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Optimal Control in Space and Time and the Management of Environmental Resources

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Optimal Control in Space and Time and the Management of Environmental Resources*

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Abstract

We present methods and tools that can be used to study dynamic environmental resource management in a spatial setting, to explore

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spatially dependent regulation, and to understand pattern formation. In particular we present the maximum principle and its use in the context of the emerging frontier of applications of optimal control of diffusive transport processes to environmental and resource economics. We show how optimal spatiotemporal control induces pattern formation, and how deep uncertainty with a spatial structure can be handled with spatial robust control methods. Finally we show how models with diffusive transport can be extended to allow for long-range effects and more general transport mechanisms.

Keywords: Diffusion, optimal control, pattern formation, robust control, hot spots, spatial externalities.

Contents

1 Introduction

Research in the field of environmental and resource economics has been conducted in a predominantly dynamic framework in which the temporal evolution of the main factors characterizing the state of the environmental system under study is explicitly taken into account. Resource management issues - both in the context of renewable resources such as fisheries, or ecosystems where many resources interact among themselves; or exhaustible resources such as fossil fuels - are analyzed in terms of dynamic models where the resource stock is a state variable that evolves in time and harvesting or extraction per unit time is a control variable. The evolution of the state variables under the influence of resource growth functions and harvesting or extraction is modeled in general by dynamical systems consisting of nonlinear ordinary differential equations (ODE).

In a similar way pollution management problems are dynamic when pollution has stock and not flow characteristics, such as accumulation of phosphorus in a lake that may cause eutrophication, accumulation of sulfur dioxide and nitrogen oxide in the atmosphere causing acid rain, or accumulation of airborne particles and pollutants from combustion creating "brown clouds." In this case the state variable is the stock of pollutant, control variables are emissions or emission abatement, and again the system's evolution is described by dynamical systems of ODEs. In the context of climate change the state of the system is described by environmental variables such as the stock of greenhouse gases (GHGs) and temperature, along with economic variables such as the stock of capital across economies, or the stock of available fossil fuels. The dynamical system describing the climate and the economy, which is an integrated assessment model (IAM), includes both climate dynamics and economic growth dynamics and is represented by nonlinear ODEs. In IAMs, control variables that relate to climate change could be emissions, mitigation, or geoengineering.

The generally accepted method for solving problems in the areas mentioned above is by using optimal control and Pontryagin's maximum principle, or dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation. The maximum principle provides valuable information through the costate variables about the shadow value of the resources or the pollution stock along optimal paths, while the dynamic programming approach is very useful in characterizing feedback controls. In any case the method to be used depends on the existing problems and the information that the solution is supposed to provide. In this paper we will focus on reviewing work in the newly emerging frontier of applications of optimal control of diffusive transport processes to environmental and climate problems in economics. This forces us to deal with optimal control problems with infinite dimensional state spaces. Fortunately in many cases there are series expansion techniques that render such apparently formidable problems quite tractable.

Variables describing the state of an environmental system such as resources (renewable or exhaustible), pollutants, GHGs, heat, and precipitation have a profound spatial dimension in addition to their temporal dimension. This is because:

(i) Resources or pollutants are harvested, extracted, emitted, or abated in a specific location or locations.

(ii) The impacts of environmental variables, whether beneficial or detrimental, have a strong spatial dimension. For example in the context of climate change, temperature at the Poles increases faster than at the equator, which is Polar amplification (Alexeev et al. (2005), Alexeev and Jackson (2013); the brown cloud is mainly associated with South Asia and the Indian Ocean; acid rain impacts can be identified around the globe in places such as Eastern Europe, Scandinavia, the United States and Canada, and China; while fisheries have crashed in different parts of the world (e.g., Peruvian coastal anchovies, cod fishery off Newfoundland).

(iii) There is transport of environmental state variables across geographical space due to natural processes. Thus in climate change, the energy balance climate models (EBCM) explicitly account for the transport of heat across the globe from warm areas to cool areas by an amount proportional to the gradient of the temperature (North et al. (1981), Wu and North (2007)). In models that combine energy balance and moisture balance (e.g. Fanning and Weaver (1996)) there is horizontal heat and moisture transport across the globe. Air-borne contaminants are transported in the atmosphere from the source of emissions due to turbulent eddy motion and wind (Weaver et al. (2001), Fanning and Weaver (1996)).

The discussion above suggests that the spatial dimension is important in environmental and resource economics. When forward-looking optimizing economics agents that take decisions regarding resource management or emissions ignore transport effects, they essentially ignore the impact of their own actions on the utility or profits of agents located at different sites. This is a spatial externality, which is not internalized. Therefore policy must involve mechanisms to internalize spatial spillovers, along with potential temporal spillovers.

While there is a huge literature related to the spatial economy in the context of New Economic Geography, the papers closest to the research area which concerns us are Desmet and Rossi-Hansberg (2010) and Desmet and Rossi-Hansberg (2012). Much less research has been done that emphasizes the spatial aspect in environmental and resource economics, although there are notable exceptions in several cases. Spatially dependent taxes have been proposed to regulate environmental externalities (Goetz and Zilberman (2000), Xabadia et al. (2004a), Xabadia et al. (2004b), Goetz and Zilberman (2007), Kyriakopoulou and Xepapadeas (2013)); pattern formation and regulation have been studied in the context of semi-arid areas models with explicit plant dispersal and transport of surface and soil water across the semi-arid areas (HilleRisLambers et al. (2001), Brock and Xepapadeas (2010)); spatial fishery models have been developed around metapopulation models that combine larval-dispersal processes and adult movements in a geographical space of multiple interconnected patches (Wilen (2007), Smith et al. (2009)). Spatial models of climate and the economy have also been developed, which combine economic growth models with climate models based on ECBM with heat transport (e.g. Brock et al. (2013)), or with a temperature function with spatial characteristics (Desmet and Rossi-Hansberg (2012)), or with regional damage functions (Nordhaus (2010), Hassler and Krusell (2012)).

The lack of substantial literature incorporating spatial issues in environmental and resource economics can be attributed to the technical difficulties involved when the mathematics of optimal control theory dealing with finite or infinite horizon problems is extended to infinite dimensional state spaces that naturally emerge when optimization takes place in spatiotemporal domains. It should be noted that dealing with infinite horizon problems seems to have been barely touched in the New Economic Geography literature, because of the need to deal with infinite dimensional spaces, which also explains their absence from the environmental economics literature. The exceptions in the environmental economics literature mentioned above try to overcome the mathematical complication by imposing a certain structure to the problem that allows simplifications and sometimes closed form solutions. However, the importance of transport phenomena in environmental economics, and the need to design regulation for internalizing spatial externalities emerging from these transport phenomena, makes it necessary to extend dynamic optimization methods into spatial settings.

In this context our paper studies dynamic optimization for the management of environmental resources in spatiotemporal settings, where spatial transport phenomena across space are explicitly taken into account. We present approaches that deal with dynamic optimization in infinite dimensional spaces which can be used as tools in environmental and resource economics, along with examples of their application. We also present methods which can be used to study the emergence of spatial patterns in dynamic optimizations models. Our methods draw on the celebrated Turing diffusion induced instability but are different from Turing's mechanism since they apply to forward-optimization models. We believe that this approach provides the tools to analyze a wide range of problems with explicit spatial structure which are very often encountered in environmental and resource economics. [1]

2 Modeling Spatial Transport Phenomena

To model transport phenomena we consider a bounded spatial domain $\mathcal{O} \subset \mathbb{R}^d$ representing geographical space. In most applications the geographical space will be one- or two-dimensional, i.e., $d = 1$ or $d = 2$. It is often assumed that the spatial domain is a circle or a torus in order to eliminate the influence of boundary conditions on outcomes related to the emergence of spatial patterns. Let $y\,(t, x)$ denote the stock or the concentration or the density of an environmental state variable at time $t \geq 0$ and spatial point $x \in \mathcal{O}$. The spatial behavior of y is modeled by assuming that the functions $y\,(t, \cdot)$ belong for all t to an appropriately chosen function space $\mathbb{H}$.[2]

The state of the system at point $s \in \mathcal{O}$ is expected to influence the state of the system at point $x \in \mathcal{O}$ through transport phenomena. For instance biomass, pollution, or heat can be transported across locations. Let us assume that $y(t, x)$ represents the concentration of a quantity of interest at point $x \in \mathcal{O}$ and time t, e.g., biomass. This stock grows locally (i.e. the local population multiplies and dies) and is transported from x to other locations while at the same time biomass from other locations is transported to x.

In this context consider, therefore, a dynamic fishery occupying an area which for simplicity is taken to be one-dimensional, i.e., a finite line segment, or a circle. Thus $y\,(t, x)$ is the biomass concentration at time t and spatial

[1] There is a substantial literature on IAMs in which "space" is taken into account (e.g. the big IAMs RICE/DICE, PAGE, FUND). General Circulation Models (GCMs) are also full of spatial transport phenomena but the models are so complicated that all the work is essentially numerical. We are trying to add value by working with models of intermediate complexity so that the reader can actually understand the forces and the mechanisms generating the results.

[2] A convenient choice is to let $\mathbb{H}$ be a Hilbert space such as the space of square integrable functions on $\mathcal{O}$, or an appropriately chosen subspace, e.g. the space of square integrable functions on $\mathcal{O} = [-L, L]$ satisfying periodic boundary conditions in order to model a circular spatial domain.

point x (see for example the models presented in sections 4.1 and 4.1). Assume that biomass located at point x moves to nearby locations and that the direction of the movement is such that biomass from locations where biomass is abundant, i.e., location of high biomass concentration, moves towards locations of low biomass concentration. This is the assumption of Fickian diffusion, or Fick's first law, and is equivalent to stating that the flux of biomass denoted by $J(t,x)$ is proportional to the gradient of the biomass concentration, i.e., the spatial derivative of concentration, or

$$J(t,x) = -D\frac{\partial y(t,x)}{\partial x} \tag{1}$$

where D is the *diffusion coefficient or diffusivity* measuring how fast biomass moves from locations of high concentration to locations of low concentration. Fickian diffusion shown by (1) can be derived by a random walk approach to diffusion in which individual particles move randomly backward and forward along a line in fixed steps (see for example Murray (2002)). In the random walk approach, the diffusion coefficient D measures how efficiently particles disperse from a high to a low density.

In a region $x_0 < x < x_1$, the rate of change of the biomass is equal to the rate of flow across the boundary plus the net amount of biomass created in the region. If the biomass grows in the region according to logistic growth $f(y(t,x)) = \rho y(x,t)\left(1 - \frac{y(x,t)}{K}\right)$, where ρ is an intrinsic growth rate and K is a carrying capacity, and is reduced by harvesting at a rate $u(t,x)$, then by writing $F(y(t,x),u(t,x)) = f(y(t,x)) - u(t,x)$, the total rate of change of biomass in the region $[x_0, x_1]$ is:

$$\frac{\partial}{\partial t}\int_{x_0}^{x_1} y(t,x)dx = \int_{x_0}^{x_1} F(y(t,x),u(t,x))dx + J(t,x_0) - J(t,x_1) \tag{2}$$

where the integral on the right hand side denotes the net biomass creation in the region, and the term $J(t,x_0) - J(t,x_1)$ denotes the rate of biomass flow across the boundary of the region. This setup is presented in Figure 1.

FIGURE 1

Fickian diffusion

Taking $x_1 = x_0 + \Delta x$ and the limit as $\Delta x \to 0$ we obtain

$$\frac{\partial}{\partial t} y(t,x) = F(y(t,x), u(t,x)) + \frac{\partial J\,(t,x)}{\partial x}. \tag{3}$$

Using (1) to obtain $\frac{\partial J(t,x)}{\partial x} = -D\frac{\partial y^2(t,x)}{\partial x^2}$, the spatiotemporal evolution of the biomass is described by the diffusion equation:

$$\frac{\partial}{\partial t} y(t,x) = D\frac{\partial^2}{\partial x^2} y(t,x) + F(y(t,x), u(t,x)). \tag{4}$$

This type of diffusion equation has been used in the examples of section 4.[3] More complex cases of diffusion, when for example D is not constant, or the spatial domain is not one-dimensional, can be modelled by the more general approach summarized in the rest of this section.

Consider a general bookkeeping equation:

$$\frac{\partial}{\partial t} y(t,x) = B(t,x) - D(t,x) + I(t,x) - O(t,x), \tag{5}$$

where B, D are births and deaths at x, I is the influx of biomass into x from other locations and O is the outflux from x to other locations. Let $\mathfrak{f}(t,x) = B(t,x) - D(t,x)$ the local growth term and $\Psi(t,x) = I(t,x) - O(t,x)$ the flux term. If no new biomass is generated in $\mathcal{O}$, i.e. $B - D = \mathfrak{f} = 0$, so the local rates of change are due to biomass being transported while the total quantity is kept constant then Ψ must have the property that $\int_{\mathcal{O}} \Psi dx = 0$. The bookkeeping equation (5) in the absence of net local birth and death rates can be expressed (invoking Gauss' divergence theorem) as

$$\frac{\partial}{\partial t} y(t,x) = -\nabla \cdot J(t,x), \tag{6}$$

where J is a vector field often called the flux of material and $\nabla \cdot$ is the divergence operator (sometimes the alternative notation div$J$ is used). In

[3] The pollution equation of section 4.3 can be easily derived by following the same approach.

Cartesian coordinates $J = (J_1, J_2, J_3)$ and

$$\nabla \cdot J = \sum_{i=1}^{3} \frac{\partial J_i}{\partial x_i}.$$

We now consider a model where the local flux of material J is given by a generalization of Fick's law, that is we consider that material flows to regions of lower density but now spatial variability with respect to all possible directions has to be taken into account and this is done using the spatial gradient operator ∇ (not to be confused with $\nabla\cdot$) that acts on a scalar field y providing a vector field defined (in Cartesian coordinates) as $\nabla y = \left(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \frac{\partial y}{\partial x_3}\right)$. The three-dimensional generalization of Fick's law is $J(t,x) = -D(t,x)\nabla y(t,x)$ and therefore the vector field is

$$J = (J_1, J_2, J_3) = (-D\frac{\partial y}{\partial x_1}, -D\frac{\partial y}{\partial x_2}, -D\frac{\partial y}{\partial x_3}). \tag{7}$$

Note that in the one-dimensional case, where y depends only on $x = x_1$, our expression for J degenerates to a scalar, $J(t,x) = -D\frac{\partial y}{\partial x}$. Substituting (7) in (6) gives us a partial differential equation (PDE) for the evolution of y of the form

$$\frac{\partial}{\partial t} y(t,x) = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left(D(t,x) \frac{\partial y}{\partial x_i}(t,x) \right)$$

where we use the shorthand notation $x = (x_1, x_2, x_3)$. If D is independent of x, then the above equation is simplified to

$$\frac{\partial}{\partial t} y(t,x) = D\Delta y(t,x),$$

where Δ is the Laplace operator defined (in Cartesian coordinates) as

$$\Delta y(t,x) = \frac{\partial^2}{\partial x_1^2} y(t,x) + \frac{\partial^2}{\partial x_2^2} y(t,x) + \frac{\partial^2}{\partial x_3^2} y(t,x).$$

Consider local growth $\mathfrak{f} = B - D$, which may depend on the current local state of the system at x and possibly a control procedure $u\,(t,x)$ which in environmental and resource economics could reflect policies such as harvesting, emissions, pollution control, or extraction . Expressing $\mathfrak{f}$ by $\mathfrak{f}(t,x) = F(y(t,x), u(t,x))$, the model becomes:

$$\frac{\partial}{\partial t} y(t,x) = \nabla \cdot (D \nabla y(t,x)) + F(y(t,x), u(t,x)), \tag{8}$$

which for constant D and a one-dimensional space reduces to (4).

In the case, for example, of a two-dimensional EBCM (e.g. North et al. (1981), or Brock et al. (2013) for an economic application of this model in the economics of climate change), the evolution of surface temperature at latitude $\theta \in [-\pi/2, \pi/2]$ and longitude $\phi \in [-\pi, \pi]$ denoted by $T(r,t)$, $r = (\theta, \phi)$ is given by

$$C(r) \frac{\partial T(t,r)}{\partial t} = h(T(t,r), \mathbf{v}) - I(r,t) + E(t) + \nabla \cdot (D(x) \nabla T(r,t)) \tag{9}$$

where $C(r)$ denotes local heat capacity, $h(T(t,r), \mathbf{v})$ denotes incoming solar radiation as a function of surface temperature and parameters $\mathbf{v}$ of the natural system, $I(r,t)$ denotes outgoing radiation, $E(t)$ denotes global emissions of GHGs from anthropogenic sources and $D(x)$, $x = \sin\theta$ represents the diffusion term for all different forms of heat transport across the globe. A common parametrization for this term is $D(x) = D_0 (1 + D_2 x^2 + D_4 x^4)$.

The possibly nonlinear function F reflects the dynamics of the system even in the absence of spatial effects. A common general choice for F is a separable form such as $F(y,u) = f(y) - Bu$. Thus $f(y)$ will denote resource dynamics and u could be a harvesting, emission or extraction function and B a mapping between function spaces, which quantifies the effect that harvesting at any point $x \in \mathcal{O}$ will have on the rate of change of the state of the system at point $x \in \mathcal{O}$. In (9) for example f is the term $h(T(t,r), \mathbf{v}) - I(r,t)$ and u is the term $E(r)$. The map B is called the control to state map and may be either local or nonlocal in space. This map can be illustrated by two examples. The first one is when we harvest from the population directly at every point at a local rate $u(t,x)$, then $(Bu)(t,x) = u(t,x)$ and $B = I$ the identity mapping, as is also the case for $E(t)$ in (9). This is the typical harvesting or extraction, or emission control problem. The second one is when we harvest at a local rate $u(t,x)$ in $\mathcal{O}_h \subset \mathcal{O}$ (i.e., we only harvest at selected points in our domain), then $(Bu)(t,x) = \mathbf{1}_{\mathcal{O}_h}(x)u(t,x)$ where $\mathbf{1}_{\mathcal{O}_h}$ is the indicator function of the subset $\mathcal{O}_h$, meaning that $(Bu)(t,x)$ vanishes if $x \neq \mathcal{O}_h$. This can be regarded as an example of protected areas within the spatial domain of interest. The dynamics of the system's state in the absence of control depend on the specific problem analyzed. If we use logistic growth in modeling the spatiotemporal evolution of biomass, our model is reduced to the celebrated Fisher-Kolmogorov-Petroskii-Piskunov (FKPP) equation which is commonly used in the natural resources literature.

Equation (8) must be complemented with an initial condition $y(0,x) = y_0(x)$, and boundary conditions. The boundary conditions will provide information about what the solution is expected to do at all times at the boundary of the spatial domain $\mathcal{O}$. Possible boundary conditions are: (i) Periodic boundary conditions; (ii) Dirichlet type boundary conditions which means specifying the concentration y on the boundary. A particular class is setting this quantity to 0 which corresponds to hostile boundaries; (iii) Neumann type boundary conditions which means specifying some coordinates of the flux vector at the boundary. A common choice is setting the flux in the normal direction equal to 0 corresponding to impermeable boundaries. In one dimension, boundary conditions for (4) could take different forms: (i) The spatial domain is a circle or $y\left(t,-L\right) = y\left(t,L\right)$ for all t, (ii) hostile boundaries or $y\left(t,-L\right) = y\left(t,L\right) = 0$ for all t, or (iii) zero flux at the boundaries or $\frac{\partial y(t,-L)}{\partial x} = \frac{\partial y(t,L)}{\partial x} = 0$ for all t.

The above formulation allows for the treatment of any number of interacting species with spatially nonhomogeneous densities. For example a spatially dependent Lotka-Volterra predator-prey model, with population densities at time t and point $x \in \mathcal{O}$, $y_1(t,x)$ and $y_2(t,x)$ respectively, takes the form

$$
\begin{aligned}
\frac{\partial}{\partial t}y_1(t,x) &= f_1(y_1(t,x), y_2(t,x)) - (B_1u_1)(t,x) + D_1\frac{\partial^2}{\partial x^2}y_1(t,x),\\
\frac{\partial}{\partial t}y_2(t,x) &= f_2(y_1(t,x), y_2(t,x)) - (B_2u_2)(t,x) + D_2\frac{\partial^2}{\partial x^2}y_2(t,x),
\end{aligned}
$$

where u_1, u_2 are the harvesting of the two species, B_1, B_2 are the relevant control to state operators, and f_1, f_2 are the functions modelling the nonlinear interactions between the species. The species may have different transport terms, as denoted by the difference in the diffusion coefficients $D_1 \neq D_2$, while more complicated transport terms may also be considered.

3 Dynamic Optimization in a Spatial Setting

Having obtained a suitable model of the general form (4), for the evolution of the state of the system subject to the choice of a particular control process u, we examine now how to choose u optimally so as to optimize an objective defined in terms of the spatial domain and the spatial and temporal evolution of the system. In environmental and resource economics the traditional objective is to maximize a general benefit or utility function that may depend on the control process and resource's stock. For example in models with harvesting, the benefit function depends on the flow of harvest and the resource stock, while in pollution control models the benefit function depends on the flow of emissions and the stock of pollutants.

A general formulation for the dynamic optimization problem in a spatial setting with forward-looking agents can be defined in terms of a local benefit (or utility) function $U(t, x, y(t, x), u(t, x))$ which assigns benefits at time t and spatial point $x \in \mathcal{O}$ given that the state of the system is $y(t, x)$ and the control exerted is $u(t, x)$. Given U we consider the following spatial optimization problem:

$$\max_{u \in \mathcal{U}} \int_{x \in \mathcal{O}} \int_0^\infty e^{-rt} U(t, x, y(t, x; u), u(t, x)) dt dx, \text{ subject to } (4)$$

where $r > 0$ is a discount rate. This problem can be interpreted as corresponding to a situation where a social planner (or regulator) chooses a control procedure in order to maximize global benefits for the whole spatial domain by taking into account spatial interactions.

This is an infinite dimensional optimization problem in which we seek to find a function $u^* \in \mathcal{U}$ (within the admissible class of controls) such that the process $y(u^*)$, i.e., the solution to (4) with $u = u^*$, is the maximizer of the above functional. There are two general approaches to this problem, which under technical conditions are equivalent. The first is a dynamic programming approach which yields a PDE for the value of the problem, called the Hamilton-Jacobi-Bellman (HJB) equation. This equation is a PDE on an infinite dimensional Hilbert space. The optimal control policy can be found in terms of the solution of this equation but the mathematical treatment, though elegant and rigorous, soon moves beyond standard expertise encountered in the economics profession. The other approach is that of the maximum principle. This is a generalization in Hilbert space of the well-known maximum principle often employed in the treatment of finite dimensional economics systems. According to this approach, we need to introduce an auxiliary process p which is called the costate. In this approach, a dynamical evolution law is derived for the costate variable p and the optimal control u^* is obtained as a functional of the process p. The infinite dimensional version of the maximum principle, though formally looking very similar to its finite dimensional counterpart, presents important mathematical intricacies which we cannot fully treat here. We will thus simply state it and use it in order to derive certain important phenomena that may arise in dynamic optimization in a spatial setting (see for example Derzko et al. (1980), Derzko et al. (1984), Brock and Xepapadeas (2008), Brock and Xepapadeas (2010)).

We begin our presentation of the maximum principle with a finite horizon version of the problem of the form

$$\max_{u\in\mathcal{U}} \int_{x\in\mathcal{O}} \int_0^T e^{-rt} U(t,x,y(t,x;u),u(t,x))dtdx + \int_{\mathcal{O}} e^{-rT}\Phi(y(T,x))dx,$$
$$\text{subject to } (4)$$

where we assume that the period over which the system is controlled is the finite interval $[0,T]$ and a value Φ for the final state of the system y is also added. We also assume that $\mathcal{U} := \{u : [0,T]\times\mathcal{O} : u(t,x) \in C \subset R,\ t \in [0,T],\ x \in \mathcal{O}\}$, where C is an appropriate subset of $\mathbb{R}$ (typically $[0,b]$).

The first step in the formulation of the maximum principle is the definition of the current value Hamiltonian function $H : [0,T]\times\mathcal{O}\times\mathbb{R}\times\mathbb{R}\times C \to \mathbb{R}$ as

$H(t,x,y,p,u) = U(t,x,y,u) + pF(t,x,y,u)$ which is now considered as a function from a subset of a finite dimensional set into $\mathbb{R}$. Next we maximize the function H over all $u \in C$ (this is treated as a finite dimensional and static optimization problem) and obtain the maximizer $u^* = \arg\max H(t,x,y,p,u)$, which is evidently a function of the remaining variables now treated as parameters. We thus denote $u^* = \mathcal{C}(t,x,y,p)$, where $\mathcal{C} : [0,T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the function which describes the variability of the maximizer of the Hamiltonian function when t, x, y, p are treated as parameters. We then define the function $H^* : [0,T] \times \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $H^*(t,x,y,p) = H(t,x,y,p,u^*)$. We summarize this first step as

$$\begin{aligned} H(t,x,y,p,u) &= U(t,x,y,u) + F(t,x,y,u)p, \\ u^*(t,x,y,p) &:= \mathcal{C}(t,x,y,p) = \arg\max_{u \in C} H(t,x,y,p,u), \\ H^*(t,x,y,p) &:= H(t,x,y,p,u^*(t,x,y,p)). \end{aligned} \tag{10}$$

The function $\mathcal{C}$ will be used in the sequel to define a feedback control rule.

We then consider the following evolution equation for the process p,

$$\begin{aligned} &\frac{\partial}{\partial t} p(t,x) = rp(t,x) - D\frac{\partial^2}{\partial x^2} p(t,x) - \frac{\partial}{\partial y} H^*(t,x,y(t,x),p(t,x)) \\ &p(x,T) = \frac{\partial}{\partial y}\Phi(y(T,x)), \end{aligned}$$

which is now a final value problem, i.e., a problem treated with a final rather than an initial condition. The third term is the derivative of the function H^* defined above, with respect to the static variable y, and then this function is calculated at $(t,x,y(t,x),p(t,x))$. It can be proved using duality arguments that if (y^*,u^*) is an optimal pair for the finite horizon system, then y^* and

u^* must solve the following system of forward-backward PDEs

$$\frac{\partial}{\partial t} y^*(t,x) = +D\frac{\partial^2}{\partial x^2} y^*(t,x) + \frac{\partial}{\partial p} H^*(t,x,y^*(t,x),p^*(t,x)), \tag{11}$$

$$y^*(0,x) = y_0(x), \tag{12}$$

$$\frac{\partial}{\partial t} p^*(t,x) = -D\frac{\partial^2}{\partial x^2} p^*(t,x) + rp^*(t,x) - \frac{\partial}{\partial y} H^*(t,x,y^*(t,x),p^*(t,x)), \tag{13}$$

$$p^*(x,T) = \frac{\partial}{\partial y}\Phi(y(T,x)) \tag{14}$$

and u^* is connected with the pair (y^*, p^*) by

$$u^*(t,x) = \mathcal{C}(t,x,y^*(t,x),p(t,x)). \tag{15}$$

An explanation for the appearance of the term $D\frac{\partial^2 p}{\partial x^2}$ in the costate equation (13) can be provided by considering a heuristic derivation of conditions (11)-(14) using a standard variational approach for deriving the necessary conditions for the maximum principle (e.g. Kamien and Schwartz (1991)). In this case the term $-D\int_{x\in\mathcal{O}} p\left(t,x\right) \frac{\partial^2 y(t,x)}{\partial x^2} dx$ which appears when the variational argument is applied should be integrated by parts twice with respect to x in order to express the second derivative of y with respect to x in terms of the derivatives of p with respect to x. This leads to the appearance of the term $-D\int_{x\in\mathcal{O}} x\left(t,x\right) \frac{\partial^2 p(t,x)}{\partial x^2} dx$ and finally to (13). For the details of this approach see for example Brock and Xepapadeas (2006).

Questions regarding sufficiency and necessity of the above maximum principle are related to the concavity properties of the Hamiltonian (see, e.g., Oksendal (2005)).

So our strategy for looking for a candidate for an optimal control and optimal path can be summarized as follows:

- ▷ Solve the static optimization problem (10).
- ▷ Solve the forward-backward parabolic problem (11)-(14) to obtain the pair of functions (y^*, p^*).
- ▷ Calculate the optimal control using (15).

A few remarks are in order at this point.

1. System (11)-(14) is a forward-backward system, the state variable (forward variable) y is treated as an initial value problem while the costate variable (backward variable) p is treated as a final value problem. The costate has an important economic interpretation since it represents the shadow value of the state at each point of time and space, i.e., it is a local dynamic shadow value.

2. Note the change in sign in front of the diffusion operator in the costate (backward) equation (13) of system (11)-(14). This is important for the well posedness of the system since a diffusion equation cannot be solved backwards in time. This is an ill-posed problem.

3. We point out that in the spatially homogeneous case, the state-costate (forward-backward) equations (11), (13) become the ODEs used in optimal control problems where only temporal variability of the state is taken into account. This can be treated with a phase plane analysis which provides a nice geometrical intuition of the optimal path as a stable saddle path.

4. There exist generalizations for systems like the Lotka-Volterra system and for more general types of transport equations.

5. There is a case where uncertainty can also be treated using the maximum principle, but that requires the introduction of a new type of stochastic differential equations called forward-backward stochastic differential equations (see Oksendal (2005)). For a generalization of the saddle point concept in this context, see Yannacopoulos (2008).

6. While equations (11)-(14) are written as an equality that holds for every $(t,x) \in [0,T] \times \mathcal{O}$, i.e., we assume that (y^*, p^*) is a classical solution, PDEs very rarely have solutions that enjoy this property. However, the maximum principle is still valid if the solutions are understood in a more generalized form, called the weak formulation, in which the functions (y^*, p^*) are considered as elements of appropriately chosen Lebesgue spaces, as is standard practice in the modern theory of PDEs.

7. In the infinite horizon case we must take the above finite horizon results appropriately to the limit $T \to \infty$. This requires some knowledge concerning the behavior of the solution as $t \to \infty$, called the transversality condition.

Loosely speaking this involves setting $\lim_{t\to\infty}\int_{\mathcal{O}} e^{-r\,t} y(t,x)p\,(t,x)\,dx = 0$ (Benveniste and Scheinkman (1982)). These transversality conditions determine the behavior of (y,p) as $t \to \infty$ and even though the phase space here is infinite dimensional, we may consider the familiar analogue of the saddle point in two-dimensional phase space.

8. There are many excellent expositions of control theory for PDEs. Unfortunately as space is limited we limit ourselves to the 21st century and from this set only cite three: Glowinski et al. (2008), which also provides a lot of important information concerning the numerical treatment of such problems; Komornik and Loreti (2005), an excellent and highly readable exposition which at the same time provides deep insight into the abstract theoretical issues; and Zuazua (2007), a valuable introduction which leads the reader to the current problems in the field.

4 The Maximum Principle in Environmental Economics: Examples

4.1 Target following in the FKPP model: Linear quadratic problems

Consider the case where the objective of the decision maker is to control the system so as to keep it as close as possible to a predetermined target y_d which may be a function of space and time. Deviations from this target are costly and so is the control procedure needed to keep the system close to the desired target. In the limit of small deviations from the target, the evolution equation may be approximated by the linearized equation around the target, and the cost functional which models the distance from the target may be assumed to be quadratic in the deviations in state and control. This leads us to a wide class of spatial optimal control problems, that of linear quadratic problems.

As a concrete example, consider a model where the spatio-temporal distribution of biomass is evolving according to an FKPP equation of the form

$$\frac{\partial}{\partial t}y(t,x) = D\frac{\partial^2}{\partial x^2}y(t,x) + \rho y(t,x)\left(1 - \frac{y(t,x)}{K}\right) - u(t,x), \tag{16}$$

where ρ is the intrinsic growth rate, K is the carrying capacity and u is the harvesting rate. Assume that the desired state of the population is a fraction γ of the carrying capacity uniformly over space, i.e. $y_d(t,x) = \gamma K$, $t \in [0,T]$, $x \in \mathcal{O}$. It can be seen that this is achieved by imposing a constant and homogeneous in space harvesting rate $u(t,x) = u_d(t,x) = \rho K\gamma(1-\gamma)$. Suppose now that for some reason, this desired equilibrium has been disrupted, and the state of the system at time $t = 0$ is equal to $y(0,x) = \gamma K + \bar{y}(0,x)$. Obviously, this spatial disruption is going to "propagate" through space and time and is going to create spatiotemporal deviations from the desired states whose exact form is given by the solution of (16). Setting $y(t,x) = \gamma K + \bar{y}(t,x)$ and $u(t,x) = \rho K\gamma(1-\gamma) + \bar{u}(t,x)$, in (16) we obtain

$$\frac{\partial}{\partial t}\bar{y}(t,x) = \rho(1-2\gamma)\bar{y}(t,x) - \bar{u}(t,x) - \frac{\rho}{K}\bar{y}(t,x)^2 + D\frac{\partial^2}{\partial x^2}\bar{y}(t,x). \tag{17}$$

Keeping the system as close as possible to the desired state and harvesting target y_d and u_d means keeping the deviations $\bar{y}$ and $\bar{u}$ as close as possible to zero. Suppose that we wish to choose $\bar{u}$ so as to minimize the functional

$$J_\epsilon = \int_{\mathcal{O}}\int_0^T e^{-rt}\left(\frac{P}{2\epsilon}\bar{y}(t,x)^2 + \frac{Q}{2}\bar{u}(t,x)^2\right) dtdx$$

for some $\epsilon > 0$, under the dynamic constraint (17). Note that by the choice of control functional to be minimized the state $\bar{y}$ for the optimal path will be kept small and of order comparable to ϵ (if J_ϵ is finite then necessarily the integral of $\bar{y}^2$ over all space and time must be of order ϵ) so the quadratic term in (17) will be small and may be neglected, keeping only the linear terms in the equation. The target following problem can thus be considered as a linear quadratic optimal control problem, i.e., a problem of minimizing

J_ϵ under the linear dynamic constraint

$$\frac{\partial}{\partial t}\bar{y}(t,x) = \rho(1-2\gamma)\bar{y}(t,x) - \bar{u}(t,x) + D\frac{\partial^2}{\partial x^2}\bar{y}(t,x).$$

The solution of this problem by construction stays ϵ- close to the desired target. This can be treated with the application of the maximum principle[4] which yields

$$\begin{aligned} u^* &= -\frac{p}{Q}, \\ H^* &= -\frac{P}{2\epsilon}y^2 + \frac{1}{2Q}p^2 + \rho(1-2\gamma)yp, \end{aligned}$$

and the optimal path is provided by the solution of the forward-backward system

$$\begin{aligned} \frac{\partial}{\partial t}\bar{y}^*(t,x) &= \rho(1-2\gamma)\bar{y}^*(t,x) - \frac{\bar{p}^*(t,x)}{Q} + D\frac{\partial^2}{\partial x^2}\bar{y}^*(t,x), \\ \frac{\partial}{\partial t}\bar{p}^*(t,x) &= r\bar{p}^*(t,x) + \frac{P}{\epsilon}\bar{y}^*(t,x) - \rho(1-2\gamma)\bar{p}^*(t,x) - D\frac{\partial^2}{\partial x^2}\bar{p}^*(t,x) \end{aligned} \tag{18}$$

with initial condition $\bar{y}^*(0,x) = \bar{y}(0,x)$ and final condition $\bar{p}^*(T,x) = 0$. The forward-backward system (18) is a special case of the general linear quadratic problem (39) described in Section 9.1 in the Appendix for the choice of parameters $c_1 = \rho(1-2\gamma)$, $c_2 = -\frac{1}{Q}$, $c_3 = \frac{P}{\epsilon}$, $c_4 = -c_1$. System (18) admits an exact solution following the procedure, described in detail in Section 9.1 in the Appendix, in terms of a Fourier series expansion, which if we assume hostile boundary is

$$\begin{aligned} \bar{y}^*(t,x) &= e^{\frac{rt}{2}}\sum_n \left(\bar{A}(n)e^{\bar{\sigma}(n)t} + \bar{B}(n)e^{-\bar{\sigma}(n)t}\right)\sin\left(\frac{n\pi}{L}x\right), \\ \bar{u}^*(t,x) &= -\frac{1}{Q}e^{\frac{rt}{2}}\sum_n \left(A(n)e^{\bar{\sigma}(n)t} + B(n)e^{-\bar{\sigma}(n)t}\right)\sin\left(\frac{n\pi}{L}x\right), \end{aligned}$$

[4]Minimizing J is equivalent to maximizing $-J$.

with

$$\bar{\sigma}(n) = \frac{1}{2}\sqrt{r^2 - 4b_n},$$
$$b_n = -\lambda_n^2 - (r + c_4 - c_1)\lambda_n + (c_1 r + c_1 c_4 - c_2 c_3),$$
$$\lambda_n = \frac{D\pi^2}{L^2} n^2$$

and the constants $\bar{A}(n)$, $\bar{B}(n)$, $A(n)$ and $B(n)$ are given explicitly by equations (43) and (44) in the Appendix.

This explicit solution highlights the spatial variability of the optimal control as well as the spatial variability of the deviation from the target and shows clearly that in the presence of transport, dealing only with the temporal aspects of the optimal control problem can be misleading. In Figure 2 we present the optimal path and the optimal control for a typical set of parameters for this problem.

FIGURE 2

Optimal path and optimal control policy for the target following example

This solution, as well as the solution of more general linear quadratic problems, leads to a spatial feedback control rule. Therefore, an important observation is that the optimal stabilization policy is space dependent, i.e., there is no way to bring the system optimally close to the desired state unless we apply different policy procedures at different spatial points. A one-size-fits-all policy (i.e., a spatially homogeneous policy) is clearly sub-optimal!

Linear quadratic control problems play an important role in the theory of spatial optimal control for the following two reasons. First, they are one of the few cases that are subject to exact analytic solutions or to semi-analytic solutions through use of the Ricatti equation, and as such offer important insight into the phenomena that may arise in dynamic optimization in spatial settings. Second, nonlinear problems may be approximated around a desired state by a properly constructed version of a linear quadratic control problem. For details on the construction of this approximation in which the linearization is ingeniously designed to take into account the effects of higher order terms in the dynamics, the reader may consult the classic work of Magill (e.g., Magill (1977a), Magill (1977b)) and the excellent exposition in Judd (1998)). For a detailed discussion of the tracking problem with a special reference to resource economics, see e.g. Brock et al. (2012d).

4.2 Renewable resource harvesting in a spatial setting

Assume that y models the spatiotemporal distribution of biomass (e.g. population of fish in a spatially distributed fishery) and u is a harvesting function. Harvesting generates revenues at location $x \in \mathcal{O}$ according to a concave revenue function $\mathsf{R} : \mathbb{R} \to \mathbb{R}$, so that $R(u(t,x))$ denotes the revenues from harvesting at time t and point $x \in \mathcal{O}$. Harvesting costs are stock dependent and modelled by a function $\mathsf{C} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that harvesting costs at time t and point $x \in \mathcal{O}$ are $\mathsf{C}(u(t,x), y(t,x))$. The stock of fish generates further environmental benefits associated with values from non-consumptive services (e.g. regulation, or existence values) modelled by a function $\mathsf{B} : \mathbb{R} \to \mathbb{R}$ such that environmental benefits at time t and point $x \in \mathcal{O}$ are $\mathsf{B}(y(t,x))$. Let the equation for the evolution of the biomass be of the form

$$\frac{\partial}{\partial t} y(t,x) = D \frac{\partial^2}{\partial x^2} y(t,x) + F(y(t,x)) - u(t,x). \qquad (19)$$

A regulator seeks to maximize discounted benefits for the whole spatial domain by choosing the harvesting function u to maximize the functional

$$J := \int_{\mathcal{O}} \int_0^T e^{-rt} [\mathsf{R}(u(t,x)) - \mathsf{C}(u(t,x)), y(t,x)) + \mathsf{B}(y(t,x))] dt dx + \\ e^{-rT} \theta \int_{\mathcal{O}} y(T,x) dx,$$

under the dynamic constraint (19) and possibly the constraint $y(t,x) \geq 0$ for all t, x, where θ is the value per unit of fish biomass at the end of the planning horizon. In order to make the algebra simpler and to provide an explicit form for the resulting forward-backward system, as we only use this example for the sake of illustration, we choose $\mathsf{R}(u) = \ln(u)$, assume that the cost function depends only on u, i.e., $\mathsf{C}(u,y) = \alpha u$, $\alpha > 0$, and we set $r = 0$.

The Hamiltonian for this problem is

$$H(t,x,y,p,u) = \mathsf{R}(u) - \mathsf{C}(u,y) + \mathsf{B}(y) + (F(y)-u)p,$$

and defines $u^* = \arg\max_u H(t,x,y,p,u)$. Then, the first order condition for maximization of H, for the specific choice of the functions R and C, yields

$$u^* = \frac{1}{p+\alpha}, \tag{20}$$

and this defines the function $\mathcal{C}$. We may then calculate H^* as

$$\begin{aligned} H^*(t,x,y,p) &= -\ln(p+\alpha) - \frac{\alpha}{p+\alpha} + \left(F(y) - \frac{1}{p+\alpha}\right)p \\ &= -\ln(p+\alpha) + F(y)p - 1 \end{aligned}$$

and therefore

$$\frac{\partial}{\partial y}H^* = \frac{d}{dy}F(y)p, \quad \frac{\partial}{\partial p}H^* = -\frac{1}{p+\alpha} + F(y).$$

The forward-backward equation for this problem becomes

$$\frac{\partial}{\partial t}y^*(t,x) = D\frac{\partial^2}{\partial x^2}y^*(t,x) + F(y^*(t,x)) - \frac{1}{p^*(t,x)+\alpha}, \tag{21}$$

$$y^*(0,x) = y_0(x), \tag{22}$$

$$\frac{\partial}{\partial t}p^*(t,x) = -D\frac{\partial^2}{\partial x^2}p^*(t,x) - F'^*(t,x))p^*(t,x), \tag{23}$$

$$p^*(T,x) = \theta. \tag{24}$$

This system can be complemented with boundary conditions, e.g., Dirichlet boundary conditions for y^* and p^* or no flux boundary conditions for y^* and p^*. Once the solution of (21)-(24) has been obtained, we may compute the optimal control using (20). Equations (21)-(24) are a fully coupled forward-backward system of diffusion equations which is also nonlinear. In general, system (21)-(24) does not admit analytic solutions and has to be treated numerically.

4.3 Pollution control in a spatial setting

Consider y as modelling the stock of a pollutant which presents spatio-temporal variability. The pollutant is assumed to diffuse in the environment and its evolution is given by the PDE

$$\frac{\partial}{\partial t}y(t,x) = u(t,x) - my(t,x) + D\frac{\partial^2}{\partial x^2}y(t,x), \tag{25}$$

where m is the natural rate of decay of the pollutant and $u(t,x)$ is the rate of pollutant emissions at time t and location x. Emissions are a by-product of production and generate benefits. At the same time the increasing concentration of pollutant is assumed to be harmful. An environmental regulator seeks to maximize discounted net benefits from production (i.e. emissions), or maximize over u the functional

$$J = \int_{\mathcal{O}}\int_0^{\infty} e^{-rt}[\mathsf{B}(u(t,x)) - \mathsf{D}(y(t,x))]dtdx,$$

where $\mathsf{B} : \mathbb{R} \to \mathbb{R}$ is a concave function modelling benefits from emissions and $\mathsf{D} : \mathbb{R} \to \mathbb{R}$ is a convex function modelling damage from the concentration of pollutants. This maximization problem has to be solved under the dynamic constraint (25).

The relevant current value Hamiltonian is $H(t,x,y,p,u) = \mathsf{B}(u) - \mathsf{D}(y) + (u - my)p$ and assuming differentiability of the first order condition yields

$$u^* = I(p),$$

where I is the inverse function of B'. This allows us to calculate the Hamiltonian H^* as a function of y and p and formulate the forward-backward system (14) with initial condition for y at $t = 0$ but with a transversality condition at infinity. For general choice of the benefit and damage functions this will lead to a nonlinear system of PDEs whose solution will determine the optimal control but which may only be treated numerically. For the sake of illustration we assume a quadratic damage function of the form $\mathsf{D}(y) = \frac{1}{2}cy^2$ and a quadratic benefit function of the form $\mathsf{B}(u) = a_1 u - \frac{1}{2}a_2 u^2$

with $0 \leq u < \frac{a_1}{a_2}$ so that we have only the increasing part of the benefit function. A straightforward calculation yields that $u = \frac{1}{a_2}(a_1 + p)$ and the Hamiltonian is of quadratic form

$$H^* = \frac{1}{2a_2}(p + a_1)^2 - m(p + a_1)y + ma_1 y - \frac{c}{2}y^2.$$

For $p < 0$, with $p\,(t, x)$ interpreted as the shadow cost of pollution at time t and location x, the forward backward system (14) takes the form

$$\begin{aligned}
\frac{\partial}{\partial t}y^*(t,x) &= +D\frac{\partial^2}{\partial x^2}y^*(t,x) - my^*(t,x) + \frac{1}{\alpha_2}p^*(t,x) + \frac{a_1}{a_2}, \\
\frac{\partial}{\partial t}p^*(t,x) &= -D\frac{\partial^2}{\partial x^2}p^*(t,x) + rp^*(t,x) - cy^*(t,x) + mp^*(t,x).
\end{aligned} \tag{26}$$

This equation admits a steady state solution which is also uniform in space given by

$$y_s = \frac{(r+m)a_1}{m(r+m)a_2 + c}, \quad p_s = -\frac{a_1 c}{m(r+m)a_2 + c}$$

which is an acceptable solution as it leads to

$$u_s = \frac{m(r+m)a_1}{m(r+m)a_2 + c} < \frac{a_1}{a_2}.$$

Therefore we will consider spatially varying perturbations around this desirable state, by expressing $y^*(t,x)$ and $p^*(t,x)$ in terms of $y^*(t,x) = y_s + y(t,x)$, and $p^*(t,x) = p_s + p(t,x)$ where now the functions $y(t,x)$ and $p(t,x)$ satisfy the linear system

$$\begin{aligned}
\frac{\partial}{\partial t}y(t,x) &= +D\frac{\partial^2}{\partial x^2}y(t,x) - my(t,x) + \frac{1}{\alpha_2}p(t,x), \\
\frac{\partial}{\partial t}p(t,x) &= -D\frac{\partial^2}{\partial x^2}p(t,x) + rp(t,x) - cy^*(t,x) + mp(t,x),
\end{aligned} \tag{27}$$

with homogeneous Dirichlet boundary conditions. It also satisfies an initial condition for $y(0,t) = \phi(x)$ and the transversality condition

$$\lim_{t\to\infty} \int_{\mathcal{O}} e^{-r\,t} y(t,x) p\,(t,x)\,dx = 0$$

for all $x \in \mathcal{O}$. The forward-backward system (27) is a special example of the general linear system treated in Section 9.2 in the Appendix, for $c_1 = -m$, $c_2 = \frac{1}{a_2}$, $c_3 = -c$, $c_4 = -c_1$. The solution which satisfies the transversality condition is found using the general procedure described in Section 9.2 by

$$\begin{aligned} y(t,x) &= e^{\frac{rt}{2}} \sum_n a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L}x\right), \\ p(t,x) &= e^{\frac{rt}{2}} \sum_n \frac{c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)}{c_2} a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L}x\right). \end{aligned}$$

where

$$\begin{aligned} \bar{\sigma}(n) &= \frac{1}{2}\sqrt{r^2 - 4b_n}, \\ b_n &= -\lambda_n^2 - (r + c_4 - c_1)\lambda_n + (c_1 r + c_1 c_4 - c_2 c_3), \\ \lambda_n &= \frac{D\pi^2}{L^2} n^2 \end{aligned}$$

and a_n are the coefficients in the Fourier expansion of ϕ as shown in the Appendix. Then the optimal path and the optimal control policy is given by

$$\begin{aligned} y^*(t,x) &= \frac{(r+m)a_1}{m(r+m)a_2 + c} + e^{\frac{rt}{2}} \sum_n a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L}x\right), \\ u^*(t,x) &= \frac{m(r+m)a_1}{m(r+m)a_2 + c} + \frac{1}{a_2} e^{\frac{rt}{2}} \sum_n \frac{c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)}{c_2} a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L}x\right). \end{aligned}$$

In Figure 3 we present a typical calculation for the optimal path and the optimal control policy for the pollution control example, displaying clearly the spatial variability.

FIGURE 3
Optimal path and optimal control policy for the pollution control example

4.4 Spatial regulation

The above examples reveal that when the environmental system is characterized by spatial transport, then regulation will in general be space dependent and will be characterized by the solution of a system of PDEs derived from the maximum principle. The spatially dependent regulation will be optimal in the sense that it will describe the optimal control process, e.g., harvesting or emissions, so the objective is maximized for the whole spatial domain. We will call this problem the socially optimal management problem (SOMP). The SOMP should be distinguished from the case in which there is an economic agent located at each spatial point that makes profit maximizing harvesting or emission decisions at this location and acts spatially myopically by ignoring spatial transport. This means that he ignores the impact of his decisions on the stocks accumulated in the other locations. We will call this problem the privately optimal management problem (POMP). At the POMP the spatiotemporal evolution of the state variable will be determined by (4) with the control process replaced by the spatially myopic control of each agent. The spatiotemporal evolution of the state variable will be different between the SOMP and the POMP, and the optimal regulation determined by the SOMP will correct for the spatial externality and induce the system to follow the optimal spatiotemporal evolution y^*.

5 Pattern Formation in Spatially Controlled Systems

A point of interest is whether pattern formation occurs in an optimally controlled spatial system, since this relates to the spatial structure of regulation. Assume that the uncontrolled spatially dependent system admits a steady state $\bar{y}(t,x) = y_h$ which is spatially homogeneous. This means that y_h is such that $f(y_h) = 0$.

Consider now the controlled system and the corresponding state-costate system of equations (11 -14) which provides the optimal path and the control. Suppose that we are interested in states of the controlled system which are small deviations from this homogeneous state y_h, i.e., we look for solutions of the form

$$y(t,x) = y_h + \epsilon \bar{y}(t,x),$$
$$p(t,x) = 0 + \epsilon \bar{p}(t,x)$$
$$u(t,x) = 0 + \epsilon \bar{u}(t,x)$$

where ϵ is a small parameter and $\bar{y}(t,x)$ are the spatio-temporal deviations around the homogeneous state (obtained for the uncontrolled system) while $\bar{p}(t,x)$, $\bar{u}(t,x)$ are the corresponding costate and control variables. We substitute this solution ansatz in (11 -14) and, taking a series expansion in the small parameter ϵ, we obtain a linear approximation of the original equation which is valid for the evolution of small deviations around the steady state. This linearized equation is of the general form

$$\frac{\partial}{\partial t}\bar{y}(t,x) = +D\frac{\partial^2}{\partial x^2}\bar{y}(t,x) + A_1\bar{y} + A_2\bar{p}(t,x), \tag{28}$$
$$\frac{\partial}{\partial t}\bar{p}(t,x) = -D\frac{\partial^2}{\partial x^2}\bar{p}(t,x) + A_3\bar{y}(t,x) + A_4\bar{p}(t,x). \tag{29}$$

The coefficients $A_1, A_2, A_3, A_4 = (r - A_1)$ arise from the linearization of (11 -14) around the point $(y,p,u) = (y_h, 0, 0)$. Without loss of generality we may assume $y_h = 0$.

Let us also assume for simplicity, and without loss of generality, that the physical domain is $\mathcal{O} = [0, L]$ and 0 and L are considered as hostile boundaries, so that $\bar{y}(t,0) = \bar{y}(t,L) = 0$ for all t. The same boundary condition must be assumed for the costate variable $\bar{p}$. We may now expand $\bar{y}$ and $\bar{p}$ into a Fourier series, and on account of the boundary conditions this Fourier series will only contain the sinusoidal terms

$$\bar{y}(t,x) = \sum_n y_n(t)\sin\left(\frac{n\pi}{L}x\right), \quad \bar{p}(t,x) = \sum_n p_n \sin\left(\frac{n\pi}{L}x\right), \quad x \in [0, L].$$

Note that the above functions automatically satisfy the boundary conditions. As the Fourier series expansion for a function is unique, we may identify the functions $\bar{y}$ and $\bar{p}$ with their Fourier coefficients, i.e. the sequences $\{y_n\}$ and

$\{p_n\}$. Substituting into the equation and assuming that $B = I$ for simplicity and then projecting along the various elements of the Fourier basis yields a system of ODEs for (y_n, p_n) of the form

$$\frac{d}{dt} y_n = (-\lambda_n + A_1) \, y_n + A_2 p_n, \tag{30}$$

$$\frac{d}{dt} p_n = A_3 y_n + (\lambda_n + r + A_4) \, p_n \tag{31}$$

where $\lambda_n = D \frac{n^2 \pi^2}{L^2}$, and this is valid for every $n \in \mathbb{N}$. Furthermore, recall that $A_4 = -A_1$ where r is the discount factor. Note that these systems are all separated. Since the Fourier basis consists of eigenfunctions of the diffusion operator, we often refer to equations (30), (31) as the spectral decomposition of the forward-backward system (28),(29). By the uniqueness of the Fourier transform these two systems are equivalent, but (30), (31) is much easier to handle as it is a countable system of ODEs (rather than a PDE) and furthermore, it is a fully decoupled system in the sense that we may solve it for each $n \in \mathbb{N}$ separately. The spatial dependence of the solution is regained from the solution of (30), (31) by resuming the relevant Fourier series. Equations (30), (31) have to be treated as a two-point boundary value problem, initial and final value problem for y_n and p_n respectively, or equivalently for the given $y_n(0)$ we must choose a $p_n(0)$ so that the desired value of $p_n(T)$ is achieved for any $n \in \mathbb{N}$ (similarly to a shooting problem). This is equivalent to picking $p_n(0)$, $n \in \mathbb{N}$, along the eigen-direction of stable manifold of the saddle point, which is our familiar visualization method for solving a control problem if we had a finite dimensional problem.

At this stage it is a good idea to eliminate p_n from system (30), (31).[5] This yields a second order equation in terms of y_n only, of the form

$$\frac{d^2}{dt^2} y_n - r \frac{d}{dt} y_n + b_n y_n = 0, \quad n \in \mathbb{N},$$

[5] This does not affect the well-posedeness of the control problem as such since p is a shadow (auxiliary) variable and its elimination from the final system bypasses several conceptual issues arising with respect to its long-term behaviour. An alternative but equivalent approach is presented in the Appendix.

where

$$b_n = -\lambda_n^2 - (r + A_4 - A_1)\lambda_n + (A_1 r + A_1 A_4 - A_2 A_3),$$

whose general solution is given as an exponential $y_n(t) = C_{1,n}e^{\sigma_1(n)t}+C_{2,n}e^{\sigma_2(n)t}$ where $\sigma_{1,2}(n)$ are the roots of the quadratic $s^2 - rs + b_n = 0$, which are explicitly given as

$$\sigma_{1,2}(n) = \frac{1}{2}r \pm \bar{\sigma}(n), \;\; \bar{\sigma}(n) = \frac{1}{2}\sqrt{r^2 - 4b_n}, \;\; n \in \mathbb{N}.$$

The long-term behavior of the linearized system is now fully determined by the behavior of these roots. If for some Fourier mode n we have that the real parts of $\sigma_{1,2}(n)$ are negative, then the effect of this term will soon die out in the Fourier series representing the solution, thus not contributing to the spatial variability of y. If for some Fourier mode n we have that one of these roots has a positive real part, then this will give rise to an exponentially increasing term in time, which will contribute to the spatial variability of y as a sinusoidal term in the Fourier series of y. Therefore, this will be a term contributing to the spatial pattern of y in the long run. However, note that not all terms corresponding to a positive real part of the roots $\sigma_{1,2}(n)$ are acceptable. The system is controlled and therefore terms whose temporal growth is excessive will be suppressed in the long run by the control procedure. A simple rule of thumb to find the acceptable terms is to note that if $\lim_{t\to\infty} e^{-rt}y(t,x)^2 \neq 0$ or $\lim_{t\to\infty} e^{-rt}p(t,x)^2 \neq 0$, then the infinite horizon linear quadratic control problem is not well posed as the functional becomes infinite. We must thus consider only solutions such that $\lim_{t\to\infty} e^{-rt}y(t,x)^2 = 0$ and $\lim_{t\to\infty} e^{-rt}p(t,x)^2 \neq 0$ and this rules out terms which grow at an exponential rate in time larger than $\frac{r}{2}$. We will call this condition a sustainability condition and we note that this is fully compatible with the Benveniste-Scheinkman transversality condition (Benveniste and Scheinkman (1982)). Therefore, the terms that will contribute to the final pattern are those with n such that $\max(Re(\sigma_1(n), \sigma_2(n)) \in [0, \frac{r}{2}]$. This observation allows us to provide a detailed analysis of the long-term spatio-temporal behavior of y.

If $r^2 - 4b_n < 0$, then the roots of the quadratic are of the form $\sigma_{1,2}(n) =$

$\frac{r}{2} \pm i\sigma_0$, so they are complex conjugates with real part exactly equal to $\frac{r}{2}$. These are temporally oscillatory modes which are marginal (in the sense of the sustainability condition) and may be associated with a Hopf type bifurcation.

If $r^2 - 4b_n > 0$, then $\sigma_{1,2}(n)$ are real, and $\sigma_1(n) > \frac{r}{2}$ thus violating the sustainability condition. On the other hand $\sigma_2(n)$ satisfies the transversality condition at infinity, leading to an acceptable solution of the form

$$y(t,x) = e^{\frac{rt}{2}} \sum_n c_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L}x\right),$$

and the long term pattern will be of the form

$$y_{pattern}(t,x) = \sum_{n\in\mathcal{N}_u} c_n \exp\left(\left(\frac{r}{2} - \bar{\sigma}(n)\right)t\right) \sin\left(\frac{n\pi}{L}x\right),$$

where $\mathcal{N}_u = \{n \in \mathbb{N} : \frac{r}{2} - \bar{\sigma}(n) > 0\}$. Therefore, all modes n such that $r - \sqrt{r^2 - 4b_n} < 0$ will not contribute in the long term pattern and if this inequality holds for every n we expect the long run equilibrium to present no spatial variability. If $b_n < 0$ this condition is certainly true and all modes are stable. On the other hand the modes n such that $r - \sqrt{r^2 - 4b_n} > 0$ will contribute to the long-term pattern and will correspond to the spatial variability of the long-run equilibrium. Therefore the condition for pattern formation in the long-run equilibrium is the existence of $n \in \mathbb{N}$ such that $r - \sqrt{r^2 - 4b_n} > 0$. Whether this condition will hold or not depends on (a) the form of the spatially homogeneous system (i.e. the coefficients A_1, A_2, A_3, A_4), (b) the discount factor r, (c) the nature of the spatial domain, i.e. the length L, and (d) the diffusion coefficient D. Clearly, there may be more than one mode which can turn unstable, depending on the combination of the relevant parameters. However, we can see that in general the modes that are likely to turn unstable are modes corresponding to small values of λ_n, i.e., to small values of n (as n becomes large, $b_n \to -\infty$ so modes with large n are expected to be stable). How small n should be depends on the ratio $\frac{D}{L^2}$ and the parameters A_1, A_2, A_3, A_4 and r, thus clarifying the interplay between the properties of the domain, the diffusion coefficient and the intrinsic dynamics of the system.

The picture is clarified even more if we assume that the spatially independent system presents a saddle point structure which implies that $A_1A_4 - A_2A_3 < 0$, so that b_n is expected to be negative for every n unless $r + A_4 - A_1 < 0$ or $A_1 > 0$ (and n is small). A small n in the Fourier series corresponds to well formed large scale patterns with respect to the spatial variable, so it corresponds to patterns which are not expected to be merely short-scale spatial fluctuations that may be swept away as an effect of diffusion. In Figure 4 we present some typical stability diagrams. On the vertical axis we display the value of the acceptable eigenvalue $r - \bar{\sigma}(n)$ for various values of n. As we move from panel (a) to panel (c), the parameter D/L^2 is increasing. In the top panel we observe that the first 3 Fourier modes are unstable whereas in the bottom panel all Fourier modes are stable. In Figure 5 we present the solution of the system for random initial condition for the cases. The top figure corresponds to the choice of parameters of panel (a) in Figure 4. We observe the evolution of a long-term spatial pattern which is compatible with the transversality condition. The second figure corresponds to the choice of parameters of panel (c) in Figure 4. We observe that no spatial pattern is formed as expected, and the system relaxes to the flat steady state.

FIGURE 4

Stability diagrams for various values of the parameter D/L^2

FIGURE 5

Pattern formation and absence of pattern formation

Summarizing:

- ▷ If $r^2 - 4b_n < 0$, then we have marginal modes leading to spatio-temporal oscillations (Hopf type instability).
- ▷ If $b_n < 0$, then we have stable modes (no pattern formation).
- ▷ If $0 \leq b_n \leq \frac{r^2}{4}$, then we have unstable modes (pattern formation).

This mechanism is reminiscent of the celebrated Turing instability mechanism but with a very important difference: It arises from a controlled system and in fact the control procedure may encourage and lead to the development of the pattern. This type of mechanism has been called *optimal diffusion induced instability* (e.g., Brock and Xepapadeas (2008), Brock et al. (2012a)). Furthermore the controlled nature of the system selects the unstable modes; note that unlike Turing instability, in our case the unstable modes are those for which $Re(\sigma_{1,2}(n)) \in (0, \frac{r}{2}]$. Optimal diffusion induced instability is a phenomenon which can occur in domains other than $[0, L]$, in dimensions higher than 1, for boundary conditions other than the Dirichlet and also for transport mechanisms other than diffusion.

6 Robust Control and Hot Spot Formation

The optimization described in the sections above is subject to noise and uncertainty, as it is very common in models of resources and environment. Uncertainties are related to sources such as limited modelling capacity of the natural system and lack of theories to anticipate thresholds; major gaps in global and national monitoring systems; the lack of a complete inventory of species and their actual distributions; and emergence of surprises and unexpected consequences. These uncertainties may impede adequate scientific understanding of the underlying mechanisms and the impacts of regulatory policies applied to ecosystems. Furthermore, uncertainty may have a spatial structure in the sense that the degree of uncertainty may vary across the locations of the natural system. A model with spatial transport and uncertainty can be written as:

$$\frac{\partial}{\partial t} y(t,x) = D\frac{\partial^2}{\partial x^2} y(t,x) + f(y(t,x)) - Bu(t,x) + \frac{\partial}{\partial t} N(t,x,\omega),$$

where the term $N(t, x, \omega)$ corresponds to statistical fluctuations around the mean observed state of the system. These fluctuations are assumed to vary both in space and time and must average to 0. Our modelling of the statistical fluctuations is of course arbitrary, however general considerations based on the central limit theorem suggest that if we consider the fluctuations (noise) to be the overall outcome of many statistically independent errors, then $N(t,x)$ is expected to have some sort of Gaussian distribution. We use a stochastic process, the Wiener process, to model this kind of uncertainty and we define $N(t,x)$ using a Fourier series expansion in space,

$$N(t,x) = \sum_n c_n w_n(t) \sin\left(\frac{n\pi}{L}x\right),$$

where $\{w_n(t)\}$ are independent standard one-dimensional Wiener processes and $\{c_n\}$ are real numbers, the choice of which characterizes N as an element of an appropriate function space (either $\mathbb{H}$ or larger). Our model for the evolution of y may take the form of an Itō differential equation as

$$\begin{aligned} dy(t,x) = & \left(D\frac{\partial^2}{\partial x^2}y(t,x) + f(y(t,x)) - Bu(t,x)\right)dt \\ & + \sum_n c_n \sin\left(\frac{n\pi}{L}x\right) dw_n(t). \end{aligned} \quad (32)$$

An important conceptual step in the modelling is knowing the distribution of the family of random variables $w(t) = \{w_n(t)\}$. Our initial "idea" about them is that they are distributed as $w(t) = N(0, It)$ where 0 is the zero element in $\mathbb{H}$ and I is the identity operator in $\mathbb{H}$. This corresponds to our belief that we can predict well the mean behavior of the system as given by equation (32). However, this is not always true and we may also have certain doubts about the validity of our mean or benchmark model, which in effect is equivalent to assuming that $w_n(t)$ is not distributed as $N(0, I\,t)$ but rather as $N(\int_0^t v(\bar{t},x)d\bar{t}, I\,t)$ where $v(t,x)$ is a general drift term. The possibility that the $w(t)$ is distributed according to a different probability law is essentially equivalent to viewing our model (32) as a whole family of models $\mathcal{M}$, each of which is identified with a choice of a function v characterizing the probability law of the term w generating the fluctuations. Since $v \in \mathbb{H}$ admits a Fourier expansion, let us assume that $\{v_n\}$ are the real numbers which are the Fourier modes of v. A deep result in stochastic analysis (Girsanov's theorem) allows us to view the whole family of models in terms of a family of Itō differential equations.[6] Assuming $f(y) = A_1 y$ to simplify (the fully nonlinear case has been treated in Brock et al. (2012a)), we obtain the spatiotemporal dynamics as the solution of the stochastic PDE

$$\begin{aligned} dy(t,x) &= \left(D\frac{\partial^2}{\partial x^2} y(t,x) + A_1 y(t,x) - Bu(t,x) \right) dt \\ &+ \sum_n \sum_m c_n v_m \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dt \\ &+ \sum_n c_n \sin\left(\frac{n\pi}{L}x\right) d\bar{w}_n(t), \end{aligned} \tag{33}$$

where $\bar{w} = \{\bar{w}_n\}$ is a sequence of standard Wiener processes. Using the Fourier decomposition of the functions y, u, v, we can view (33) as a countable set of ordinary Itō type equations of the form

$$dy_n = (-\lambda_n y_n + A_1 y_n - Bu_n + \sum_m C_{mn} v_m)dt + c_n d\bar{w}_n, \;\; n \in \mathbb{N}, \tag{34}$$

where

$$C_{mn} = \sum_\ell c_\ell \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{\ell\pi}{L}x\right) dx.$$

The system of Itō differential equations (34) can be understood as the spectral form of the original stochastic PDE (33) (in a form completely analogous to the deterministic case of Section 5), which is an equivalent formulation but easier to work with.

For particular choices of the family of models we are interested in, for example assuming models for which $w_n(t) = \int_0^t v_n(\bar{t})d\bar{t} + \bar{w}_n(t)$ for every

[6] According to Girsanov's theorem we may consider our model as not under the probability law $\mathbb{P}$ (which corresponds to the Itō stochastic differential equation (32) in which the fluctuations are introduced by the Wiener process w) but under the new probability law $\mathbb{Q}$ which is related to the law $\mathbb{P}$ via a Radon-Nikodym derivative of exponential form. The new model (33) involves a linear correction in the drift term involving the process v and now the fluctuations are introduced by the process $\bar{w}$ which is a standard Wiener process under the probability law $\mathbb{Q}$ (see Brock et al. (2012a) for details and references therein). Note that even if F is nonlinear, the above procedure is still valid and introduces a linear correction in the drift term.

$n \in \mathbb{N}$, the infinite matrix (C_{nm}) may be diagonal and (33) simplifies to

$$dy_n = (-\lambda_n y_n + A_1 y_n - B u_n + c_n v_n)dt + c_n d\bar{w}_n, \;\; n \in \mathbb{N}. \tag{35}$$

If $v_n = 0$ for every $n \in \mathbb{N}$, then the state dynamics in (35) correspond to the benchmark model. This situation where a single probability model - or a unique prior - is sufficient to analyze the phenomenon and formulate decision rules can be identified as the case of pure risk or measurable uncertainty in which the decision maker is able to assign probabilities to outcomes. If v_n is not zero, however, we have at the same time statistical fluctuations and lack of knowledge concerning the "true" statistical model describing the random fluctuations. We will call this situation *deep uncertainty or ambiguity.* This is a situation where the decision maker operates in the realm of many models - or multiple priors. Under ambiguity the decision maker does not have the ability to determine a precise probability structure for the physical or the economic model, or to put it differently, to measure uncertainty using a single probability model.

Note that this uncertainty has a spatial character as well, since v_n is in general a function of n and time and through the Fourier series representation the variability in n corresponds to variability in the spatial variable x. Therefore, this approach allows for differentiated levels of uncertainty across space, a situation which is very realistic for a number of applications. For example, in the context of resource economics, imagine modelling a spatially extended fishery. Our knowledge of the parameters of the model comes from statistical estimation techniques and as such provides accuracy as long as a sufficient number of measurements of a relevant quantity are available. For example, in the context of the FKPP model, estimation of the growth rate can be derived from measurements of the levels of the population. There are regions of the fishery in which collection of measurements is not easy, so there is lack of sufficient data leading to poor validation of the model, hence enhanced uncertainty concerning the statistical fluctuations in the vicinity of this region. For a detailed discussion of the issue of spatial uncertainty in fishery modelling, see e.g., Brock et al. (2012d). Similar situations arise for instance in climate change modelling (see e.g. Xepapadeas and Yannacopoulos (2013)).

Decision making under ambiguity can be studied in the context of Wald (1950) who suggested that a maxmin solution could be a reasonable solution to a decision problem, where an a priori probability distribution does not exist or is not well known to the researcher. One way to approach the maxmin solution is to use the idea of least favorable prior (LFP) decision theory, as developed by Gilboa and Schmeidler (1989), which results in maxmin expected utility theory and represents an axiomatic foundation of Wald's criterion.

In the context of minimax expected utility we consider two players, the decision maker who optimizes an objective, for example minimizes the distance from a target, and a second adversarial agent (let us call her Nature) who actually chooses the model from a family of models $\mathcal{M}$. The decision maker is not completely ignorant concerning the true model, she has a feeling of which fluctuations are acceptable or not, and therefore Nature is constrained to choose her model from within this universe of acceptable models. As an example of that, consider the case of a fishery again. As part of the uncertainty which affects management we can consider the weather conditions. The decision maker will consider a large number of scenarios apart from some very extreme ones which simply cannot happen, such as for instance snow in August or a heat wave in February in the Mediterranean (or the opposite in fisheries near Chile). One of the many ways of modelling the universe of acceptable scenarios (or probability laws) is using the concept of Kuhlback-Leibler entropy between two probability distributions. In the present context we will focus on the probability distribution of the Wiener process w and consider two probability laws (models/measures) for w, $\mathbb{P}$ and $\mathbb{Q}$. According to the first, $w \sim N(0, It)$, whereas according to the second $w \sim N(\int_0^t v dt, It)$. A (pseudo)-distance between the two laws is the Kuhlback-Leibler entropy defined as

$$\mathcal{H}(\mathbb{P} \mid \mathbb{Q}) = \int_\Omega \ln \left(\frac{d\mathbb{P}}{d\mathbb{Q}} \right) d\mathbb{P},$$

where $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is the Radon-Nikodym derivative of the two measures. We consider thus that the allowed universe of models are the models such that a properly temporally discounted and spatially weighted version of this distance (or a symmetric version of it) is less than some allowed value, which is what we call hereafter an entropy ball. This is because for the class of models considered it can be shown (see e.g. Brock et al. (2012a)) that the entropic constraint is equivalent to a quadratic constraint which in spectral form yields

$$\mathbb{E}\left[\sum_n \int_0^T v_n(t)^2 dt\right] \leq H,$$

where $\mathbb{E}$ denotes the expectation operator under $\mathbb{Q}$.

We now consider the problem of targeting a particular desired state y_h for a resource stock under deep uncertainty in the context of a linear quadratic model (see Brock et al. (2012a) or Brock et al. (2012d) for a specific application in resource economics). The problem can be formulated in terms of a robust control problem, e.g., Hansen and Sargent (2008), as

$$\min_{\{u_n\}} \max_{\{v_n\}} \mathbb{E}\left[\int_0^T e^{-rt} \sum_n (Py_n(t)^2 + Qu_n(t)^2 - Rv_n(t)^2) dt\right].$$

The first two terms are control and deviation costs and the third is a penalty term related to the entropic constraint, so that R can be regarded as a Lagrange multiplier, hereafter called the robustness parameter. It can be shown (Hansen and Sargent (2008)) that low values of R reflect strong concerns about deep uncertainty and model misspecification, while $R \to \infty$ implies that we trust the benchmark model. This is a stochastic differential game which, by employing Fourier decomposition, has been transformed into an infinite dimensional but separable problem which only involves the temporal variable, and all information on spatial variability can be regained by recomposing the Fourier series. Therefore, the robust control problem is equivalent to the following sequence of decoupled problems:

$$\min_{\{u_n\}} \max_{\{v_n\}} \mathbb{E}\left[\int_0^T e^{-rt}(Py_n(t)^2 + Qu_n(t)^2 - Rv_n(t)^2) dt\right], \quad n \in \mathbb{N},$$

subject to the constraints (35). Each of these problems is a standard linear quadratic stochastic differential game for real valued stochastic processes which can be easily solved in terms of the Hamilton-Jacobi-Bellman-Isaacs formulation. We omit details and provide the final solution.[7] The optimal path is given in terms of the solution of the Itō equation

$$dy_n = \left(-\lambda_n + A_1 - \frac{B^2}{2Q} M_n + \frac{c_n^2}{2R} M_n \right) F_n y_n + c_n d\bar{w}_n, \;\; n \in \mathbb{N},$$

where M_n is the root of the algebraic quadratic equations

$$\left(\frac{c_n^2}{2R} - \frac{B^2}{2Q} \right) M_n^2 + (-2\lambda_n + A_1) M_n + 2P = 0, \;\; n \in \mathbb{N} \tag{36}$$

and the optimal policy is given by the feedback rules

$$u_n = \frac{BM_n}{2Q} y_n, \;\; v_n = \frac{c_n M_n}{2R} y_n, \;\; n \in \mathbb{N}.$$

The above not only provides us with an explicit solution of the optimal control strategy for solving the tracking problem for the spatially extended system under ambiguity, but also allows some valuable qualitative insight into the interaction of control, uncertainty and spatial variability. If the robustness parameter is low enough, which implies very strong ambiguity aversion, then (36) may not have a real solution for a mode n. In this case regulation breaks down. We will call this case a hot spot of type I. In Brock et al. (2012a) it is shown, using a more general model, that with location specific entropic constraints, very strong ambiguity aversion for a specific location may cause global regulation to break down. This location is a type I hot spot. A type II hot spot emerges when ambiguity aversion in a specific location induces pattern formation through the optimal diffusion induced instability mechanism. Finally if the objective of the regulator (tracking cost) under robust control and ambiguity aversion in a specific location is very high relative to the cost when $R \to \infty$, which is the case of no concerns about model misspecification, then we have the emergence of a hot spot of

[7] The interested reader may consult Brock et al. (2012a) and references therein.

type III.

To summarize, we classify hot spots into 3 different types:

▷ Hot spot of type I: Breakdown of control procedure and thus breakdown of global regulation as a result of the interaction of ambiguity aversion, spatio-temporal uncertainty and endogenous spatio-temporal dynamics of the system.

▷ Hot spot of type II: Global regulation is possible but spatial pattern formation (similar to Turing instability) occurs as a result of it.

▷ Hot spot of type III: Global regulation is possible but the cost of robustness becomes high relative to the cost of regulation that disregards any concerns about model misspecification.

A detailed study of the occurrence of the above types of hot spots (with detailed estimates for the relevant parameter ranges) can be found in Brock et al. (2012a). In general hot spots of type I are expected to appear in systems where $R \to 0$ (a fact which is related to the loss of concavity for the value function of the game). Hot spots of type II occur if there exist Fourier modes with n such that

$$\frac{1}{2}(r - \sqrt{K_n}) \leq -\lambda_n + A_1 \leq \frac{1}{2}(r + \sqrt{K_n}),$$
$$K_n = r^2 + 8P\left(\frac{c_n^2}{2R} - \frac{B^2}{2Q}\right) \geq 0.$$

This condition determines the modes that may become unstable, and thus characterizes the spatial structure of the pattern. Note that both control and uncertainty play an important role in the emergence of unstable patterns and the robust control procedure may either have a stabilizing effect with respect to generation of spatial instability or the opposite. The robust control procedure is expected to facilitate the onset of instabilities if $\frac{Q}{R} > \frac{B^2}{c_n^2}$, and this is certainly true in the case where $R \to 0$.

Robust control methods in economics have been receiving increasing attention over the last decade. General approaches can be found for example in Hansen and Sargent (2001b), Hansen et al. (2006), Hansen and Sargent (2008). The general methodology has been applied to areas such as macroeconomics (e.g. Hansen and Sargent (2001a), Onatski and Williams (2003), Hansen and Sargent (2003), Leitemo and Söderström (2008)), finance (e.g. Maenhout (2004), Maenhout (2006)), or environmental and resource economics (e.g. Roseta-Palma and Xepapadeas (2004), Vardas and Xepapadeas (2010), Asano (2010), Athanassoglou and Xepapadeas (2012)). Robust control methods either in a temporal or in a spatiotemporal setup seems to provide a prominent area for further research.

Other types of uncertainty may be also taken into account. One type of uncertainty that is often important is uncertainty with respect to model parameters, e.g. the exact value of the diffusion coefficient D. Then a robust control procedure can be formulated as a min-max approach where the controller minimizes over u (assuming D known) and then nature chooses D over a set of possible values in such a manner as to maximize the "loss" of the controller (see for example Armaou and Christofides (2001), El-Farra and Christofides (2001)). These problems may be treated using versions of the Hamilton-Jacobi-Bellman-Isaacs equation (e.g. Bardi and Capuzzo-Dolcetta (2008)).

7 Generalizations and Extensions

In this section we discuss several directions in which our results may be generalized and extended. While diffusion has been a dominant model in resource economics, it may not be an adequate representation in certain cases. This is because diffusion relates to local transport, i.e. only quantities from a small neighborhood $\mathcal{N}_x$ around x can affect what is happening in x. In practice, this is not necessarily so. Long-range effects may be important. For instance, if we are interested in the transport of avian species, then a diffusion approximation may not be satisfactory as birds may fly from rather distant places to affect the location under consideration. Also, when economic quantities are studied, long-range spatial externalities may become important. Clearly, a generalization of the diffusion model in the state equation is called for.

A general model can take the form of

$$\frac{\partial}{\partial t} y(t,x) = f(y(t,x), Y(t,x)) + Bu + (\mathcal{T}_1 y)(t,x),$$

where $f(y(t,x), Y(t,x))$ represents local growth for y at $x \in \mathcal{O}$, Bu is the effect of controls u on the system and $(\mathcal{T}_1 y)(t,x)$ is a transport term which quantifies how y migrates from one spatial point to the other and the overall contribution of such activity to the change of the rate of the system at x. As the term $(\mathcal{T}_1 y)(t,x)$ presents the net transport, i.e. inflow minus outflow, it will have to average to 0 over the whole space $\mathcal{O}$. Apart from direct transport we also allow for spatial externalities which have an effect on the local growth f, and are represented by $Y = (\mathcal{T}_3 y)(t,x)$, which is a non-local term modelling how activity at neighboring site $s \in \mathcal{O}$ affects the growth of the state variable at site $x \in \mathcal{O}$. The effects of control may also be nonlocal. This is a reasonable assumption when controlling the system at site $s \in \mathcal{O}$ has effects on the state of the system at site $x \in \mathcal{O}$ through the inter-connectivity of the system. For example, fishing activity at one location affects, through transport phenomena, the stock of biomass in other locations. In this case $(Bu)(t,x) = (\mathcal{T}_2 u)(t,x)$. A general model for such nonlocal effects can be to define

$$(\mathcal{T}_i y)(t,x) := \int_{s \in \mathcal{O}} w_i(x-s) y(t,s) ds, \ \ i = 1,2,3$$

where $w_i : \mathcal{O} \to \mathbb{R}$ is a function, called a kernel function, which quantifies the effect that activity at site $s \in \mathcal{O}$ is expected to have on activity at site $x \in \mathcal{O}$. The kernel formulation is general enough to include positive and negative externalities, localized transport processes related to standard diffusion models, long-range transport effects, discrete structures, etc. Therefore our general class of models will be of the form

$$\frac{\partial}{\partial t} y(t,x) = f(y(t,x), (\mathcal{T}_3 y)(t,x)) + (\mathcal{T}_2 u)(t,x) + (\mathcal{T}_1 y)(t,x), \tag{37}$$

where spatial effects are assumed to be present in transport phenomena (the $\mathcal{T}_1$ term), in the externality effects of the control procedure (the $\mathcal{T}_2$ term) and in the externality effects in growth (the $\mathcal{T}_3$ term). A large variety of spatial models falls within the general class of models (37).

Kernels can be used to describe positive spatial externalities (e.g., productivity enhancement) and negative spatial externalities (e.g., congestion). Bell-shaped and inverted bell-shaped kernels have been used to model positive and negative spatial externalities respectively (Murray (2002), Papageorgiou and Smith (1983), Krugman (1996)), which can be further combined to produce composite externalities as for example in Figure 6. It can be shown that our general formulation (37) covers and extends the general class of diffusion models, in the sense that under specific assumptions the kernel representation of spatial transport leads to the diffusion approximation (for a discussion of this approximation in a biological context see, e.g., Murray (2002)).

FIGURE 6
Composite externality - two hump kernel

An important class of spatial models are purely discrete models. This corresponds to the case where economic activity can be thought of as located at discrete sites or cells centered at points $\{x_1, \cdots, x_N\} \subset \mathcal{O}$. Then the state of the system can be thought of as a sum of delta functions $y(t,x) = \sum_{i=1}^{N} y_i(t)\delta_{x_i}(x)$ and similarly for u so that the functions $y(t,x)$ and $u(t,x)$ can be replaced by vectors $y(t) = (y_1(t), \cdots, y_N(t))$ and $u(t,x) = (u_1(t), \cdots, u_N(t))$ in $\mathbb{R}^N$. When the kernel operators act on these localized functions, they are replaced by

$$(\mathcal{T}_1 y) = ((\mathcal{T}_1 y)_1, \cdots, (\mathcal{T}_1 y)_N),$$

where

$$(\mathcal{T}_1 y)_i = \sum_{j=1}^{N} w_{1,ij} y_j,$$

and $w_{1,ij} = w_1(x_i - x_j)$ for $i, j = 1, \cdots, N$. Similarly for the other kernels. Therefore, the action of the integral operators $\mathcal{T}_i$, $i = 1, 2, 3$ can be realized as matrix multiplication with the $N \times N$ matrices W_1, W_2, W_3 with elements $W_r = (w_{r,ij})$, $r = 1, 2, 3$ and $i, j = 1, \cdots, N$, and thus the integrals understood as sums. Within the general class of such models, (37) becomes a system of ODEs of the form

$$\frac{d}{dt}y_i(t) = f(y_i(t), \sum_{j=1}^{n} w_{3,ij} y_j(t), \sum_{j=1}^{N} w_{2,ij} u_j(t))) + \sum_{j=1}^{N} w_{1,ij} y_j(t), \tag{38}$$

where $i = 1, \cdots, N$. In this model we have included the nonlocal effects of the control term in the function f (this is always possible by proper redefinition of f).

This class of models may also be considered an approximation of the spatial continuous models such as (37), by a discretization procedure (see Xepapadeas and Yannacopoulos (2013)).

8 Concluding Remarks

The purpose of the paper was to review and present tools and methods that can be used to study dynamic environmental resource management in a spatial setting, to explore spatially dependent regulation, and to understand pattern formation. Most of the results presented here hold true for more general spatio-temporal evolution laws, and many applications and extensions of this methodology to areas outside environmental economics are possible and have already been performed. For example in Petracou et al. (2013), a general model of the form (38) has been constructed in order to account for human migration patterns by taking into account discrete choice theory and economic factors in the relevant transport operator. This model has been coupled with an economic growth model in order to predict future spatial patterns in the economy as an effect of labor migration. In Brock et al. (2012b) a general discrete model of the form (38) has been proposed as an attempt to understand the emergence of spatial patterns in the economy as a result of the interplay between spatial knowledge externalities and adjustment costs. In Xepapadeas and Yannacopoulos (2013), a spatially extended nonlinear climate model has been approximated by a discrete approximation and robust control results concerning optimal mitigation policies have been provided, using techniques from the theory of viscosity solutions. There is also a growing literature in which the tools presented here are applied to problems of spatial economic growth (e.g., Camacho and Zou (2004), Camacho et al. (2008), Boucekkine et al. (2009), Brito (2011), Brock et al. (2012c), Boucekkine et al. (2013a), Boucekkine et al. (2013b),). Optimal control in a spatiotemporal context is relatively new in environmental economics, but we believe it is an approach that can provide significant new insights into many important issues.

9 Appendix: Solution of the forward backward system

9.1 Finite horizon problems

The forward-backward PDEs for the problems related to linear state equations and quadratic objective functionals encountered in Section 4 reduce to the general form

$$\begin{aligned} \frac{\partial y}{\partial t} &= c_1 y + c_2 p + D\frac{\partial^2}{\partial x^2} y \\ \frac{\partial p}{\partial t} &= c_3 y + (r + c_4)p - D\frac{\partial^2}{\partial x^2} y \end{aligned} \tag{39}$$

where in general c_1 and c_4 are related by the condition $c_1 + c_4 = 0$ (as follows by assuming in equation (14) that H^* is a quadratic function). Furthermore, we also have some information concerning the signs of the coefficients, since H^* must be a concave function (for a maximization problem). The system of equations (39) must be solved with initial condition $y(0, x) = \phi(x)$, final condition $p(T, x) = 0$ and homogeneous boundary conditions of Dirichlet type.

This type of systems can be solved in a rather algorithmic fashion following the procedure sketched below:

STEP A. We look for solutions in terms of Fourier series of the form

$$\begin{aligned} y(t, x) &= \sum_n y_n(t) \sin\left(\frac{n\pi}{L} x\right) \\ p(t, x) &= \sum_n p_n(t) \sin\left(\frac{n\pi}{L} x\right). \end{aligned} \tag{40}$$

Any function (of sufficient regularity) satisfying these boundary conditions has an expression of this type, so it is a matter of choosing the right coefficients y_n and p_n so as to satisfy the PDEs. We do not worry about initial and final conditions yet.

STEP B. We expand the initial condition $\phi : [0, L] \to \mathbb{R}$ in a Fourier series of the above form in terms of

$$\phi(x) = \sum_n a_n \sin\left(\frac{n\pi}{L}x\right), \quad a_n = \frac{2}{L}\int_0^L \phi(x) \sin\left(\frac{n\pi}{L}x\right) dx.$$

STEP C. Substituting[8] the series (40) into the system of equations (39) and letting $\lambda_n = \frac{D\pi^2}{L^2}n^2$ we obtain for every $n \in \mathbb{N}$ a system of ODEs for y_n and p_n of the form

$$\begin{aligned} y_n' &= (c_1 - \lambda_n)y_n + c_2 p_n, \\ p_n' &= c_3 y_n + (r + c_4 + \lambda_n)p_n, \\ y_n(0) &= a_n, \;\; p_n(T) = 0. \end{aligned} \tag{41}$$

STEP D. We now solve for any $n \in \mathbb{N}$ the eigenvalue problem for the matrix

$$C(n) = \begin{pmatrix} c_1 - \lambda_n & c_2 \\ c_3 & r + c_4 - \lambda_n \end{pmatrix}$$

which is equivalent to the solution of the quadratic equation

$$\sigma^2(n) - (r + c_1 + c_4)\sigma(n) + [(c_1 - \lambda_n)(r + c_4 + \lambda_n) - c_2 c_3] = 0$$

which on account of the special symmetry of the system becomes

$$\sigma^2(n) - r\sigma(n) + b_n = 0$$

where $b_n = -\lambda_n^2 - (r + c_4 - c_1)\lambda_n + (c_1 r + c_1 c_4 - c_2 c_3)$. This yields two

[8]Noting (a) that $D\frac{\partial^2}{\partial x^2} \sin\left(\frac{n\pi}{L}x\right) = -\lambda_n \sin\left(\frac{n\pi}{L}x\right)$ where $\lambda_n = D\frac{n^2\pi^2}{L^2}$ and (b) that $\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = 0$ if $n \neq m$.

solutions:

$$\sigma_{1,2}(n) = \frac{r}{2} \pm \bar{\sigma}(n), \quad \bar{\sigma}(n) = \frac{1}{2}\sqrt{r^2 - 4b_n}, \tag{42}$$

with corresponding eigenvectors

$$E_{1,2}(n) = \left(\frac{c_2}{c_1 - \lambda_n - \sigma_{1,2}(n)}, 1 \right)'.$$

STEP E. The general solution of the system of ODEs (41) in step C is given for every $n \in \mathbb{N}$

$$\begin{aligned} y_n(t) &= A(n)\frac{c_2}{c_1 - \lambda_n - \sigma_1(n)} e^{\sigma_1(n)t} + B(n)\frac{c_2}{c_1 - \lambda_n - \sigma_2(n)} e^{\sigma_2(n)t}, \\ p_n(t) &= A(n)e^{\sigma_1(n)t} + B(n)e^{\sigma_2(n)t}, \end{aligned}$$

where $A(n)$ and $B(n)$ are constants to be determined.

STEP F. For every $n \in \mathbb{N}$ determine $(A(n), B(n))$ by the initial and final conditions, i.e., by the solution of the linear system

$$\begin{aligned} &A(n)\frac{c_2}{c_1 - \lambda_n - \sigma_1(n)} + B(n)\frac{c_2}{c_1 - \lambda_n - \sigma_2(n)} = a_n, \\ &A(n)e^{\sigma_1(n)T} + B(n)e^{\sigma_2(n)T} = 0. \end{aligned}$$

This readily yields

$$\begin{aligned} A(n) &= \frac{\left(c_1 - \lambda_n - \frac{r}{2}\right)^2 - \bar{\sigma}(n)^2}{\left(c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)\right) - \left(c_1 - \lambda_n - \frac{r}{2} - \bar{\sigma}(n)\right) e^{2\bar{\sigma}(n)T}} \frac{a_n}{c_2}, \\ B(n) &= -A(n)e^{2\bar{\sigma}(n)T}. \end{aligned} \tag{43}$$

STEP G. The optimal state is then given by

$$y(t, x) = e^{\frac{rt}{2}} \sum_n \left(\bar{A}(n)e^{\bar{\sigma}(n)t} + \bar{B}(n)e^{-\bar{\sigma}(n)t} \right) \sin\left(\frac{n\pi}{L} x \right)$$

where

$$\bar{A}(n) := \frac{\left(c_1 - \lambda_n - \frac{r}{2}\right)^2 - \bar{\sigma}(n)^2}{\left(c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)\right)^2 - \left[\left(c_1 - \lambda_n - \frac{r}{2}\right)^2 - \bar{\sigma}(n)^2\right] e^{2\bar{\sigma}(n)T}} a_n,$$

$$\bar{B}(n) := -\frac{\left[\left(c_1 - \lambda_n - \frac{r}{2}\right)^2 - \bar{\sigma}(n)^2\right] e^{2\bar{\sigma}(n)T}}{\left[\left(c_1 - \lambda_n - \frac{r}{2}\right)^2 - \bar{\sigma}(n)^2\right] - \left(c_1 - \lambda_n - \frac{r}{2} - \bar{\sigma}(n)\right)^2 e^{2\bar{\sigma}(n)T}} a_n, \tag{44}$$

whereas the optimal control is proportional to

$$p(t,x) = e^{\frac{rt}{2}} \sum_n (A(n)e^{\bar{\sigma}(n)t} + B(n)e^{-\bar{\sigma}(n)t}) \sin\left(\frac{n\pi}{L}x\right).$$

Note that for the cases of interest it usually holds that $\bar{\sigma}(n) \in \mathbb{R}$ so that the above solution is the usual saddle point structure found in temporal control problems, but now we have one "copy" of such a structure for every mode $n \in \mathbb{N}$. The factor $e^{\frac{rt}{2}}$ takes care of discounting effects. Note that the above solutions are formal Fourier series whose convergence must be studied, and that depends on the nature of the coefficients a_n of Fourier expansion of the initial condition. Such expansions may also form the starting point for the construction of generalized (weak) solutions.

9.2 Infinite horizon problems

The forward-backward PDEs for the problems related to linear state equations and quadratic cost functionals in infinite horizon encountered in Section 4 reduce to the general form (39) with initial condition $y(0,x) = \phi(x)$ but now the final condition is replaced by the transversality condition

$$\lim_{t\to\infty} \int_{\mathcal{O}} e^{-r\,t} y(t,x) p\,(t,x)\, dx = 0.$$

The problem is still complemented with homogeneous boundary conditions of Dirichlet type.

Steps A, B, C, D, E are identical to the finite horizon case. The major difference here comes in the determination of the constants $A(n)$ and $B(n)$ so that the transversality condition is satisfied.

STEP F$'$. We inspect carefully for each $n \in \mathbb{N}$ the characteristic exponents $\sigma_{1,2}(n)$ as given by (42). If for every $n \in \mathbb{N}$, $\bar{\sigma}(n) \in \mathbb{R}$ then $\sigma_1(n) > \frac{r}{2}$ and $\sigma_2(n) < \frac{r}{2}$, so that the part of the solution related to $e^{\sigma_1(n)t}$ is not compatible with the transversality condition. Therefore, the general solution satisfies the transversality condition only if $A(n) = 0$, so that the acceptable solution is

$$\begin{aligned} y_n(t) &= B(n)e^{\frac{rt}{2}} \frac{c_2}{c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)} e^{-\bar{\sigma}(n)t}, \\ p_n(t) &= B(n)e^{\frac{rt}{2}} e^{-\bar{\sigma}(n)t}, \end{aligned}$$

where $B(n)$ are constants to be determined. These are determined by the initial condition for y, and it is easily seen that the solution is

$$\begin{aligned} y_n(t) &= a_n e^{\frac{rt}{2}} e^{-\bar{\sigma}(n)t}, \\ p_n(t) &= \frac{c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)}{c_2} a_n e^{\frac{rt}{2}} e^{-\bar{\sigma}(n)t}. \end{aligned}$$

STEP G$'$. The optimal state is then given by

$$y(t,x) = e^{\frac{rt}{2}} \sum_n a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L} x\right),$$

whereas the optimal control is proportional to

$$p(t,x) = e^{\frac{rt}{2}} \sum_n \frac{c_1 - \lambda_n - \frac{r}{2} + \bar{\sigma}(n)}{c_2} a_n e^{-\bar{\sigma}(n)t} \sin\left(\frac{n\pi}{L} x\right).$$

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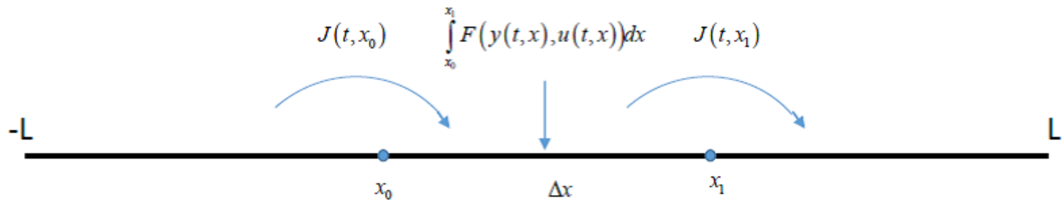

Figure 1: Fickian diffusion

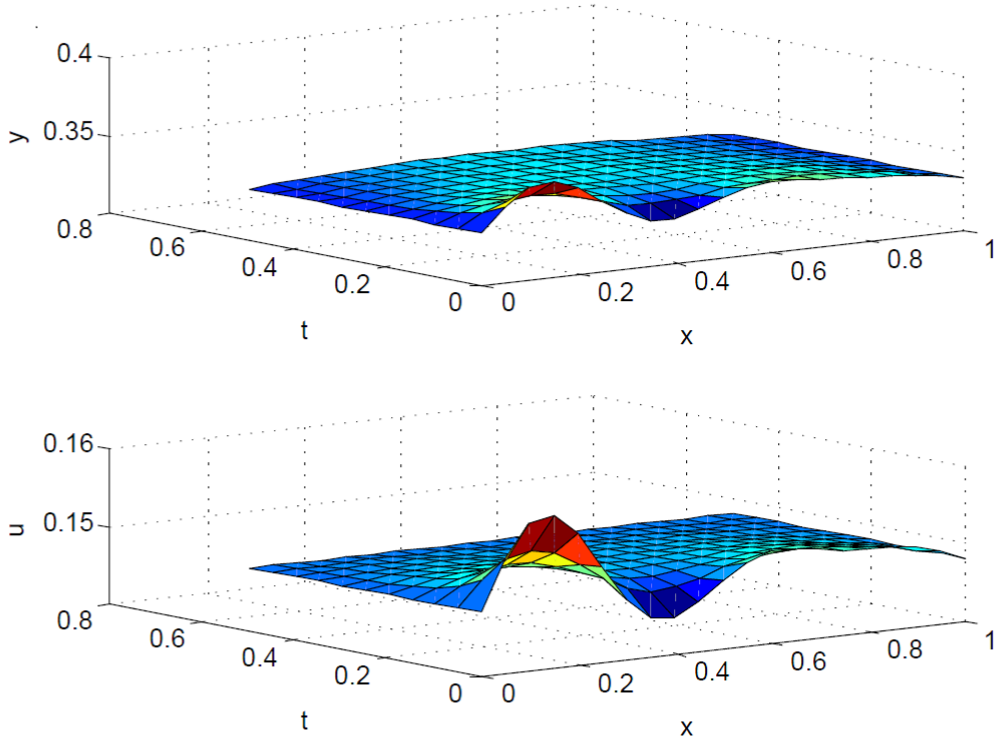

Figure 2: Optimal path and optimal control policy for the target following example

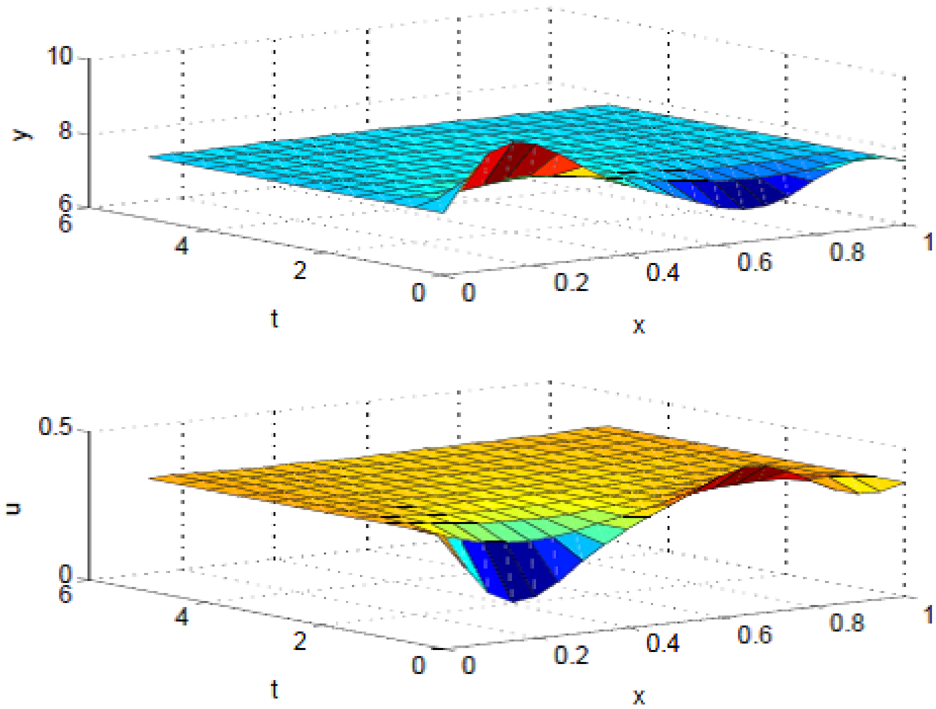

Figure 3: Optimal path and optimal control policy for the pollution control example

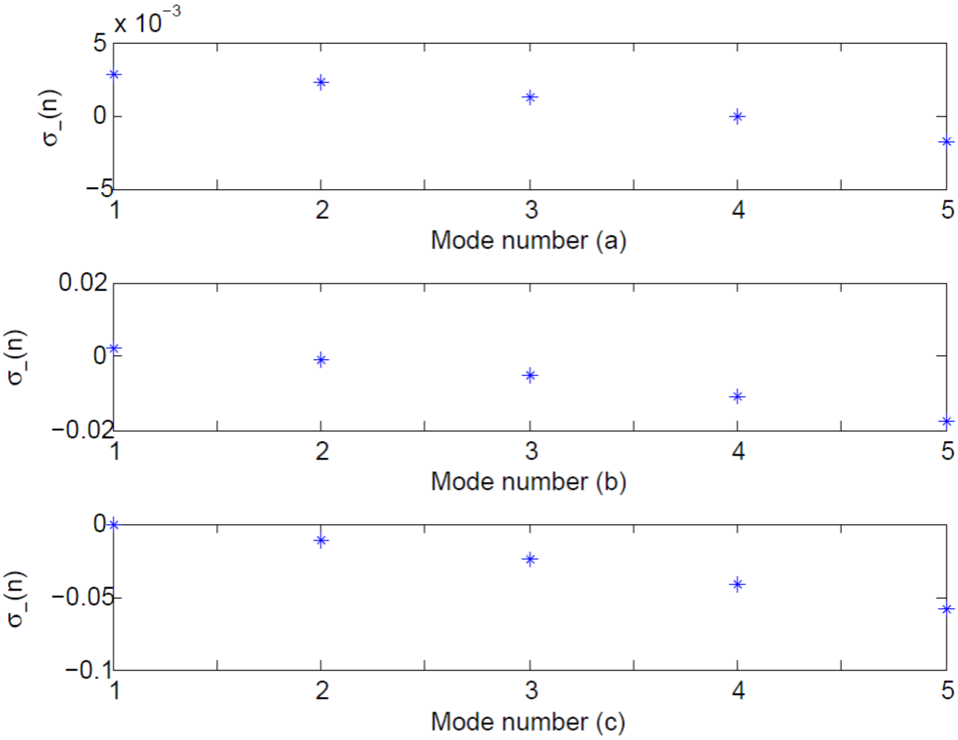

Figure 4: Stability diagrams for various values of the parameter D/L^2

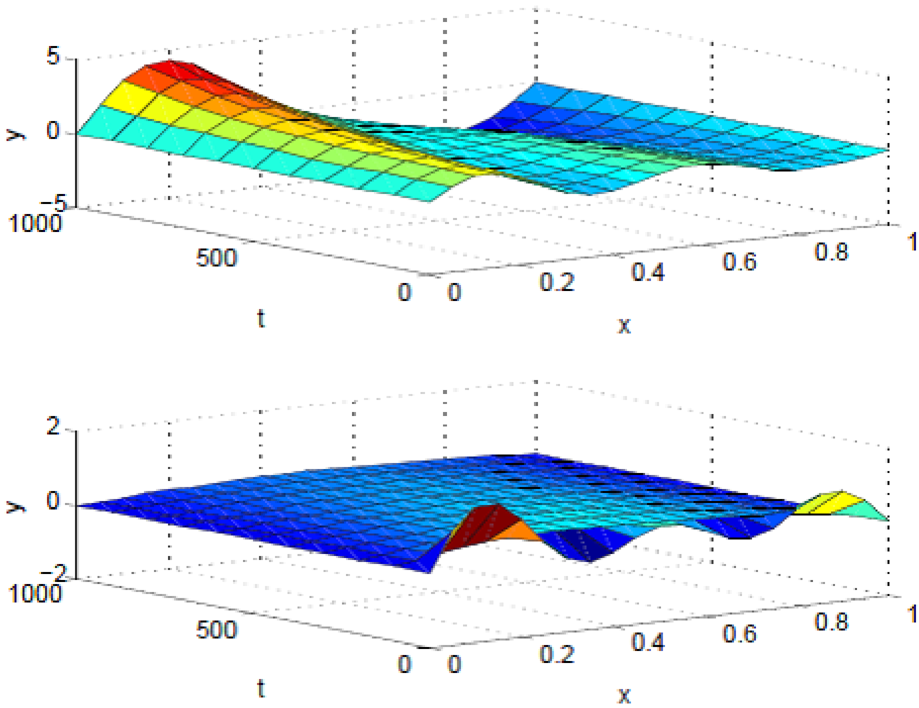

Figure 5: Pattern formation and absence of pattern formation

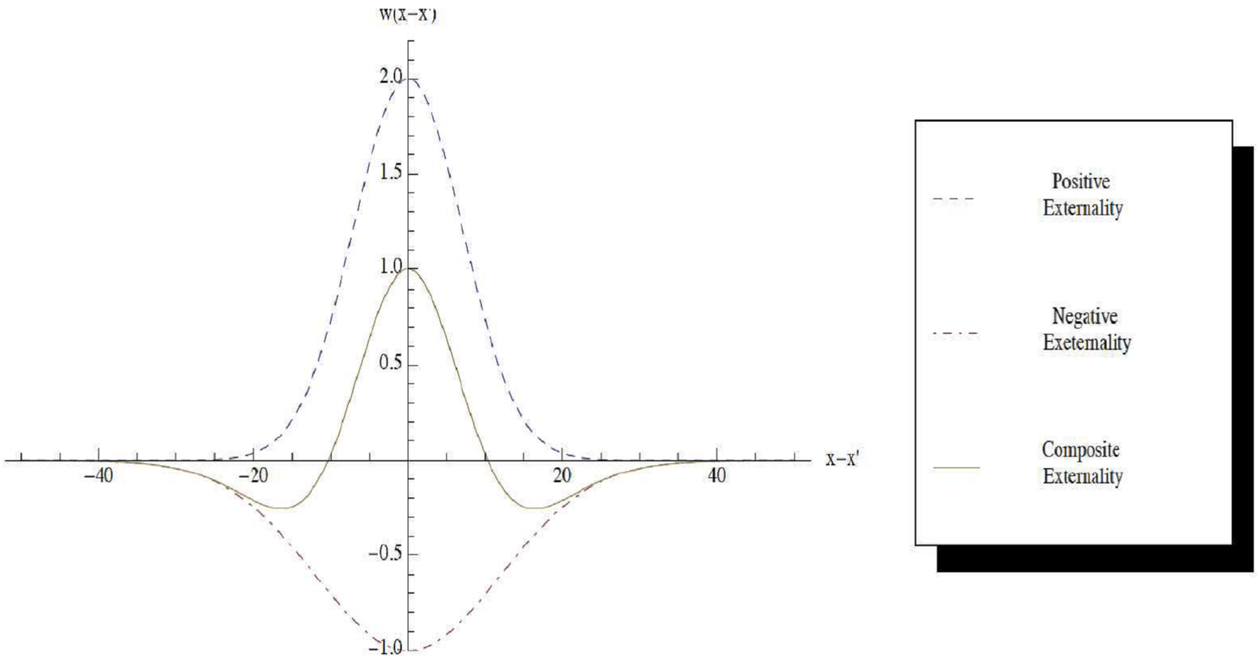

Figure 6: Composite externality - two hump kernel