_\nu)^{-1}$ *compact or if it is strongly monotone then (b) holds but the convergence is strong.*

(d) Assume that Property $\mathcal{P}$ *holds. Then* $x(t) \rightharpoonup s$ *as* $t \to \infty$*, where* s *is the (unique) spatially homogeneous solution of* $\mathsf{A}_\nu s = 0$ *(weak global non-agglomeration result). If* $(I + \mathsf{A}_\nu)^{-1}$ *is compact or* A_ν *is strongly monotone, then* $x(t) \to s$ *in* $\mathbb{H}$ *as* $t \to \infty$ *(strong global non-agglomeration result).*

The conditions of compactness of $(I+\mathsf{A}_\nu)^{-1}$ may hold for the case where K is an unbounded operator, as for example the Laplacian. The condition of strong monotonicity of A_ν may hold if for example we assume λ to depend on x, i.e., if we assume a faster than linear decay of capital stock. Under such assumptions we may guarantee strong global non-agglomeration results. The weak global non-agglomeration result precludes the occurrence of large scale spatial structures, but does not preclude the occurrence of small scale structures. Small scale structures are not interesting from the economic theory point of view since they may be regarded as small scale spatial fluctuations, similar to noise, which do not add to long range patterns.

The restriction of considering only bounded solutions of (19) (i.e.,solutions that satisfy the condition that $\sup_{t \in \mathbb{R}_+} ||x(t)|| < \infty$) is not too serious, and in fact it is very natural, since we are considering a controlled system. For the optimally controlled system the saddle path structure restricts us to the stable manifold, which is the class of solutions of interest to economic theory and correspond to bounded solutions.

What would the situation be for the REE? In this case, the operator A_ν is not necessarily maximal monotone because of the concavity of f as in the SO case. This means that the general asymptotic result of Theorem 2 cannot be directly applied, and so we may not preclude the occurrence of agglomerations in the REE. There are cases of course where A_{RE} can be maximal monotone. In these cases Theorem 2 is still applicable, and global non-agglomeration holds. However, this will impose further restrictions on both

f and K. For instance, using a perturbation result for maximal monotone operators (see e.g. Barbu (2010), Corollary 2.6 p. 44), which states that the sum of a maximal monotone and a bounded, demicontinuous, monotone operator is still a maximal monotone operator, we can express A_{RE} as $\mathsf{A}_{RE} = \mathsf{A}_{SO} + \mathsf{B}$, where B is defined by $\mathsf{B}(x) := \alpha^{-1}\mathsf{K}\partial_X f(x, \mathsf{K}x)$ and conclude that A_{RE} is K (which is in fact a condition on the spillover effects operator) and that $\langle \partial_X f(x_2, \mathsf{K}x_1) - \partial_X f(x_2, \mathsf{K}x_2), x_1 - x_2 \rangle \geq 0$ for any $x_1, x_2 \in \mathbb{H}$. This second condition may hold for instance if the production function is of the form $f(x, X) = X\Phi(x)$ where Φ is a concave function. However, for general production functions and kernels, as for instance the case in which we have decreasing returns with respect to the spatial externality, this property is not expected to hold and agglomerations may emerge as the linear theory presented in the next section indicates. The existence of a unique spatially homogeneous steady state for the REE case, if the condition $\mathcal{P}$ holds can be deduced by a simple modification of the relevant proof for the SO case.

At this point we summarize and comment upon our global results as stated in the above theorem and proposition, focusing on their economic meaning.

For strictly concave production functions f, if the steady state equation $\mathsf{A}_{SO}x = 0$ admits a flat solution then all bounded solutions of the time dependent system will finally tend weakly to that flat solution as $t \to \infty$. Thus agglomeration is not a socially optimal outcome in this case. The uniqueness of the solution of $\mathsf{A}_{SO}x = 0$ precludes the existence of any steady state other than the flat steady state as long as total spillover effects are the same across all sites of the spatial domain. Then the socially optimal spatial distribution of economic activity is the uniform distribution in space. This is always true in the case of periodic boundary conditions, when α is independent of z. This result suggests that monotone operator methods can be used to obtain asymptotic convergence results in settings where the state space is infinite dimensional and where the results are not so dependent upon a small discount rate on future payoffs.[9]

When the long-run behavior of the REE and the SO are compared, we

[9]It should be noted that we obtain global asymptotic stability results for the SO, which are independent of r, whereas some results in classical turnpike theory obtain global asymptotic stability when r is close enough to zero.

note that:

(i) Convergence to the REE steady state is not guaranteed by the strict concavity of the production function, as in the SO case, but depends on the structure of production and the spatial externality;

(ii) If a unique globally stable REE steady state exists it will be flat. Hence for both the REE and SO the unique steady state is the flat steady state.[10]

(iii) If (ii) is not true, a more complex behavior is expected in the REE. In this case, multiple REE steady states cannot be eliminated, and a potential agglomeration at the RE equilibrium takes the form of instability of the flat steady state.[11]

Therefore to study the emergence of agglomeration at the REE in terms of the stability of a RE equilibrium flat steady state, we turn to local analysis.

5 Agglomeration Emergence and Local Spillover Induced Instability

We now focus on the spatiotemporal evolution the state of the RE system for initial conditions close to the spatially homogeneous steady state $\bar{x}$, in the case where the operator A_{RE} is not a maximal monotone operator. Our strategy is to study the evolution of small perturbations around this spatially homogeneous steady state $\bar{x}$ and consider the question of whether small fluctuations are likely to be attenuated or suppressed by the Euler equation. These fluctuations will have a spatio-temporal structure. Assuming we are close to the initial spatially homogeneous steady state allows us to consider initial conditions for the Euler equation of the form $x(0, z) = \bar{x} + \epsilon\hat{x}(0, z)$ and then consider solution of the Euler equation $x(t, z)$ corresponding to this initial condition which we will express as $x(t, z) = \bar{x} + \epsilon\hat{x}(t, z)$. Substituting this ansatz into the nonlinear Euler equation, and assuming ϵ to be small,

[10]The REE and SO steady states will in general be different from each other, which calls for spatially dependent economic policy if the SO steady state is to be attained.

[11]Note that we work on a symmetric space throughout so that any agglomeration that arises is due to endogenous forces, not to any kind of exogenous imposition of structure on the problem.

using the smoothness assumptions, we may linearize the system around the spatially homogeneous steady state and obtain a linear evolution equation that will provide an approximation for the spatiotemporal behavior of the fluctuations $\bar{x}(t, z)$. While there results are local they provide us with an indication of the behavior of the system as far as agglomeration formation is concerned.

If the initial spatial inhomogeneity is suppressed in time, then the flat steady state is stable, thus leading to a new flat steady state and no agglomeration. If it grows in time, then this corresponds to a precursor instability, which may lead to pattern formation and agglomeration in the long run. We call this type of instability, which in the context of economic geography may lead to equilibrium agglomeration, *optimal spillover induced instability.* Furthermore, the spatial dependence of the growing solutions will provide us with an idea of the spatial structure of the possible agglomeration pattern.

As in this section we focus on the REE case, when A_{RE} is not maximally monotone, we are absolutely free to relax positivity conditions on K, and we may also consider cases where $\partial_{xX} f < 0$, i.e. cases that the marginal product of private capital is decreasing with respect to the spatial externality. Although this is not possible with a Cobb-Douglas technology, it is possible with a CES technology which means that the linear stability of the flat steady state can be analyzed under more general conditions.[12] However, for the sake of illustration we will also present the corresponding linearized theory for the SO case, eventhough Theorem 2 provides a complete description in the full nonlinear case, so as to show that the result of the absence of long run agglomeration patterns for the SO case is compatible with the predictions of the linear theory.

The linearized stability of the flat equilibrium is determined by the spectral theory of the following linear operators:

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[12]For a CES production function $f(x, X) = A\left[\beta x^{-\theta} + (1-\beta) X^{-\theta}\right]^{-\frac{\gamma}{\theta}}$, $A > 0$, $\theta \geq -1$, $0 < \beta < 1$, with degree of homogeneity γ, and elasticity of substitution $\hat{\sigma} = \frac{1}{1+\theta}$ we have:

$$\partial^2_{xX} f = (\gamma + \theta)(1-\beta)\beta\gamma A x^{-(1+\theta)} X^{-(1+\theta)} \left[\beta x^{-\theta} + (1-\beta) X^{-\theta}\right]^{-2-\frac{\gamma}{\theta}}, \qquad (20)$$

The condition for $\partial^2_{xX} f < 0$ can be expressed in terms of the elasticity of substitution between x and X as $\hat{\sigma} > \frac{1}{1-\gamma}$. Thus a negative cross partial requires decreasing returns to scale and $\hat{\sigma} > 1$.

Definition 5. For a flat steady state $\bar{x}$, let

$$\begin{aligned} s_{11} &:= \alpha^{-1}\partial^2_{xx} f(\bar{x}, \mathsf{K}\bar{x}) < 0, \\ s_{22} &:= \alpha^{-1}\partial^2_{XX} f(\bar{x}, \mathsf{K}\bar{x}) < 0, \\ s_{12} &:= \alpha^{-1}\partial^2_{xX} f(\bar{x}, \mathsf{K}\bar{x}), \end{aligned}$$

and define the linear bounded operators $\mathsf{L}_\nu : \mathbb{H} \to \mathbb{H}$ by[13]

$$\begin{aligned} \mathsf{L}_{RE}\hat{x} &:= s_{11}\hat{x} + s_{12}\mathsf{K}\hat{x} \\ \mathsf{L}_{SO}\hat{x} &:= s_{11}\hat{x} + 2s_{12}\mathsf{K}\hat{x} + s_{22}\mathsf{K}^2\hat{x}. \end{aligned}$$

A typical linearization argument shows that these operators govern the behavior of spatio-temporal perturbations, $\hat{x}$, from the flat steady state $\bar{x}$: Inserting the ansatz $x = \bar{x} + \epsilon\hat{x}$ into (19) and expanding in ϵ, we obtain the linearized equation for the evolution of the perturbation $\hat{x}(t, z)$ as follows:

$$\hat{x}'' - r\hat{x}' + \mathsf{L}_\nu\hat{x} = 0, \quad \nu = RE,\ SO. \tag{21}$$

From the point of view of pattern formation, a spectral decomposition of (21) may provide detailed results concerning the onset and development of instability.

Proposition 1. *Let $\{\mu_j\}$ be the eigenvalues of operator K and $\{\phi_j\}$ the corresponding eigenfunctions. Then,*

(a) An arbitrary initial perturbation of the flat steady state of the form

$$\hat{x}(0, z) = \sum_j a_j\phi_j(z), \quad \hat{x}'(0, z) = \sum_j b_j\phi_j(z),$$

evolves under the linearized system (21) to

$$\hat{x}_\nu(t, z) = \sum_j c_{\nu,j}(t)\phi_j(z)$$

where $\{c_{\nu,j}(t)\}$ is the solution of the countably infinite system of ordi-

[13]By the standard theory of integral operators, K^2 is in turn an integral operator.

nary differential equations

$$c''_{\nu,j} - rc'_{\nu,j} + \Lambda_{\nu,j}c_{\nu,j} = 0, \quad \nu = RE, SO, \quad j \in \mathbb{N} \tag{22}$$
$$c_{\nu,j}(0) = a_j, \;\; c'_{\nu,j}(0) = b_j$$

where

$$\Lambda_{RE,j} = s_{11} + s_{12}\mu_j$$
$$\Lambda_{SO,j} = s_{11} + 2s_{12}\mu_j + s_{22}\mu_j^2.$$

(b) *There are three possible types of dynamic behavior, depending on the values of* $\Lambda_{\nu,j}$*:*

1. *If* $\Lambda_{\nu,j} < 0$, *then* $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ *where* $\sigma_1 < 0 < \frac{r}{2} < \sigma_2$ *(saddle path behavior).*
2. *If* $0 < \Lambda_{\nu,j} < \left(\frac{r}{2}\right)^2$, *then* $c_{\nu,j}(t) = \bar{A}_j e^{\sigma_1 t} + \bar{B}_j e^{\sigma_2 t}$ *where* $0 < \sigma_1 < \frac{r}{2} < \sigma_2$ *(unstable solutions).*
3. *If* $\left(\frac{r}{2}\right)^2 < \Lambda_{\nu,j}$, *then* $c_j(t) = e^{\frac{r}{2}t}\left(\bar{A}_j \cos(\sigma t) + \bar{B}_j \sin(\sigma t)\right)$, $\sigma \in \mathbb{R}$

and $\bar{A}_j$, $\bar{B}_j$ *are constants related to the initial conditions.*

The above general form of the linearized solutions clarifies possible pattern formation patterns that may arise from small perturbations of the flat steady state. Assume for example that for every mode ϕ_j we have that $\Lambda_{\nu,j} < 0$. Then, according to the standard saddle path arguments, the control procedure will lead the state of the system to the stable manifold and we will observe exponential decay of the initial perturbation $\hat{x}(0, z)$ to 0. That is, the system will return to the flat steady state. Therefore, in case $\Lambda_{\nu,j} < 0$ for all j, we do not expect spatially varying patterns in the long run. In all other cases, we lose the saddle path property in the linearized system. Such cases may destabilize the system and take it away from the flat equilibrium state $\bar{x}$. How this is done depends on the type of initial perturbation. If the initial perturbation contains modes for which $0 < \Lambda_{\nu,j} < \left(\frac{r}{2}\right)^2$, then for generic initial conditions we will have the linear combination of two (increasing) exponentials, one with rate larger than $r/2$ and one with rate smaller than $r/2$. Clearly such a general combination will not satisfy the transversality condition (8) and its general validity is doubtful. However, for particular initial conditions (such that $\bar{B}_j = 0$) the remaining part satisfies the transversality condition and will lead to pattern formation with a mechanism that resembles Turing instability. Note that for a randomly selected initial perturbation $\hat{x}(0,z)$, it is not expected that $\bar{B}_j = 0$ so this mechanism for pattern formation will lead to patterns as long as the initial perturbations from the flat steady state are carefully selected. One the other hand, if the initial perturbation contains modes j such that $\left(\frac{r}{2}\right)^2 < \Lambda_{\nu,j}$, then we obtain a spatio-temporal pattern which satisfies the transversality conditions (8) for any choice of initial conditions. Therefore, for "generic" initial perturbations from the flat steady state, we obtain patterns which are compatible with the transversality conditions and correspond to temporal growth accompanied with temporal oscillations. This can be compared to a Turing-Hopf-type pattern formation mechanism. We then obtain spatio-temporal growing, oscillatory patterns, which may correspond to the onset of spatio-temporal cyclic economic behavior.

To summarize:

- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} < 0$ will die out and the system will converge to the flat steady state – no possible agglomeration is expected.
- The perturbations from the flat steady state which contain modes ϕ_j such that $\Lambda_{\nu,j} > 0$ will turn unstable and lead to possible potential agglomeration spatial patterns, either monotone in time or oscillatory in time.

Remark 4. This instability can be contrasted with the celebrated Turing instability mechanism (Turing, 1952), which leads to pattern formation in biological and chemical systems. The important differences here are that: (a) in our model the instability is driven not by the action of the diffusion operator (which is a differential operator) but rather by a compact integral operator that models geographical spillovers, and (b) contrary to the spirit of the Turing model, here the instability is driven by optimizing behavior, so it is the outcome of forward-looking optimizing behavior by economic agents and not the result of reaction diffusion in chemical or biological agents. It is the optimizing nature of our model which dictates precisely the type of unstable modes which are "accepted" by the system, in the sense that they are compatible with the long-term behavior imposed on the system by the policy maker. In some sense local spillover induced instability is a mixture of Turing- and Hopf-type instabilities.

We now turn to the comparison of stability of the RE and the SO flat equilibrium. A relevant question is under what conditions we might expect possible agglomeration to emerge.

Proposition 2.

(a) At the RE equilibrium, agglomeration is possible for the modes for which $s_{11} + s_{12}\mu_j > 0$, *i.e. the modes for which*

If $s_{12} > 0$, *then* $\mu_j > -\frac{s_{11}}{s_{12}} > 0$,

If $s_{12} < 0$, *then* $-\frac{s_{11}}{s_{12}} < \mu_j < 0$.

(b) At the SO equilibrium, since the production function is strictly concave, agglomeration is never possible.

The above proposition confirms and further clarifies the results of global analysis. The SO has a locally-stable flat steady state which by the global analysis is unique. Thus no agglomeration is possible at the SO with strictly concave production function. However, the flat steady state of the RE system can be locally unstable and this locally instability may induce agglomeration. Thus theorem 2 and Proposition 2 allow us to link the global nonlinear picture of the system with the linearized picture we obtain in this section using the Turing-type analysis.

We can distinguish between two cases of potential agglomeration emergence depending on the sign of s_{12}. (i) If $s_{12} > 0$, which is the case for a Cobb Douglas technology or a CES technology with $\partial^2_{xX} f > 0$, and K is a positive operator (positive spillovers) then agglomeration is possible only if $\mu_1 = ||\mathsf{K}|| \geq \frac{|s_{11}|}{s_{12}}$. The economic intuition behind this is that agglomeration can occur if the spillover effects (as measured by the eigenvalues of operator K) are strong enough as compared to the ratio $\frac{|s_{11}|}{s_{12}}$, which is determined by the production function and gives the relative strength of diminishing returns with respect to complementarity effects. (ii) If $s_{12} < 0$, which is the case a CES technology with $\partial^2_{xX} f < 0$ (or equivalently decreasing returns to scale and elasticity of substitution grater than 1), then agglomeration is possible only if $-\frac{s_{11}}{s_{12}} \leq \mu_j < 0$. This is possible if the integral operator characterizing spatial spillovers is positive at a flat steady state or $(\mathsf{K}\bar{x})(t,z) = \bar{x}\int_{\mathcal{O}} w(z-s)\,ds > 0$, indicating an overall positive spatial externality if private capital is the same across the spatial domain, but some individual spillovers are negative indicating damage to production at (t,z) from usage of capital at (t,s). A numerical example of such a kernel is provided in section 6.2. This result about potential agglomeration at the REE does not depend on the returns to scale of the production function but on the structure of the spatial externality. Thus agglomeration may emerge in the REE of a Ramsey type growth model with decreasing returns to scale and marginal product of private capital declining with respect to the spatial externality.

Remark 5. Note the important qualitative difference between the case considered in this paper (where K is a symmetric bounded compact operator) as compared to models commonly used in biology or chemistry - including Turing's own seminal contribution (where K is a symmetric unbounded operator, e.g., $\mathsf{K} = -\Delta$, the Laplacian). In the latter case the spectrum of operator K is unbounded ($|\mu_j| \to \infty$). This means that if K is an unbounded symmetric positive operator, there will always be a mode which turns unstable.

An important special case is that of periodic boundary conditions. In this case the eigenfunctions and the eigenvalues of operator K are obtained very easily in terms of the Fourier basis, thus leading to very general and easy to implement results. The main general results in this case are summarized in Proposition 3 (where the symmetry of the kernel is explicitly used). Proposition 3 is elementary and is only included here as a reminder, and for completeness of the paper.

Proposition 3 (Periodic Boundary Conditions)**.** *Assume periodic boundary conditions, i.e.,* $\mathbb{H} = L_{per}(\mathcal{O})$, $\mathcal{O} = [-L, L]$. *Then,*
(a) The eigenfunctions of operator K *are the Fourier modes* $\phi_n(z) = \cos(n\pi z/L)$, $n \in \mathbb{N}$ *with corresponding eigenvalues* $W_n = \int_{-L}^{L} w(z)\,\phi_n(z)dz$.
(b) The action of operator K *on a flat state returns a flat state,* $\mathsf{K}\bar{x} = \bar{x}\int_{-L}^{L} w(z)dz$.

Therefore, the onset of instability (i.e., the particular modes which are likely to become unstable) is determined by the Fourier expansion of the kernel w. For specific classes of kernels the calculation of the eigenvalues can be made explicit, thus leading to detailed results on the unstable modes and the shape of the patterns created near the onset of the instability (in the linear regime). The general result is that if certain modes are to become unstable, these will be the low modes, since W_n is expected to decay to 0 as $n \to \infty$.

6 An Illustrative Example: Potential Agglonerations with Cobb-Douglas and CES Technologies

6.1 A Cobb-Douglas technology

In this section we provide an illustrative example of the general theory provided in this paper, using a Cobb-Douglas type production function with spillover effects. We assume periodic boundary conditions and for simplicity we assume the spillovers to be described by exponential kernels. The reason for this choice is twofold: First, kernels of this type were employed by Krugman (1996) in his modelling of spillover effects and second, this type of kernels allows for some closed form expressions for the spectrum of the integral operator K.

We illustrate our general theoretical results through a detailed analysis of the modes that may turn unstable, and some results for the evolution of possible agglomeration patterns obtained by numerical analysis of the system.

6.1.1 The primitives of the economy and the Euler-Lagrange equation

Consider a standard Cobb-Douglas production function

$$f(x, X) = C_0 x^a X^b, \quad a + b < 1,$$

where C_0 is a constant. This production function takes into account local effects (modelled by the x contribution) and nonlocal effects (modelled by the X contribution) in the production.

The nonlocal effects are modelled by the integral operator K, defined by the composite exponential kernel function

$$w(z)=\sum_{i=1}^{N} C_i\,\exp(-\gamma_i|z|),\;\;\gamma_i\geq 0,\;\;C_i\in\mathbb{R}. \tag{23}$$

The coefficients γ_i give a measure of the spatial decay of spillover effects. The larger γ_i is, the faster the spillover effects are decaying as distance increases. These effects may be positive or negative depending on the sign of the corresponding coefficient C_i.

The operators A_{RE} and A_{SO} become

$$\begin{aligned}\mathsf{A}_{RE} &:= -\alpha^{-1}\,(a\,C_0\,x^{a-1}\;(\mathsf{K}x)^b-\lambda),\\ \mathsf{A}_{SO} &:= -\alpha^{-1}\,(a\,C_0\,x^{a-1}\;(\mathsf{K}x)^b+bC_0\mathsf{K}\left(x^a\,(\mathsf{K}x)^{b-1}\right)-\lambda)\end{aligned}$$

and the steady-state equations are nonlinear integral equations.

Under the assumption of periodic boundary conditions, the action of operator K on a flat state $\bar{x}$ renders a flat state. Since $\mathsf{K}\bar{x}=W\,\bar{x}$ where W is known (see next section for a closed form expression), we have that the flat steady state for the REE is the solution of the nonlinear algebraic equation

$$aW^b\bar{x}^{a+b-1}-\bar{\lambda}=0,$$

whereas for the SO it is the solution of

$$aW^b\bar{x}^{a+b-1}+bw^b\bar{x}^{a+b-1}-\bar{\lambda}=0,$$

where $\bar{\lambda}=\frac{\lambda}{C_0}$. These immediately yield

$$\begin{aligned}\bar{x}_{RE} &= \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{a+b-1}}W^{-\frac{b}{a+b-1}},\\ \bar{x}_{SO} &= \left(\frac{\bar{\lambda}}{a+b}\right)^{\frac{1}{a+b-1}}W^{-\frac{b}{a+b-1}}.\end{aligned}$$

Note that $\bar{x}_{SO}\geq\bar{x}_{RE}$ since $b>0$ and $a+b<1$. In the case where spillovers play no role ($b=0$), these steady states coincide.

6.1.2 The spectrum of operator K

The spectrum of operator K, for periodic boundary conditions, can be calculated analytically, in closed form using Proposition 3.

Consider first the case where $N = 1$ in (23). Then, setting $\gamma_1 = \gamma$ and $C_1 = C$, a straightforward application of Proposition 3 shows that the eigenfunctions of K are the Fourier modes $\phi_n = \cos\left(\frac{n\pi z}{L}\right)$ with eigenvalues

$$\mu_n(\gamma) = W_n = 2C\frac{L^2\gamma}{n^2\pi^2 + \gamma^2 L^2}(1 - \mathsf{e}^{-\gamma L}\cos(n\pi)),$$

for $n \in \mathbb{N}$. Furthermore, $W = \mu_0 = W(0)$.

Consider now a composite kernel as in (23), for $N \neq 1$. Clearly K is a linear combination of operators generated by simple exponential kernels. Using the linearity of the operators it is clear that the Fourier modes ϕ_n, for $n \in \mathbb{N}$, are eigenfunctions of K with corresponding eigenvalue

$$M_n = \sum_{i=1}^{N} C_i \mu_n(\gamma_i), \tag{24}$$

and $W = M_0$.

6.1.3 Local spillover induced instability

As already mentioned, our general results preclude emergence of agglomerations in the SO case. However, agglomeration emergence, via local spillover induced instability, is possible under certain conditions for the REE problem.

The emergence of this instability is governed by the linearized system

$$x^{''} - rx^{'} + \mathsf{L}_{RE}x = 0, \tag{25}$$

with initial conditions $x(0,z)$, $x^{'}(0,z)$ where x now denotes a small perturbation from the flat steady state $\bar{x} := \bar{x}_{RE}$ and L_{RE} is the linear operator defined by

$$\mathsf{L}_{RE}x := s_{11}x + s_{12}\mathsf{K}x,$$

where s_{11}, s_{12} are the constants

$$s_{11} = a(a-1)C_0 W^b \, \bar{x}^{a+b-2},$$
$$s_{12} = a \, b \, C_0 \, W^{b-1} \, \bar{x}^{a+b-2}.$$

The knowledge of the eigenvalues in combination with Proposition 1 allows us to determine unstable modes and draw some general conclusions.

Consider first the case where the kernel consists of a single exponential ($N = 1$) with $\gamma_1 = \gamma$ and $C_1 = C$. Then, the instability condition yields that a mode n is unstable if

$$\frac{a-1}{b} + \frac{L^2\gamma^2}{n^2\pi^2 + \gamma^2 L^2}(1 - e^{-\gamma L}\cos(n\pi)) \geq \frac{1}{ab}\left(\frac{r}{2}\right)^2 W^{-b}\bar{x}^{2-a-b}.$$

The expression $I(n) = \frac{L^2\gamma^2}{n^2\pi^2+\gamma^2 L^2}(1 - e^{-\gamma L}\cos(n\pi))$ attains its maximum value for $n = 0$ so that

$$\frac{a-1}{b} + \frac{L^2\gamma^2}{n^2\pi^2 + \gamma^2 L^2}(1 - e^{-\gamma L}\cos(n\pi)) \leq \frac{a-1}{b} + 1 - e^{-\gamma L} \leq \frac{a+b-1}{b} \leq 0,$$

since $a + b < 1$ so that the instability condition is never satisfied and all modes are stable.

The situation is more interesting when composite kernel functions are taken into account. Consider for example a kernel function as in (23), with $N = 2$ and $\gamma_1 = 0.3$, $C_1 = 2$, $\gamma_2 = 0.1$, $C_2 = -0.75$. In this case the influence of neighboring locations on local state is a weighted average of the state at neighboring locations, but the influence from nearby locations is positive, while the influence is negative from relatively more distant locations. The kernel function chosen is similar to the one employed by Krugman (1996) in the modelling of a market potential function. The shape of the composite kernel function is shown in the left panel of Figure 1.

The operator K associated with the specific composite spillover is positive. Furthermore, the spectrum of this operator is available in closed form (see equation (24)) so that using the spectrum, we can readily check for the existence of unstable modes, for which spillover induced instability is possible. Since the spectrum depends on the parameters a, b and L in the right panel of Figure 1, we plot the number of unstable modes that emerge as a function of the ratio $\frac{1-a}{b}$ and the length of the domain L. We remark

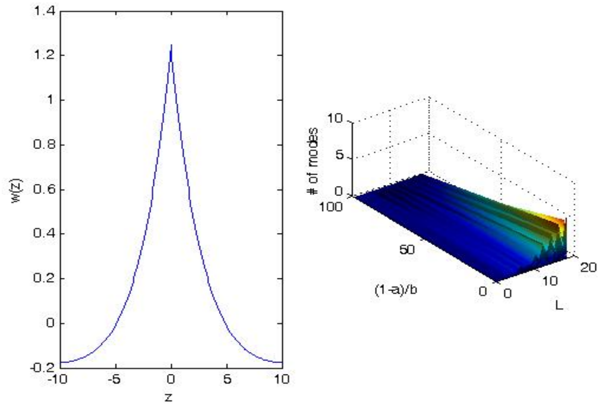

Figure 1: The composite kernel

first that low modes may turn unstable (as a result of the monotonicity of the spectrum and its boundedness; recall that K is a compact operator). That means that when, e.g., one mode is unstable, this is the first mode corresponding to $n = 1$, when two modes are unstable, this means that these are the modes $n = 1$ and $n = 2$, etc. One observes from our results that as L increases, more modes become unstable, as the increased length of the system can then accommodate more modes. This will lead to different spatial patterns that emerge as the length of the domain increases. The spatial complexity increases with the increase of L, and this is essentially a bifurcation phenomenon.

To display the wealth and variety of possible spatio-temporal agglomeration patterns that may emerge, we solve numerically the linearized equation (25) for the chosen kernel, and for the parameter values $a = 0.8$, $b = 0.1$, for various choices of L. We assume a small random perturbation from the flat steady state and we leave this perturbation evolving by solving (25), in order to determine the $\bar{x} + x(t, z)$ which is an approximation of the optimal path for the full system. The results are shown in Figure 2. In the first panel, where $L = 10$, no modes can become unstable and the perturbation dies out, and we revert to the flat steady state and no agglomeration occurs. In the second panel, $L = 15$ and according to our analysis only the first mode $n = 1$ is unstable. This leads to an agglomeration pattern whose spatial structure resembles that of the first eigenfunction of operator K. In the third panel, $L = 18$ and in this case two modes, $n = 1$ and $n = 2$, are

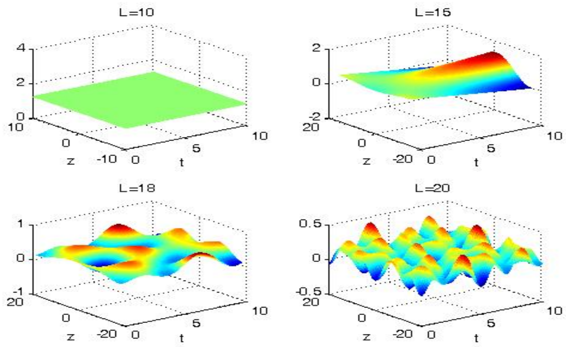

Figure 2: $\Lambda_{RE,2}$ and spillover induced instability

unstable, a fact that leads to agglomeration patterns with spatial structure resembling linear combinations of the first two eigenfunctions of operator K. Finally, in the fourth panel, $L = 20$ and more modes become unstable leading to the occurrence of more complex spatio-temporal agglomeration patterns.

6.1.4 The existence of spatial patterns in the fully nonlinear model

We close this section by showing that the agglomeration patterns predicted by the linear stability analysis exist in the fully nonlinear case, by using a variational argument based on the mountain pass argument. We consider the steady-state equation $\mathsf{A}_{RE}x = 0$, which in this case is a nonlinear integral equation of the form

$$ax^{a-1}\left(\mathsf{K}x\right)^{b} = \bar{\lambda} \tag{26}$$

which, as we saw earlier on, has a unique flat steady-state solution

$$\bar{x}_{RE} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{a+b-1}} W^{-\frac{b}{a+b-1}}.$$

We solve (26) in terms of $X = \mathsf{K}x$ and rewrite this integral equation as

$$\mathsf{K}x = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} x^{\frac{1-a}{b}}.$$

We write $x = \bar{x}_{RE} + v$ and define the new variable

$$u = (v + \bar{x}_{RE})^{\frac{1-a}{b}}. \tag{27}$$

Using (27), (26)assumes the equivalent form

$$\mathsf{K}u^{\frac{b}{1-a}} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} u \tag{28}$$

which is in the standard form of a Hammerstein nonlinear integral equation. Since $a + b < 1$, we see that $\frac{b}{1-a} < 1$, so this is a sublinear Hammerstein equation. Clearly it admits the trivial solution $u = 0$ which is not acceptable on economic grounds since it leads to a flat steady state which is negative. Furthermore, it also admits a unique nontrivial flat solution $\bar{u}$ such that

$$\bar{u} = \left(\frac{\bar{\lambda}}{a}\right)^{\frac{1-a}{b(a+b-1)}} W^{-\frac{1-a}{a+b-1}}$$

which by undoing the transformation of variables (27) is easily seen to coincide with $\bar{x}_{RE}$.

The question that arises is whether the Hammerstein equation (28) admits other solutions apart from these two. If it does, then this must be a non-flat solution, which corresponds to agglomeration. This is a fully nonlinear agglomeration pattern. By the linear analysis performed in the previous section, this fully nonlinear pattern must in the appropriate linear limit coincide with the patterns predicted by the linear stability analysis.

There is a well-developed theory concerning the solution of such Hammerstein equations. This theory, which is based on the powerful techniques of critical point theory, allows detailed results on the number and nature of nontrivial solutions. The following proposition provides an answer to the question we have set.

Proposition 4. *There exists a Λ_* such that for $\left(\frac{\bar{\lambda}}{a}\right)^{\frac{1}{b}} > \Lambda_*$ the Hammerstein equation (28) admits at least two nontrivial solutions. Since one of these is the unique flat steady state $\bar{u}$, the other corresponds to a nonlinear agglomeration pattern.*

The proof of the proposition follows the proof of Theorem 7 of Faraci and Moroz (2003) with minor modifications and is omitted for the sake of brevity. The critical value Λ_* can be estimated in terms of the primitives of the problem. A full account of the nonlinear bifurcation theory for the steady states is beyond the scope of the present paper and will be addressed in future work.

6.2 Local instability and potential agglomerations with a CES technology

We consider the CES production function

$$f(x, X) = A\left[\beta x^{-\theta} + (1-\beta) X^{-\theta}\right]^{-\frac{\gamma}{\theta}}.$$

A flat steady state will be a solution $\bar{x} : \partial_x f(\bar{x}, \mathsf{K}\bar{x}) = \lambda$. Then, following Proposition 2, linear instability of this flat steady state that may lead to agglomerations requires that $0 < s_{11} + s_{12}\mu_j < \left(\frac{r}{2}\right)^2$. To examine this possibility we consider the two cases that may emerge with a CES technology: (i) $s_{12} > 0$, $\mu_j > 0$, and (ii) $s_{12} < 0$, $\mu_j < 0$. We assume $r = 0.03$, $\eta = 0.03$ so that $\lambda = 0.06$, and $A = 1$ and $\beta = 0.3$. For case (i) we use $\theta = -0.05$,

$$w(z) = 0.025 + 2\exp(-0.5|z|) - 0.525\exp(-0.11|z|),$$

$L = 50$, resulting in $W(z) = \int_{-L}^{L} w(z)\,dz = 0.993556$ and largest positive eigenvalue at mode $j = 4$, $\mu_4 = 4.8581$. Assuming $\gamma = 0.307$, we obtain $\bar{x} = 1.861$, $s_{11} = -0.0280338$, $s_{12} = 0.00581464$, $s_{11} + s_{12}\mu_j = 0.000214 < \left(\frac{r}{2}\right)^2$. Thus we have a potential agglomeration inducing instability at mode 4. For case (ii) we use $\theta = -0.8$,

$$w(z) = 0.045 + 2\exp(-0.5|z|) - 0.525\exp(-0.11|z|) - 0.3\exp(-0.3|z|),$$

$L = 50$, resulting in $W(z) = \int_{-L}^{L} w(z)\,dz = 0.993556$ and negative eigenvalue at mode $j = 1$, $\mu_1 = -1.26697$. Assuming $\gamma = 0.45$, we obtain $\bar{x} = 4.38108$, $s_{11} = -0.0112285$, $s_{12} = -0.00337185$, $s_{11}+s_{12}\mu_j = 0.0000897736 < \left(\frac{r}{2}\right)^2$, which implies that we have a potential agglomeration inducing instability at mode 1. This empirical example confirms our theoretical results about the possibility of agglomeration at the REE of a Ramsey type model with a CES technology, decreasing returns to scale and marginal productivity of private capital increasing or decreasing with respect to the spatial externality.

7 Discussion, Extensions and Concluding Remarks

This paper develops a fairly general approach to the study of infinite dimensional, infinite horizon, Intertemporal recursive dynamic optimization models in continuous spatial settings. Using the theory of maximal monotone operators for global analysis and spectral theory of compact operators for local analysis, we studied the spatiotemporal long-run behavior of the rational expectations equilibrium and the social optimum associated with a Ramsey-Fisher-type optimal growth model. We show, in the context of global analysis, that strong concavity of the production function implies convergence of the SO to a unique flat steady state, while similar convergence of the RE equilibrium requires stronger conditions. The possibility of a potential agglomeration at the RE equilibrium induced by instability of the flat steady state led us to local analysis. In the local analysis, we derived conditions for local stability and spillover-induced instability associated with the RE equilibrium which could signal agglomeration emergence.

Agglomerations may emerge, with any type of returns to scale (increasing or decreasing) and with the marginal productivity of private capital increasing or decreasing with respect to the spatial externality. This is a fairly general result indicating the importance of the network structure of the spatial externality relative to the properties of the aggregate production function.

Since both our local and global conditions depend on primitives such as the strength of diminishing returns and complementarities in the production function, the characteristics of the spatial spillovers and the spatial geometry, our results are easily interpretable and potentially testable.

Our global results can be associated with general turnpike theorems in infinite dimensional spaces, which means that they allow the study of the long-run behavior of dynamic systems that include explicit spatial interactions - e.g. spatial spillovers - among economic agents. The global asymptotic stability of the SO, obtained by using monotone operator theory, can be related to global asymptotic stability results obtained by Scheinkman (1978) about the stability of separable Hamiltonians in finite dimensional settings. From (17) and (18) we can write our problem in terms of separable

Hamiltonians as:

$$
\begin{aligned}
&\mathcal{H}_\nu(p,x,\check{X}) = \mathcal{H}^1_\nu(p) + \mathcal{H}^2_\nu(x,\check{X}) \ ,\nu = RE, SO \\
&\mathcal{H}^1_{RE}(p) := \max_{x'} \left\{ -\frac{\alpha}{2} \left(x'\right)^2 + p\,x' \right\}, \\
&\mathcal{H}^1_{SO}(p) := \max_{x'} \left\{ \int_{\mathcal{O}} \left[-\frac{\alpha}{2} \left(x'\right)^2 + p\,x' \right] dz \right\} \\
&\mathcal{H}^2_{RE}(x,\check{X}) := f(x, X^e) - \lambda x, \\
&\mathcal{H}^2_{SO}(x,\check{X}) := \int_{\mathcal{O}} \left[(f(x, \mathsf{K}x) - \lambda x \right] dz.
\end{aligned}
$$

This implies that the global long-run behavior of infinite dimensional optimization problems that give rise to Hamiltonians which are separable in the above sense, can be analyzed using the theory of monotone operators developed in this paper, while local results can be obtained using spectral theory.

For example, the well-known problem of investment theory of the firm with convex adjustment costs (Lucas 1967a,b), which has been analyzed in terms of separable Hamiltonians by Scheinkman (1978), can be extended using our methods in a spatial setting. In a simplified set-up, consider a large number of firms which occupy a spatial domain $\mathcal{O}$, sell a homogeneous output at an exogenous price, face quadratic adjustment costs with respect to net investment, and experience knowledge spillovers which generate spatial interactions among them. The REE and SO problems can be written as:

$$
\begin{aligned}
\text{REE:} \quad &\max_{x'} \int_0^\infty e^{-r\,t} \left(l f(x, X^e) - q\left(x' + \eta x\right) - \frac{\alpha}{2}(x')^2 \right) dt \\
\text{SO:} \quad &\max_{x'} \int_0^\infty \int_{\mathcal{O}} e^{-r\,t} \left(l f(x, \mathsf{K}x) - q\left(x' + \eta x\right) - \frac{\alpha}{2}(x')^2 \right) dz\, dt,
\end{aligned}
$$

where l is the exogenous output price, and q is the unit price of capital, both assumed independent of time. In this case the nonlinear operators of

Definition 5 become

$$\mathsf{A}_{RE}x := -\alpha^{-1}(f_x(x,\check{X}) - q\,(\eta - r)), \quad \check{X} = \mathsf{K}x,$$
$$\mathsf{A}_{SO}x := -\alpha^{-1}(f_x(x,\check{X}) + \mathsf{K}f_X(x,\check{X}) - q\,(\eta - r)), \quad \check{X} = \mathsf{K}x.$$

and Theorem 1 suggests that the first order necessary condition for problems REE and SO are, respectively,

$$x^{''} - rx^{'} - \mathsf{A}_\nu x = 0, \quad \nu = RE, SO.$$

Therefore with a strictly concave production function, the flat steady state of SO is globally assymptotically stable, independent of the value of r, and no agglomeration is possible at the SO. On the other hand by Proposition 2, agglomeration might be possible at the REE.

The factor that could potentially differentiate the spatial structure of the SO and REE in the long run is the way in which the optimizing agent takes into account the spatial spillover along with the structure of the spatial spillovers. As seen from Proposition 2, the important quantity for potential agglomerations at the REE is the ratio $-s_{11}/s_{12}$ and its relation to the eigenvalues of operator K, when the optimizing agent treats the spillover X as exogenous and takes into account only its interactions with x which is reflected in s_{12}. At the SO the optimizing agent - e.g., the social planner - by treating X as endogenous, takes into account the diminishing returns of the spillover, in addition to the interactions s_{12}. With a strictly concave production function, this diminishing returns implies a maximal monotone operator A_{SO} and equivalently $\Lambda_{SO,j} < 0$ for all modes j in local analysis, and thus no agglomeration at the SO. Therefore it is the full internalization of the spatial externality that prevents the emergence of agglomerations.

We feel that the methods developed in this paper provide insights about the spatial structure of dynamic economic models that could provide a direct link between economic geography and optimal growth. Future research could be directed towards the further study of complexities underlying the RE equilibrium, the impact of increasing returns, and the explicit introduction of capital movement across space in pursuit of higher returns.

8 Appendix: Proofs

8.1 Proof of Theorem 1

We use the notation $F(x,X) = f(x,X) - \lambda x$ to rewrite the functional to be maximized as

$$\begin{aligned} J(x,x') &:= J_1(x) + J_2(x') \qquad (29)\\ &:= \int_0^\infty \int_{\mathcal{O}} e^{-rt} F(x(t,z), \mathsf{K}x(t,z)) dz dt - \frac{\alpha}{2} \int_0^\infty \int_{\mathcal{O}} e^{-rt} (x'(t,z))^2 dz dt. \end{aligned}$$

Since f is a concave function so is F. If f is strictly concave, then so is F.

To be in line with the standard theory of the calculus of variations, we consider the equivalent problem of minimizing the functional $\bar{J} = -J$. We also use the notation $\bar{J}_1(x) = -J_1(x)$, $\bar{J}_2(x') = -J_2(x')$ and

$$\bar{F}(x,X) = -F(x,X) = \lambda x - f(x,X).$$

Clearly, $\bar{F}$ is a (strictly) convex function since f is (strictly) concave. Therefore, we will treat the problem of minimizing

$$\begin{aligned} \bar{J}(x,x') &:= \bar{J}_1(x) + \bar{J}_2(x') \qquad (30)\\ &:= \int_0^\infty \int_{\mathcal{O}} e^{-rt} \bar{F}(x(t,z), \mathsf{K}x(t,z)) dz dt + \frac{\alpha}{2} \int_0^\infty \int_{\mathcal{O}} e^{-rt} (x'(t,z))^2 dz dt, \end{aligned}$$

Finally we use the notation $x_n \to x$ for strong convergence in $\mathbb{H}$ and $x_n \rightharpoonup x$ for weak convergence in $\mathbb{H}$.

Our proof of existence will use the direct method of the calculus of variations and requires the lower semicontinuity of the functional $\bar{J} : L^2(0,\infty;\mathbb{H}) \to \mathbb{R}$. To this end we need to recall a general result from Fonseca and Leoni (2007) and prepare a lemma.

Theorem 3 (Theorem 7.5 of Fonseca and Leoni (2007, p. 492))**.** *Let $g : \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and define the functional $\bar{\Phi}(u,v) := \int_{\mathcal{O}} g(z,u(z),v(z))dz$. Also, let $p,q \in [1,\infty)$ and assume that $g(z,u,v) \geq -C(|u|^p + |v|^q) - w(z)$, $C \geq 0$ and $w \in L^1(\mathcal{O})$. The functional $\bar{\Phi}$ is lower semicontinuous with respect to weak convergence of u in $L^p(\mathcal{O})$ and strong convergence[14] of v in $L^q(\mathcal{O})$, if and only if the following three properties hold:*

[14]Meaning that if $u_n \rightharpoonup u$ in $L^p(\mathcal{O})$ and $v_n \to v$ in $L^q(\mathcal{O})$, then $\bar{\Phi}(u,v) \leq \liminf_n \bar{\Phi}(u_n,v_n)$.

(i) $u \mapsto g(z, u, v)$ *is convex for all* $z \in \mathcal{O}$ *and for all* $v \in \mathbb{R}$,

(ii) $g(z, u, v) \geq a(z) + b(x, v)u - c|v|^q$, $c > 0$ *and* $a \in L^1(\mathcal{O})$.

(iii) For any two sequences $\{v_n\}$ *(converging weakly in* $L^p(\mathcal{O})$*) and* $\{u_n\}$ *(converging strongly in* $L^q(\mathcal{O})$*) and such that* $\sup_n \Phi(u_n, v_n) < \infty$, *then the sequence* $|b(\cdot, v_n(\cdot))|^{p'}$ *where* p' *is the conjugate exponent of* p, *is equi-integrable.*

The proof of the theorem which relies on convexity, that allows us to use Mazur's lemma (or stated slightly differently ensures the equivalence of the closure in the strong topology with the closure in the weak topology) so as to account for the weak convergence in one of the variables of the functional $\bar{\Phi}$, can be found in the classical monograph of Fonseca and Leoni (2007) (see also Berkovitz (1974)). We will need a special case of this theorem, and in particular the case where $p = q = 2$ as in our setting $\mathbb{H} = L^2(\mathcal{O})$.

Lemma 1. *The functional* $\hat{J}_1 : \mathbb{H} \to \mathbb{R}$, *defined by*

$$\hat{J}(u) := \int_{\mathcal{O}} (\lambda u(z) - f(u(z), \mathsf{K}u(z)))dz, \quad \lambda > 0,$$

is weakly lower semicontinuous, i.e., $u_n \rightharpoonup u$ *in* $\mathbb{H}$ *implies that* $\liminf_n \hat{J}_1(u_n) \geq \hat{J}_1(u)$.

Proof: Consider a sequence u_n such that $u_n \rightharpoonup u$ in $\mathbb{H}$. Since $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ is a compact operator, there exists a subsequence of u_n (denoted the same for simplicity) such that $v_n := \mathsf{K}u_n \to \mathsf{K}u$ in $\mathbb{H}$. Since f is concave (bounded above by an affine function), the function $\bar{F}$ satisfies properties (i)-(iii) of Theorem 3 for $p = q = 2$ (in fact for this choice (iii) is immediate). Applying Theorem 3 for $g = \bar{F}$ and $p = q = 2$ leads to the weak lower semicontinuity of $\hat{J}_1$. *QED*

We are now ready to present the proof of Theorem 1.

PROOF OF THEOREM 1: We only provide the proof for $\nu = SO$ as the RE case is similar.

(a) Consider a sequence (x_n, x'_n), $n \in \mathbb{N}$ such that $J(x_n, x'_n) \to M$ where $M = \sup J(x, x')$. Clearly this is a minimizing sequence for $\bar{J}$. The real valued sequence $J(x_n, x'_n)$ is bounded, so that by the properties[15] of F there

[15] In particular that $\bar{F} := -F$ considered as a function of two real variables (x, y) is bounded below for every $(x, y) \in \mathbb{R} \times \mathbb{R}$.

exists a constant C such that $\int_0^\infty \int_{\mathcal{O}} e^{-rt}(x'(t,z))^2 dzdt < C$, that is x_n' is a bounded sequence in $\mathbb{V} := L^2((0,\infty), e^{-rt}dt; \mathbb{H})$, where $\mathbb{H} = L^2(\mathcal{O})$. Since the measure $\mu = e^{-rt}dt$ is such that $\mu(\mathbb{R}^+) < \infty$ and $\mathbb{H}$ is a separable Hilbert space, we conclude that $\mathbb{V}$ is reflexive, so that by the Eberlein-Schmulian weak compactness result there exists $y \in \mathbb{V}$ and a subsequence of x_n' (denoted the same for simplicity) with the property that $x_n' \rightharpoonup y$ in $\mathbb{V}$, i.e., with the property that

$$\int_0^\infty \int_{\mathcal{O}} e^{-rt}(x_n'(t,z))\phi(t,z)dzdt \to \int_0^\infty \int_{\mathcal{O}} e^{-rt}(y_n(t,z))\phi(t,z)dzdt, \;\; \forall \;\; \phi \in \mathbb{V}.$$

We then set $\bar{x}(t) := x_0 + \int_0^t y(s)ds$ so that $y(t) = \bar{x}'(t)$ in $\mathbb{H}$, a.e. in $t \in \mathbb{R}_+$, and[16] it can be seen that $x_n(t) \rightharpoonup \bar{x}(t)$ in $\mathbb{H}$ uniformly on compact subsets of $(0,\infty)$ and $x_n \rightharpoonup \bar{x}$ in $L^2((0,\infty), e^{-rt}dt; \mathbb{H})$.

We note that the functional $\bar{J}_2 : \mathbb{V} \to \mathbb{R}$ is in fact the norm of $\mathbb{V}$, so that by the weak lower semicontinuity of the norm, we have that $\liminf_n \bar{J}_2(x_n') \geq \bar{J}_2(\bar{x}')$. We now pick any $t \in (0,\infty)$. By an application of Lemma 1 to the sequence $x_n(t) \in \mathbb{H}$ we obtain that $\liminf_n \hat{J}_1(x_n(t)) \geq \hat{J}_1(\bar{x}(t))$, so that multiplying by e^{-rt} and integrating over all $t \in (0,\infty)$ leads to the result that $\liminf_n \bar{J}_1(x_n) \geq \bar{J}_1(\bar{x})$. We therefore, conclude that $\liminf_n \bar{J}(x_n, x_n') \geq \bar{J}(\bar{x}, \bar{x}')$, and since (x_n, x_n') is the minimizing sequence for $\bar{J}$ we conclude that $\bar{x}$ is a minimizer for $\bar{J}$, hence a maximizer for J.

(b) We now consider the functional J, defined in (29), as a functional of $u = x'$ and $x = x_0 + \int_0^t u(s)ds$ (still denoted as J). The first order necessary condition will be of the form $(DJ, \phi) = 0$ where D denotes the Gâteaux derivative and ϕ is a test function in $\mathbb{H}$.[17] We proceed to the determination of the Gâteaux derivative. To this end, fix any direction $v \in \mathbb{H}$, define $u_\epsilon = u + \epsilon v$, $\bar{v} = \int_0^t v(s)ds$ and calculate

$$\frac{d}{d\epsilon} J(u_\epsilon)\bigg|_{\epsilon=0} = \int_0^\infty \int_{\mathcal{O}} e^{-rt} \left(\partial_x f(x, \mathsf{K}x) + \mathsf{K}^* \partial_X f(x, \mathsf{K}x)\bar{v} - \lambda \bar{v} - \alpha u v\right) dz\, dt$$

where K^* is the adjoint of operator K, and $\mathsf{K}^* = \mathsf{K}$ by symmetry. In the above calculation we have used the smoothness Assumption 3 that allows the use of the Lebesgue dominated convergence theorem, in order to interchange the

[16]By proper choice of the test functions $\phi \in \mathbb{V}$, e.g. $\phi(s,z) = \mathbf{1}_{(0,t)}(s)e^{rs}\psi(z)$, where $\mathbf{1}_{(0,t)}$ is the characteristic function of the interval $(0,t)$ and ψ is any function in $\mathbb{H} = L^2(\mathcal{O})$.

[17]This is assuming we treat the problem over the whole of $\mathbb{H}$, otherwise it is replaced by a variational inequality of similar form.

limit defining the derivative with integration so as to reach the stated result. Since $v = \bar{v}'$, by integration by parts over t and using the transversality condition, the first order condition becomes

$$\int_0^\infty \int_{\mathcal{O}} e^{-rt} \left(\partial_x f(x, \mathsf{K}x) + \mathsf{K}^* \partial_X f(x, \mathsf{K}x) - \lambda + \alpha u' - r\alpha u\right) \bar{v} dz\, dt = 0.$$

This must be true for all v therefore for all $\bar{v}$ which implies that the first order condition becomes

$$\partial_x f(x, \mathsf{K}x) + \mathsf{K}^* \partial_X f(x, \mathsf{K}x) - \lambda + \alpha u' - r\alpha u = 0,$$

(a.e.) and keeping in mind that $u = x'$, we reach the stated result. *QED*

8.2 Proof of Theorem 2

We recall the following definitions.

Definition 6. Let $\mathsf{A} : \mathbb{H} \to \mathbb{H}^*$ a (nonlinear) operator.

(i) A is called monotone if $\langle \mathsf{A}(x_1) - \mathsf{A}(x_2), x_1 - x_2 \rangle \geq 0$ for every $x_1, x_2 \in \mathbb{H}$.

(ii) A is called maximal monotone if it is monotone and does not admit any monotone extension, i.e. its graph cannot be properly contained in the graph of any other monotone operator.

(iii) A is called strongly monotone if there exists $c > 0$ such that $\langle \mathsf{A}(x_1) - \mathsf{A}(x_2), x_1 - x_2 \rangle \geq c \|x_1 - x_2\|$.

The following theorem regarding the long term behavior of second order evolution equations in Hilbert spaces, due to Rouhani and Khatibzadeh (2009) plays in important role in the proof of Theorem 2.

Theorem 4 (Rouhani and Khatibzadeh (2009, 2010))**.** *Let $\mathsf{A} : H \to \mathbb{H}$ be a maximal monotone (possibly multivalued) operator, $c \geq 0$ and consider the second order evolution equation*

$$u''(t) - cu'(t) \in Au(t), \quad a.e.\, t \in (0, +\infty),$$
$$u(0) = u_0, \quad \sup_{t>0} \|u(t)\| < +\infty.$$

Then, $u(t) \rightharpoonup p$ as $t \to \infty$ where $p \in \mathsf{A}^{-1}(0)$. If furthermore, $(I + \mathsf{A})^{-1}$ is compact or A is strongly monotone, then this convergence is strong.

This is a combination of Theorem 3.3 in Rouhani and Khatibzadeh (2009) p. 4372, and Theorems 4.1, 4.2, in Rouhani and Khatibzadeh (2010) p. 1303. The proof of the theorem relies on detailed a priori estimates for the solution of the evolution equation and the use of a nonlinear ergodic theorem in order to guarantee the existence of the limiting behavior in the weak sense. Maximal monotonicity then plays an important role in the identification of the weak limit as an element of $\mathsf{A}^{-1}(0)$.

We are now ready to provide the proof of Theorem 2.

Proof of Theorem 2 (a) A solution of the operator equation $\mathsf{A}_\nu x = 0$ is a maximizer of the problem,

$$\max_{x \in H} I(x) = \max_{x \in \mathbb{H}} \int_{\mathcal{O}} (f(x(z), (\mathsf{K}x)(z) - \lambda x(z))dz.$$

If a solution exists, it is going to be unique by the strict concaviity of f. We now consider the existence. By economic considerations we only need to restrict our attention to solutions of this problem which are bounded. To see this consider, $\bar{x}(z) = \bar{x}$ for every $z \in \mathcal{O}$ (i.e. a spatially uniform state) where $\bar{x}$ is chosen such that $f(\bar{x}, \bar{w}\bar{x}) - \lambda \bar{x} < 0$ and observe that, $I(\bar{x}) < 0$ and by the properties of f, there exist functions $x \in \mathbb{H}$ such that $I(x) > 0$. Restricting to bounded maximizers, we may proceed with the direct method of the calculus of variations without the need of coercivity conditions. Note that the problem in consideration is equivalent to the problem of minimization of the functional $\hat{J}$ defined in Lemma 1, and the lower semicontinuity result presented there leads to the existence.

(b) The operator A_ν is the Gâteaux derivative of the lower semicontinuous convex functional $\hat{J}_1 : \mathbb{H} \to \mathbb{R}$, defined in Lemma 1 and as such is a maximal monotone operator (the subdifferential of a lower semicontinuous functional is a maximal monotone operator, and since the functional in question is Gâteaux differentiable, the Gâteaux derivative coincides with the subdifferential which is now single valued). We now use Theorem 4 to conclude that $x(t) \rightharpoonup p$, where $p \in \mathsf{A}_\nu^{-1}(0)$ as $t \to \infty$. But since by (a) $\mathsf{A}_\nu(x) = 0$ admits a unique solution, this allows us to conclude that $x(t)$ converges weakly in $\mathbb{H}$ as $t \to \infty$ to the unique solution of $\mathsf{A}_\nu(x) = 0$.

(c) If the extra conditions of compactness of $(I + \mathsf{A}_\nu)^{-1}$ or strong monotonicity of A_ν hold, then, again by Theorem 4 we may conclude that this convergence is strong.

(d) If the operator equation $\mathsf{A}_\nu(x) = 0$ admits a solution which is spatially uniform, i.e., $x(z) = s$ for any $z \in \mathcal{O}$, then, by property $\mathcal{P}$, $(\mathsf{K}x)(z) = \bar{w}s$ and a direct substitution yields that s must be the solution of the algebraic equation stated $\partial_x f(s, \bar{w}s) + \bar{w}\partial_X f(s, \bar{w}s) - \lambda = 0$, where $\bar{w} := \int_{\mathcal{O}} w(z - z_0)dz_0$. If this algebraic equation admits a solution $s > 0$, for the chosen values of the parameters $\bar{w}$ and λ, then the state of the system converges weakly in $\mathbb{H}$ to this spatially homogeneous state. The function $g : \mathbb{R}_+ \to \mathbb{R}$ defined by $g(s) := \partial_x f(s, \bar{w}s) + \bar{w}\partial_X f(s, \bar{w}s) - \lambda$ is continuous, and by the properties of the production function $\lim_{s\to 0^+} g(s) > 0$, whereas $\lim_{s\to\infty} g(s) < 0$. Therefore, there exists an s such that $g(s) = 0$. By (a) and (b), $x(t) \rightharpoonup s$ in $\mathbb{H}$ as $t \to \infty$. This can be interpreted as a non-agglomeration result in the following sense,

$$\int_{\mathcal{O}} x(t, z)\phi(z)dz \to \int_{\mathcal{O}} s\phi(z)dz, \;\; \forall\, \phi \in \mathbb{H} := L^2(\mathcal{O}),,$$

as $t \to \infty$, which is a weak non-agglomeration result. If the extra conditions on the operator A_ν are imposed, then, we obtain the strong non-agglomeration result in the sense that $\|x(t) - s\| \to 0$ as $t \to \infty$, i.e.,

$$\int_{\mathcal{O}} |x(t, z) - s|^2 dz \to 0, \;\; \text{as } t \to \infty.$$

This concludes the proof. *Q.E.D.*

8.3 Proof of Proposition 1

Proof of Proposition 1: (a) Since $\mathsf{K} : \mathbb{H} \to \mathbb{H}$ is a compact operator, by Fredholm theory we know that the spectrum of K consists only of the point spectrum (i.e. only of the eigenvalues $\{\mu_j\}$ of operator K). Furthermore, the spectrum is at most a countable set, and if it is not finite the only accumulation point for the sequence $\{|\mu_j|\}$ is 0. Since K is a bounded self-adjoint operator, its spectrum is also bounded and real, and the eigenfunctions $\{\phi_j\}$ corresponding to the eigenvalues $\{\mu_j\}$ may be chosen so as to form an orthonormal set in $\mathbb{H}$. This set is complete in $\overline{Ran(\mathsf{K})} \subset \mathbb{H}$.[18] If additionally K has the property of strict positivity then the spectrum is contained in a bounded subset of $\mathbb{R}_+$ and $\{\phi_j\}$ is complete in $\mathbb{H}$.

Using the basis of $\overline{Ran(\mathsf{K})} \subseteq \mathbb{H}$ defined by the eigenfunctions $\{\phi_j\}$ of

[18] By *Ran* we denote the range of the operator K, while the overline denotes closure.

operator K, we can obtain a spectral decomposition of (21) as follows: We perform a Galerkin approximation of the solution of (21) using the set $\{\phi_j\}$. We consider the sequence of functions $\hat{x}_n(t,z) = \sum_{i=1}^{n} c_{\nu,i}(t)\phi_i(z)$ and we insert this into (21). Projecting along ϕ_j, $j = 1, \ldots, n$ we obtain the system of second order ODEs

$$c''_{\nu,j} - rc'_{\nu,j} + \Lambda_{\nu,j} c_{\nu,j} = 0, \quad \nu = RE, SO, \quad j = 1, \ldots, n \tag{31}$$

with $\Lambda_{\nu,j}$ as given in the statement of the proposition. Assume that the initial conditions $\hat{x}(0), \hat{x}'(0) \in \overline{Ran(\mathsf{K})} \subset \mathbb{H}$. By the completeness of the orthonormal basis $\{\phi_n\}$ there exists an expansion $\hat{x}(0,z) = \sum_j a_j \phi_j(z)$, $\hat{x}'(0,z) = \sum_n b_n \phi_n(z)$ where the series converge in $\mathbb{H}$. Therefore, solving system (31) with initial conditions $c_{\nu,j}(0) = a_j$, $c'_{\nu,j}(0) = b_j$, we obtain an approximation of the solution in terms of the Galerkin expansion. The Galerkin expansion transforms the infinite dimensional systems (21), which characterize RE and SO equilibria respectively, into a countable set of finite dimensional problems (22), each problem corresponding to a mode $j = 1, ..., n$. Using a priori estimates and weak convergence arguments, we may pass to the limit as $n \to \infty$ in a standard fashion. The conditions for agglomeration emergence can be determined by looking at the exact solution of (22) for each mode j.

(b) The solutions are characterized by the roots of the characteristic polynomial $\sigma^2 - r\sigma + \Lambda_{\nu,j} = 0$. The roots are easily found to be

$$\sigma_{1,2} = \frac{r}{2} \pm \left(\left(\frac{r}{2}\right)^2 - \Lambda_{\nu,j} \right)^{1/2}.$$

A quick inspection shows that if $\Lambda_{\nu,j} < 0$, then $\sigma_1 < 0$ and $\sigma_2 > \frac{r}{2}$ which is the usual saddle point stability. Furthermore, if $0 < \Lambda_{\nu,j} < (\frac{r}{2})^2$ then we obtain two real eigenvalues $0 < \sigma_1 < \frac{r}{2} < \sigma_2$. Finally in the case $\Lambda_{\nu,j} \geq (\frac{r}{2})^2$,

$$\sigma_{1,2} = \frac{r}{2} \pm i \left(\left| \left(\frac{r}{2}\right)^2 - \Lambda_{\nu,j} \right| \right)^{1/2}$$

so that we have modes growing with exponential growth rate $\frac{r}{2}$.[19] *Q.E.D.*

[19] This is compatible with the well posedness of the functional J, and/or with the transversality conditions.

8.4 Proof of Proposition 2

Proof of Proposition 2: (a) The modes that will be unstable will be these modes for which $\Lambda_{RE,,j} > 0$. Since, $s_{11} < 0$ this will imply that $s_{12}\mu_j > 0$ and large enough, i.e. the unstable modes will be those modes for which the sign of μ_j is the same as the sign of s_{12}. For $s_{12} > 0$ the unstable modes will be those modes with $\mu_j > 0$ and large enough such that $\mu_j > -\frac{s_{11}}{s_{12}}$ while for $s_{12} < 0$ the unstable modes will be those with $\mu_j < 0$ and such that $\frac{s_{11}}{s_{12}} < \mu_j < 0$.

(b) $\Lambda_{SO,j} \geq 0$ implies $-|s_{11}| + 2s_{12}\mu_j - |s_{22}|\mu_j^2 > 0$, i.e., μ_j must lie between the two real roots of this quadratic polynomial. However, strong concavity of the production function implies $s_{11}s_{22} - s_{12}^2 > 0$, therefore $\Lambda_{SO,j}$ keeps the sign of $-|s_{22}|$ for all values of μ_j so that $\Lambda_{SO,j} < 0$ for all j, which implies stability. *Q.E.D.*

8.5 Proof of Proposition 3

Proof of Proposition 3: (a) For every $x \in \mathbb{H}$ there exists a Fourier expansion in terms of Fourier series as $x(z) = \sum_{\ell=-\infty}^{\infty} x_\ell \exp(i\ell\pi z/L)$ with x_ℓ given by $x_\ell = \frac{1}{2L}\int_{-L}^{L} x(z)\exp(i\ell\pi z/L)dz$, where the convergence is in the $L^2(\mathcal{O})$ sense. A similar expansion exists for the kernel function w, $w(z) = \sum_{m=-\infty}^{\infty} w_m \exp(im\pi z/L)$. The condition for x to be real is $x_\ell^* = -x_{-\ell}$ where $*$ denotes the complex conjugate. To verify that the eigenfunctions are the Fourier modes, it suffices to observe that

$$(\mathsf{K}\phi_n)(z) = \frac{1}{\sqrt{2L}}\sum_\ell w_\ell \exp(i\ell\pi z/L)\int_{-L}^{L}\exp(i(n-\ell)\pi s/L)ds$$
$$= \sqrt{2L}\sum_\ell w_\ell \exp(i\ell\pi z/L)\delta_{n,\ell} = \sqrt{2L}w_n \exp(in\pi s/L) = W_n\phi_n(z)$$

where $W_n = 2Lw_n$. This calculation shows that the Fourier basis are eigenfunctions of K with eigenvalue $\mu_n = W_n$ at mode n. Note that this set of eigenfunctions forms a complete basis of $\mathbb{H}$. The symmetry of the kernel shows that only the cosine part of the eigenfunctions corresponds to nontrivial eigenvalues.

(b) The action of K on the flat state $\bar{x}$ is as follows:

$$\mathsf{K}\bar{x} = \bar{x}\int_{-L}^{L}\sum_{\ell} w_\ell \exp(i\ell\pi(z-s)/L)ds = 2Lw_0\bar{x},$$

therefore the flat state generates spillovers which remain uniformly distributed in space. *QED*

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