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Robust Control of a Spatially Distributed Commercial Fishery

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Abstract We consider a robust control model for a spatially distributed commercial fishery under uncertainty, and in particular a tracking problem, i.e. the problem of robust stabilization of a chosen deterministic benchmark state in the presence of model uncertainty. The problem is expressed in the form of a stochastic linear quadratic robust optimal control problem, which is solved analytically. We focus on the emergence of breakdown from the robust stabilization policy, called hot spots, and comment upon their significance concerning the spatiotemporal behaviour of the system.

1 Introduction

An important issue in understanding ecosystems and designing efficient management rules with the purpose of preventing collapse and secure long-term sustainable productivity, is their spatial and temporal structure. The study of the emergence and the properties of regular spatial or spatiotemporal patterns which can be found in abundance in nature, such as for example stripes or spots on animal coats, ripples in sandy desserts, vegetation patterns in arid grazing systems or spatial patterns of fish species, has drawn much attention in natural sciences (e.g. Murray (2003)).

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Thus, in the management of natural resources and the regulation of pollution it seems natural to analyze mechanisms causing spatiotemporal patterns to arise, and to design regulatory policies with spatial characteristics. In renewable resource economics, modelling with spatial-dynamic processes Smith et al. (2009) has been used to study issues such as harvesting in metapopulation models governed by discrete spatial-dynamic processes, design of optimal policies in a spatiotemporal domain, marine or terrestial reserve policies, or bioinvasions (see, e.g., Sanchirico and Wilen (1999), Wilen (2007)). Pattern formation and spatially dependent policies in renewable resource management have been also studied in the context of optimal control of reaction diffusion spatiotemporal systems (Brock and Xepapadeas (2008), Brock and Xepapadeas (2010)). In spatial pollution regulation the main objective is the internalization of the pollution externality through spatially dependent taxes (see, e.g., Goetz and Zilberman (2000), Goetz and Zilberman (2000)), while spatial analysis has also been used to study water pricing in which the concept of a spatial distribution is combined with a two-stage optimal control problem (Xabadia et al. (2004)).

Another issue which is of considerable interest in resource management is decision making when the decision maker is trying to make good choices when she regards her model not as the correct one but as an approximation of the correct one, or to put it differently, when the decision maker has concerns about possible misspecifications of the correct model and wants to incorporate these concerns into the decision-making rules (e.g., Salmon (2002), Hansen and Sargent (2001), Hansen et al. (2006), Hansen and Sargent (2008), JET (2006)).

The purpose of the present paper is to study the regulation of a commercial fishery following the classic model of commercial fishing (Smith (1969)) with explicit spatial dependence where spatial interconnections in economic and biological variables are captured by local and non-local spatial effects. In this model the regulator has concerns about possible misspecifications of the spatiotemporal evolution of the phenomenon. That is, the regulator regards her model as an approximation of the correct spatiotemporal dynamics and seeks spatially dependent regulation that performs well under the approximating model. In this context we try to study how a regulator could design optimal spatiotemporal robust control for this fishery, how hot spots, which are location where the qualitative properties of the system change along with the structure of the regulation, may emerge, and what implications they might have for regulation.

The contribution of this approach is that it allows us to study in a unified model the optimal regulation of spatially interconnected distributed parameter fishery when concerns about model misspecification vary across the spatial domain. We follow Hansen et al. (2006) or Hansen and Sargent (2008), and regard concerns about model misspecification to imply that the regulator distrusts her model and wants robust decisions over a set of possible models that surround the regulator's approximating or benchmark model, and which are difficult to distinguish with finite data sets. The robust decisions are obtained by introducing Nature, a fictitious "adversarial agent". Nature promotes robust decision rules by forcing the regulator, who seeks to maximize profits from the commercial fishery over an entire spatial domain, to explore the fragility of decision rules with respect to departures from the benchmark model. A robust decision rule to model misspecification means that lower bounds to the rule's performance are determined by Nature – the adversarial agent – who acts as a minimizing agent when constructing these lower bounds. Hansen et al. (2006) show that robust control theory can be interpreted as a recursive version of max-min expected utility theory (Gilboa and Schmeidler (1989)).

In our model, considering the spatial domain of the fishery as a ring of cells, the regulator is trying to determine an optimal level of harvesting per vessel in each spacial cell. This harvesting level can be used, for example, to set up a quota system in each site of the fishing area. The regulator's objective could be either the maximization of discounted profits over the whole ring, or the minimization deviations (or the cost of deviations) from target harvesting and biomass levels in each ring, by taking into account biomass diffusion as well as stock, congestion, and productivity externalities

The regulator is however uncertain regarding the true statistical distribution of the state of the system. This means that the regulator has concerns regarding the specification of biomass dynamics in each cell, and depending on her scientific knowledge, she might trust a benchmark model of the fishery more or less depending on the specific cell. For a large enough ring, this assumption - which implies spatially differentiated degrees of model uncertainty - seems plausible, and it is related to a localized in space entropy constraint of the spatially varying interconnected systems. In this context we derive optimal robust harvesting rules for each site and identify conditions under which concerns about model misspecification at specific site(s) could cause regulation to break down or to be very costly. We call sites associated with these phenomena hot spots. We are also able to identify spatial hot spots where the need to apply robust control induces spatial agglomerations and breaks down spatial symmetry. From the theory point of view this is a new source for generating spatial patterns as compared to the classic Turing diffusion induced instability (Turing (1952)) which belongs to the recently identified family of optimal diffusion or spatial-spillover-induced instabilities (Brock and Xepapadeas (2008), Brock and Xepapadeas (2010), Brock et al. (2012)).

Distributed parameter models result in optimal control problems in infinite dimensional spaces. By using Fourier methods and exploiting the property of spatial invariance of a class of linear quadratic problems, we are able to obtain closed form solutions to these infinite dimensional problems which reveal important information on the qualitative features of the optimal policy, possible deviations from it or breakdowns as well as its dependence on the choice of model. Furthermore, by obtaining a linear quadratic approximation around a deterministic optimal trajectory of a nonlinear distributed parameter robust control problem of a commercial fishery, the tracking problem of keeping the controlled trajectory under uncertainty and concerns about model misspecification close to the optimal deterministic trajectory, representing the ideal benchmark model of the fishery manager, and comment upon hot spot formation and their importance.

2 Modelling a Fishery with Spatial Interactions

2.1 A spatial profit maximization fishery model

We consider a commercial fishery occupying an area that consists of a circular ring of N cells or sites on a finite lattice, so that space can be considered as the finite group of integers modulo N, $\mathbb{Z}_N$. The state of the system is quantified in terms of the fish biomass in each cell, x_n, and the number of vessels or firms fishing in each cell V_n, $n \in \mathbb{Z}_N$ (see Figure 1).

Let $x_n(t)$ denote biomass at time $t \geqslant 0$ and cell $n \in \mathbb{Z}_N$. Fish biomass moves from cell to cell. The movements if there are strictly local can be described by classic diffusion with diffusion coefficient $D > 0$, which means that fish move from cells of high biomass concentration to adjacent cells of low biomass concentration. In this case the spatial movement can be modelled using the discrete Laplacian by a term $D[x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)]$. More general spatial interactions across locations can be modelled by an influence "kernel" (or rather a discretized version of an influence kernel) which can be represented in terms of a matrix $\mathsf{A} = (\alpha_{nm}) \in \mathbb{R}^{N\times N}$. The entry α_{nm} provides a measure of the influence of the biomass of the system at point m to the biomass concentration of the system at point n. If there is no movement of biomass across cells then $\mathsf{A} = \alpha_{nm} = \delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta. If only next neighborhood movements are possible then α_{nm} is non-zero only if m is a neighbor of n. Such an example is the discrete Laplacian, and matrix A in this case has a general form

$$\mathsf{A} = D \begin{pmatrix} 1 & -2 & 1 & 0 \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \cdots & 1 & -2 & 1 \end{pmatrix}.$$

This can be considered as the discretization of the Laplace operator $\mathsf{A} = D\frac{\partial^2}{\partial z^2}$, in case the space is considered as continuous e.g. the interval $[-\pi, \pi]$.

Let $V_n(t)$ denote the number of identical vessels or firms operating at cell n of the ring, and $h_n(t)$ the harvest rate at cell n per unit time. Thus total harvesting at cell n is $h_n(t)V_n(t)$. The temporal evolution of biomass of the fishery is subject to statistical fluctuations (noise), which is introduced into the model via stochastic factors (sources)[1], modelled in terms of a stochastic process $w = \{w_n\}$, $n \in \mathbb{Z}_N$, which is considered as a vector valued Wiener process on a suitable filtered probability space

[1] There is uncertainty concerning the state of the system (i.e. the true figures for the biomass) which is represented in terms of the vector valued stochastic process w. These common factors affect the state of the biomass x at the different sites. Each factor has a different effect on the state of the biomass on each particular site; this will be modelled by a suitable correlation matrix. It is not of course necessary that the number of factors is the same as the number of sites in the system however, without loss of generality we will make this assumption and assume that there is one factor or source of uncertainty related to each site. This assumption can be easily relaxed.

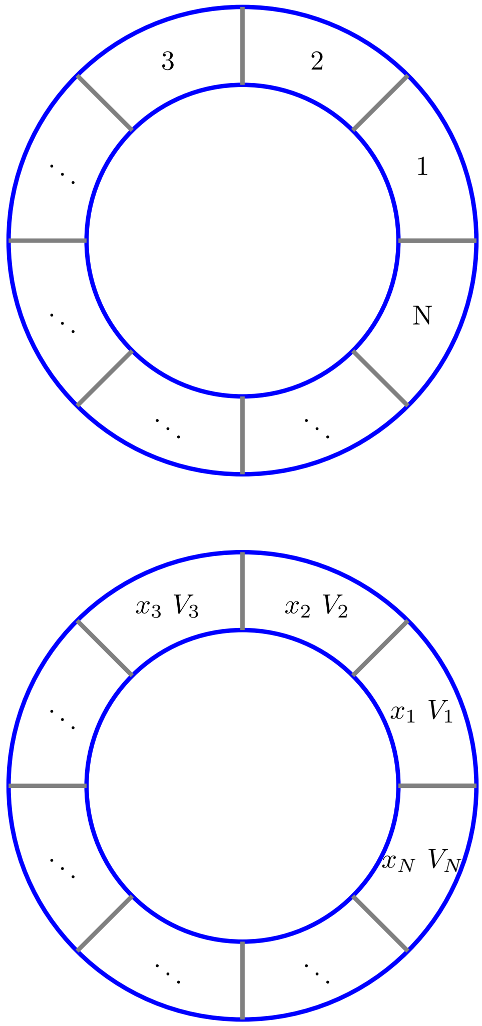

Fig. 1 The circular fishery and the relevant state variables.

$(\Omega, \{\mathscr{F}_t\}_{t\in\mathbb{R}_+}, \mathscr{F}, \mathbb{P})$ (see e.g., Karatzas and Shreve (1991)). The introduction of noise turns the biomass for a fixed time t into an $\mathbb{R}^N$-valued random variable, thus $x(\cdot)$ is an $\mathbb{R}^N$-valued stochastic process. We assume that this stochastic process is the solution of a stochastic differential equation:

$$\begin{aligned} dx_n(t) &= \left[f(x_n(t)) + \sum_m \alpha_{nm} x_m(t) - h_n(t) V_n(t) \right] dt + \sum_m s_{nm} dw_m, \\ x_n(0) &= x_{0,n}, \ n, m \in \mathbb{Z}_N. \end{aligned} \tag{1}$$

In the above equation $f(x)$, $x \geqslant 0$, is the recruitment rate or growth function for the fishery. This function has the properties that there exist three values $\underline{x}$, $\bar{x}$ and x^0 with $0 \leqslant \underline{x} < x^0 < \bar{x}$, such that $f(\underline{x}) = f(\bar{x}) = 0$, $f'(x^0) = 0$, $f''(x^0) < 0$. An example of such a function is a quadratic function which models logistic growth. It is assumed that the parameters of model (1) are chosen so that positivity of solutions is guaranteed (i.e. noise levels are assumed to be small and have a weak effect). Furthermore, for the rest of the paper $\sum_m$ will be used as a shorthand for $\sum_{m\in\mathbb{Z}_N}$.

The last term of (1), describing the fluctuations of the biomass due to the stochasticity, is understood in the sense of the Itô theory of stochastic integration. In compact form it can be represented by a finite matrix $\mathsf{S} = (s_{nm})$ with elements s_{nm} indicating how the uncertainty at site m is affecting the uncertainty concerning the biomass of the fishery at site n. The matrix $\mathsf{S} = (s_{nm})$ can be thought of as the spatial autocorrelation operator for the system. Thus the evolution of the system can be written in a compact form as:

$$dx = [\mathsf{F}(x) + \mathsf{A}x - y]\, dt + \mathsf{S} dw \tag{2}$$

where we have used the vector notation

$$\begin{aligned} x &= (x_1, \cdots, x_N)^{tr}, \\ w &= (w_1, \cdots, w_n)^{tr}, \\ y &= (h_1 V_1, \cdots, h_N V_N)^{tr}, \\ \mathsf{F}(x) &= (f(x_1), \ldots, f(x_N))^{tr}, \end{aligned}$$

and $\mathsf{A}, \mathsf{S} : \mathbb{R}^N \to \mathbb{R}^N$ are linear operators, representable by finite matrices with elements $\{\alpha_{nm}\}$, $\{s_{nm}\}$, respectively. We will also use the notation $y = h \otimes V$ for the vector which is defined by componentwise multiplication of the vectors h, V.

The cost per vessel operating at a cell n for harvesting rate h is determined by a cost function $c(h_n(t), x_n(t), C_n(t), P_n(t))$. This is a function of the harvesting rate; the biomass level at the specific cell, $x_n(t)$ which reflects recourse stock externalities; and the number of other vessels operating in the neighborhood of the cell n, which reflect two types of externalities: crowding or congestion externalities and productivity or knowledge externalities. Crowding externalities, which are negative (cost increasing), and productivity externalities, which are positive (cost reducing), are non-local effects, which are modeled by spatial kernels as:

$$
\begin{aligned}
C_n(t) &= \sum_m c_{nm} V_m(t) =: (\mathsf{C}V)_n(t), \\
P_n(t) &= \sum_m \gamma_{nm} h_m(t) =: (\Gamma h)_n(t)
\end{aligned} \tag{3}
$$

where $\mathsf{C}, \Gamma : \mathbb{R}^N \to \mathbb{R}^N$ are linear operators, representable by finite matrices with elements c_{nm}, γ_{nm}, respectively. We assume that: (i) $\frac{\partial c}{\partial h} > 0$, $\frac{\partial^2 c}{\partial h^2} \geqslant 0$; (ii) $\frac{\partial c}{\partial x} < 0$, which implies resource stock externalities; (iii) $\frac{\partial c}{\partial C} > 0$, which implies crowding externalities due to congestion effects. We assume that an increase in vessels in a given cell will always increase costs, that is $\frac{\partial c}{\partial C} > 0$. This kernel formulation in the cost function means that vessels not only in cell n but also near cell n could create congestion effects and increase operating costs of the vessels operating in cell n; and (iv) $\frac{\partial c}{\partial P} < 0$, which implies knowledge or productivity externalities because harvesting that takes place near cell n helps the development of harvesting knowledge in n and reduces operating costs.

Assuming that harvested fish is sold at an exogenous price $\mathscr{P}$, which is homogeneous over the whole ring, profit per vessel at n is defined as:

$$
\pi_n(t) = \mathscr{P} h_n(t) - c\left(h_n(t), x_n(t), (\mathsf{C}V)_n(t), (\Gamma h)_n(t)\right). \tag{4}
$$

Vessels are attracted to cell n if profits per vessel at this site are higher than the average profit over the whole spatial domain. Vessels can be attracted to the ring from locations outside the ring if profits are positive in cells of the ring, so the number of vessels in the ring does not need to be conserved[2]. Assuming that the rate of growth of vessels in each cite is proportional to the difference between the profit per vessel at n with the average profit per vessel over the whole lattice, the evolution of vessels in each site is described by:

$$
\begin{aligned}
&\frac{d}{dt} V_n(t) = \phi \left(\pi_n(t) - \frac{1}{N} \sum_m \pi_m(t) \right) V_n(t), \\
&V_n(0) = V_{0,n},
\end{aligned} \tag{5}
$$

where $\phi > 0$ measures the speed of adjustment and is set equal to one without loss of generality. Note that equation (5), though not an Itô stochastic differential equation, is now a random differential equation since x is a stochastic process.

A regulator is trying to determine in each cell an optimal level of harvesting per vessel, h_n. This harvesting level can be used, for example, to set up a quota system in each cell of the ring. The regulator's objective is the maximization of discounted profits over the whole ring by taking into account biomass diffusion as well as stock, congestion and knowledge externalities[3]. The regulator's objective is therefore

---

[2] To simplify we ignore transportation costs.

[3] To simplify the interpretation of results and the analysis, we do not include existence values for the biomass.

$$\max_{\{h_n(t)\}} \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\sum_n V_n(t)\pi_n(t)\right)dt\right], \quad \text{subject to (2) and (5),} \tag{6}$$

where the per vessel profit π_n is given by (4). It is clear that the state of the system is characterized by the biomass x and the vessel distribution V, and we will use the notation $X = (x,V)^{tr}$ where $X \in \mathbb{R}^{2N\times 1}$.

2.2 Misspecification concerns

We now assume that the regulator has concerns regarding the specification of biomass dynamics in each cell, which can be modelled as follows: Assume that there is some uncertainty concerning the "true" statistical distribution of the state of the system. This corresponds to a family of probability measures $\mathscr{Q}$ such that each $Q \in \mathscr{Q}$ corresponds to an alternative stochastic model (scenario) concerning the state of the system. From Girsanov's theorem $\bar{w}_n(t) = w_n(t) - \int_0^t v_n(s)ds$ is a Q-Brownian motion for all $n \in \mathbb{Z}$, where the drift term v_n may be considered as a measure of the model misspecification at lattice site n, where $v = (v_1,\cdots,v_N)^{tr}$ is an $\mathbb{R}^N$-valued stochastic process which is measurable with respect to the filtration $\{\mathscr{F}_t\}$ satisfying the Novikov condition $\mathbb{E}\left[\exp(\int_0^T \sum_n v_n^2(t)dt)\right] < \infty$. Thus, Girsanov's theorem (see e.g. Karatzas and Shreve (1991)) shows that the adoption of the family $\mathscr{Q}$ of alternative measures concerning the state of the system, leads to a family of differential equations for the biomass

$$\begin{aligned} dx_n(t) = &\left[f(x_n(t)) + \sum_m \alpha_{nm}x_m(t) - h_n(t)V_n(t) + \sum_m s_{nm}v_m\right]dt \\ &+ \sum_m s_{nm}d\bar{w}_m, \;\; n,m \in \mathbb{Z}_N, \\ x_n(0) = &\, x_{0,n}, \end{aligned} \tag{7}$$

parameterized by the information drift v. In (7) x indicates the state of the system when the measure[4] Q corresponding to the information drift v and the control procedure $h = (h_1,\cdots,h_N)^{tr}$, which will be denoted by Q_v, is adopted. This is an Ornstein Uhlenbeck equation which in compact form can be expressed as

$$dx = [\mathsf{F}(x) + \mathsf{A}x - h \otimes V + \mathsf{S}v]\,dt + \mathsf{S}d\bar{w}. \tag{8}$$

The regulator's problem when there are concerns about model misspecification is solved under the adoption of the measure Q, related to the drift v, i.e. it is solved under the dynamic constraints (7) and (5). This will provide a solution leading to a value function $\mathscr{V}(X;v)$; corresponding to the maximum discounted profits over the

[4] We will identify a model by a probability measure.

whole spatial domain obtained for the model Q_v under the optimal harvesting effort, given that the system had initial state $X = (x(0),V(0)) = (x,V)$. Being uncertain about the true model, the decision maker will opt to choose the strategy that will work in the worst case scenario; this being the one that minimizes $V(X;v)$ - the maximum over all h having chosen v - over all possible choices for v. Therefore, the robust control problem to be solved is of the general form

$$\begin{aligned}&\mathscr{V}(X) = \max_h \min_v J(h,v),\\&\text{subject to (7) and (5),}\end{aligned} \tag{9}$$

where

$$J(h,v) = \mathbb{E}_{Q_v}\left\{\int_0^\infty \mathrm{e}^{-rt}\left[\sum_n V_n(t)\,\pi_n(t) + \sum_n \theta_n (v_n(t))^2\right] dt\right\}.$$

The vector $\theta = (\theta_1, \cdots, \theta_N)^{tr}$ corresponds to the weight assigned to concerns related to model misspecification in a local sense (differentially in space). To clarify this point, we refer to Brock et al. (2012), as by a simple modification of the arguments in this work it can be shown that robust optimization problems of the form

$$\begin{aligned}&\sup_h \inf_{Q\in\mathscr{Q}} \mathbb{E}_Q\left[\int_0^\infty \mathrm{e}^{-rt}\sum_n V_n(t)\pi_n(t)dt\right],\\&\text{subject to } \mathscr{H}(\mathbb{P}_n \mid Q_n) < H_n,\ n\in\mathbb{Z}_N,\end{aligned} \tag{10}$$

and the dynamic constraints (7) and (5), can be written as equivalent to (9) where now the vector $\theta \in \mathbb{R}^N_+$ plays the role of a Lagrange multiplier associated with the constraints in (10). In (10) by $\mathscr{H}(\mathbb{P}_n \mid Q_n)$ we denote the Kullback-Leibler entropy of the marginal probability measures $\mathbb{P}_n$ and Q_n (i.e. the probability measures $\mathbb{P}$ and Q respectively, averaged over all possible states of the noise over the remaining sites). The localized entropic constraints mean that the regulator is only considering models in each cell (i.e., measures Q_n) whose deviation in terms of the relative entropy from the "true" model in the cell (i.e., the measure $\mathbb{P}_n$) is less than H_n.

The introduction of the local entropic constraints means that the concern of the policy maker about uncertainty on site n is quantified by H_n, the smaller H_n is the less model uncertainty she is willing to accept for site n, given her information about this site. This assumption is not unreasonable as certain cells may be considered as more crucial than others therefore specific care should be taken for them.

In the robust control problem the minimizing adversarial agent - Nature - chooses a $\{v_n(t)\}$ while $\theta_n \in (\underline{\theta}_n, +\infty]$, $\underline{\theta}_n > 0$, is a penalty parameter restraining the maximizing choice of Nature. As noted above θ_n is associated with the Lagrange multiplier of the entropy constraint at each site. In the entropy constraint H_n is the maximum misspecification error that the decision maker is willing to consider given the existing information about the system at site n[5]. The lower bound $\underline{\theta}_n$ is a so-called breakdown point beyond which it is fruitless to seek more robustness because the adversarial (i.e. the minimizing) agent is sufficiently unconstrained so that she/he can push the criterion function to $-\infty$ despite the best response of the minimizing agent. Thus when $\theta_n < \underline{\theta}_n$ for a specific site robust control rules cannot be attained. In our terminology this site is a candidate for a "nucleus" of a hot spot since misspecification concerns for this site will break down robust control for the whole spatial domain. On the other hand when $\theta_m \to \infty$ or equivalently $H_m = 0$ there are no misspecification concerns for this site and the benchmark model can be used. The effects of spatial connectivity can be seen in this extreme example. The spatial relation of site m with site n could break down regulation for both sites. If site m was spatially isolated from n there would have been no problem with regulation at m.

2.3 Robust stabilization of a desired optimal state

Problem (9) is a non linear robust control problem. The full nonlinear problem, eventhough accessible to either abstract analysis or numerical treatment, will not allow an analysis in terms of closed form expressions and as such will obscure our main interest in this paper, which is to show the existence of hot spots and spatial pattern formation. To illustrate these points we will instead choose to work in terms of a linear quadratic approximation of the full nonlinear problem, which allows a rather detailed analytical treatment. However, rather than taking a linear quadratic local approximation of the full problem (9) we choose an alternative approach. This alternative approach is related to a tracking problem, which allows the decision maker to "correct" her benchmark policy in such a way as to optimally make up for possible mispecifications of the model. Tracking problems have been addressed by the control theory community and find important applications in a variety of problem in mathematical, environmental and financial economics (see e.g., Leizarowitz (1985), Artstein and Leizarowitz (1985), Leizarowitz (1986)).

In this section we formulate a related linear quadratic robust control problem which is associated with a stabilization policy, under the effect of noise and uncertainty with respect to the nature of this noise, which allows the decision maker to keep the system as close as possible to a desired optimal state of (9). We assume that the desired optimal state is the one that corresponds to the deterministic version of the model, i.e. the case where there is no noise present. Let us call this state $(x^{(0)}, V^{(0)})$ and assume that it is supported by the optimal control procedure $h^{(0)}$. The triple $(x^{(0)}, V^{(0)}, h^{(0)})$ is thus the solution of the deterministic optimal control problem

[5] If the decision maker can use physical principles and statistical analysis to formulate bounds on the relative entropy of plausible probabilistic deviations from her/his benchmark model, these bounds can be used to calibrate the parameters H_n (Athanassoglou and Xepapadeas (2012)).

$$\max_h \int_0^\infty e^{-rt} \sum_n V_n(t)\pi_n(t)dt,$$
subject to
$$\dot{x}_n = \sum_m a_{nm}x_m - f(x_n) - V_n h_n, \ n \in \mathbb{Z}_N \tag{11}$$
$$\dot{V}_n = \phi\left(\pi_n - \frac{1}{N}\sum_m \pi_m\right)V_n, \ n \in \mathbb{Z}_N.$$

This is the idealized problem that the fishery manager wants to solve. The solution of that, furnishes the "best" she can do to optimize her profit, given the capabilities of the fishery, in the absence of unforeseen event (i.e. noise).

The solution $x^{(0)}$ is determined by the deterministic Pontryagin principle, associated with the Hamiltonian function

$$H(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) = \sum_n V_n\pi_n + \sum_n \mathfrak{p}_n\left(\sum_m a_{nm}x_m - f(x_n) - V_n h_n\right) + \sum_n \bar{\mathfrak{p}}_n\left[\phi\left(\pi_n - \frac{1}{N}\sum_m \pi_m\right)V_n\right]$$

where $\mathfrak{p} = (\mathfrak{p}_1,\cdots,\mathfrak{p}_N)^{tr}$, $\bar{\mathfrak{p}} = (\bar{\mathfrak{p}}_1,\cdots,\bar{\mathfrak{p}}_N)^{tr}$ are the adjoint variables associated with the state variables $x = (x_1,\cdots,x_N)^{tr}$ and $V = V_1,\cdots,V_N)^{tr}$ respectively. The solution of the benchmark optimal control problem is reduced to the solution of the system of differential equations

$$\begin{aligned}
&\frac{\partial H}{\partial x_n}(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) - \dot{\mathfrak{p}}_n - r\mathfrak{p}_n = 0, \ n \in \mathbb{Z}_N \\
&\frac{\partial H}{\partial V_n}(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) - \dot{\bar{\mathfrak{p}}}_n - r\bar{\mathfrak{p}}_n = 0, \ n \in \mathbb{Z}_N \\
&\frac{\partial H}{\partial \mathfrak{p}_n}(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) - \dot{x}_n = 0, \ n \in \mathbb{Z}_N \\
&\frac{\partial H}{\partial \bar{\mathfrak{p}}_n}(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) - \dot{V}_n = 0, \ n \in \mathbb{Z}_N \\
&\frac{\partial H}{\partial h_n}(x,V,\mathfrak{p},\bar{\mathfrak{p}};h) = 0, \ n \in \mathbb{Z}_N
\end{aligned} \tag{12}$$

where the last set of equations is an optimality condition. The solution $(x^{(0)}, V^{(0)}, h^{(0)})$ of this system gives the optimal benchmark path. In general this system has a solution which is spatially dependent, i.e., $x_n(t) \neq x_m(t)$ for $n \neq m$. However, it may also have solutions which are uniform in space. For example in the case of diffusive coupling, $\sum_m a_{nm} = 0$, $\sum_m \beta_{nm} = 0$ and $\sum_m \gamma_{nm} = 0$ the system (12) may admit a solution which is uniform in space, i.e. a solution $\{x_n^0(t), V_n^0(t)\}$ such that $x_n^0(t) = x^0(t)$, $V_n^0(t) = V^0(t)$ for all $n \in \mathbb{Z}$. Similarly for the optimal control $h^{(0)}$. Furthermore, we may assume that these equations have a stationary uniform in space solution, i.e., a solution that is time independent. While this assumption is not necessary for the development of the proposed model, it simplifies the exposition and will be adopted. It should be stressed though that a general theory for time dependent as spatially nonhomogeneous $x^{(0)}, V^{(0)}, h^{(0)}$ can be formulated and the necessary modifications are technical.

However, true life is often far from the idealized model, that the manager has in mind. This means that the manager should be adept to sidetrack from the idealized optimal control procedure $h^{(0)}$ as an effect of unforeseen circumstances, modeled here by noise. An important question is the following: Can we design optimally a corrective policy $h^{(1)}$ which will take into account the effects of noise so that the true system keeps as close as possible to the idealized optimal state $(x^{(0)}, V^{(0)})$ as provided by the solution of the optimal control problem (11)?

Assume that we have the nonlinear problem (1), subject to weak additive noise. The problem is subject to model uncertainty (with respect to the nature of the noise term) which may be modelled in terms of a drift v so that applying Girsanov's theorem we obtain the family of models

$$\begin{aligned} dx_n &= \left[f(x_n) + \sum_m \alpha_{nm} x_n - h_n V_n + \varepsilon \sum_m s_{nm} v_n \right] dt + \varepsilon \sum_m s_{nm} dw_m, \\ dV_n &= \phi \left(\pi_n - \frac{1}{N} \sum_m \pi_m \right) V_n dt, \ n \in \mathbb{Z}_N. \end{aligned} \tag{13}$$

This family of models will give the "observed" state of the system (x, V). The system is still subject to a control procedure h, and it is our aim to choose h so that the actual state of the system (x, V) is kept as close as possible to the ideal profit maximizing state $(x^{(0)}, V^{(0)})$ with h as close as possible to $h^{(0)}$.

Since the noise is assumed to be weak we may consider as a zeroth order approximation to (13) (i.e., the solution setting $\varepsilon = 0$) the deterministic optimal path $(x^{0)}, V^{0)})$. Let us consider perturbations of $\{x, V, h, v\}$ around this reference solution, i.e. let us consider solutions of the above problem of the form

$$\{x, V, h, v\} = \{x^0, V^0, h^0, 0\} + \varepsilon \{x^1, V^1, h^1, v^1\}$$

where now $\{x, V, h, v\}$ are subject to uncertainty and are solutions of the stochastic biomass equation (13) with ε a small parameter. The terms $(x^{(1)}, V^{(1)})$ quantify the divergence of the actual state of the system from the ideal profit maximizing optimal state $(x^{(0)}, V^{(0)})$, the fishery manager would like to follow. This deviation is in general going to be spatially dependent; this spatial dependence will depend on the interaction between the dynamics of the system and noise. We still allow the manager a control procedure $(h^{(1)}, v^{(1)})$, this is considered as the correction procedure on top of the pre-planned ideal optimal control procedure $h^{(0)}$ where $v^{(1)}$ takes care of model uncertainty which will be chosen so as to minimize deviation from the ideal plan of action $(x^{(0)}, V^{(0)}, h^{(0)})$. As we shall see this correction procedure can be chosen in terms of a feedback control procedure, whereby the corrections are determined upon observation of the deviation from the ideal desired state.

We linearize the state equations around the state $s^{(0)} := \{x^{(0)}, V^{(0)}, h^{(0)}, v^{(0)}\}$ to obtain to first order in ε that

$$dx^{(1)} = [A^{(1)}x^{(1)} + A^{(2)}V^{(1)} + B^{(1)}h^{(1)} + Sv^{(0)}]dt + Sd\bar{w}$$
$$dV^{(1)} = [A^{(3)}x^{(1)} + A^{(4)}V^{(1)} + B^{(2)}h^{(1)}]dt$$

where $x^{(1)}, V^{(1)}, h^{(1)}, v \in \mathbb{R}^N$ and $A^{(i)}, B^{(j)}$, $i = 1, \cdots, 4$, $j = 1, 2$ are $\mathbb{R}^{N\times N}$ matrices with elements

$$A^{(1)}_{nm} = f'(x^{(0)}_n)\delta_{nm} + a_{nm}$$
$$A^{(2)}_{nm} = -h^{(0)}_n \delta_{nm}$$
$$A^{(3)}_{nm} = -\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial x}\right)_n \delta_{nm} + \frac{1}{N}\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial x}\right)_m$$
$$A^{(4)}_{nm} = \phi \left(\pi^{(0)}_n - \frac{1}{N}\sum_k \pi^{(0)}_k\right)\delta_{nm} - \phi V^{(0}_n \left(\frac{\partial c_0}{\partial C}\right)_n \beta_{nm} + \frac{1}{N}\phi V^{(0)}_n \left(\sum_k \beta_{km}\left(\frac{\partial c_0}{\partial C}\right)_k\right),$$
$$B^{(1)}_{nm} = -V^{(0)}_n \delta_{nm},$$
$$B^{(2)}_{nm} = \phi V^{(0)}_n \left(\mathscr{P} - \left(\frac{\partial c_0}{\partial h}\right)_n\right)\delta_{nm} - \phi V^{(0)}_n \gamma_{nm}\left(\frac{\partial c_0}{\partial P}\right)_n - \frac{1}{N}\phi \mathscr{P} V^{(0)}_n + \frac{1}{N}\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial h}\right)_m$$
$$+ \frac{1}{N}\phi V^{(0)}_n \left(\sum_k \gamma_{km}\left(\frac{\partial c_0}{\partial P}\right)_k\right)$$

where by $\left(\frac{\partial c_0}{\partial z}\right)_n$, $z = h, x, C, P$, in the above we mean that the respective partial derivatives are calculated at the state $s^{(0)}$ and at site n. Assuming that the zeroth order state is spatially uniform, and assuming also that the interaction kernels have the property that $\sum_m \beta_{nm} = 0$, $\sum_m \gamma_{nm} = 0$ (diffusive coupling) the above expressions can simplify considerably to

$$A^{(1)}_{nm} = f'(x^{(0)}_n)\delta_{nm} + a_{nm}$$
$$A^{(2)}_{nm} = -h^{(0)}_n \delta_{nm}$$
$$A^{(3)}_{nm} = -\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial x}\right)\delta_{nm} + \frac{1}{N}\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial x}\right)$$
$$A^{(4)}_{nm} = \phi \left(\pi^{(0)}_n - \frac{1}{N}\sum_k \pi^{(0)}_k\right)\delta_{nm} - \phi V^{(0}_n \left(\frac{\partial c_0}{\partial C}\right)\beta_{nm},$$
$$B^{(1)}_{nm} = -V^{(0)}_n \delta_{nm},$$
$$B^{(2)}_{nm} = \phi V^{(0)}_n \left(\mathscr{P} - \left(\frac{\partial c_0}{\partial h}\right)_n\right)\delta_{nm} - \phi V^{(0)}_n \gamma_{nm}\left(\frac{\partial c_0}{\partial P}\right) - \frac{1}{N}\phi \mathscr{P} V^{(0)}_n + \frac{1}{N}\phi V^{(0)}_n \left(\frac{\partial c_0}{\partial h}\right).$$

We may now express the linearized system in compact form as the stochastic control system

$$dX = [\mathbb{A}X + \mathbb{B}u + \mathbb{S}v]dt + \mathbb{S}d\bar{w}, \tag{14}$$

where

$$\mathbb{A} := \begin{pmatrix} A^{(1)} & A^{(2)} \\ A^{(3)} & A^{(4)} \end{pmatrix},\ \mathbb{B} := \begin{pmatrix} B^{(1)} \\ B^{(2)} \end{pmatrix},\ \mathbb{S} := \begin{pmatrix} S \\ 0 \end{pmatrix},$$

where 0 is the $N \times N$ zero matrix and $X = (x^{(1)}, V^{(1)})^{tr}$, $u = h^{(1)}$, $v = v^{(1)}$. It is clear that $X \in \mathbb{R}^{2N\times 1}$, $u, v, w \in \mathbb{R}^{N\times 1}$, $\mathbb{A} \in \mathbb{R}^{2N\times 2N}$, $\mathbb{B}, \mathbb{S} \in \mathbb{R}^{2N\times N}$. It should be noted that matrices $\mathbb{A}$ and $\mathbb{B}$ incorporate stock, congestion and productivity externalities in the linearized dynamics

We now consider the problem of controlling the linearized system by proper choice of the control procedure u so that the system is kept as close as possible and at the minimum possible cost at the zeroth order desired steady state $s^{(0)}$. Ideally, we would like to choose $u = 0$ and keep $X = 0$ at all times, as this would correspond to keeping the system to the profit maximizing state $(x^{(0)}, V^{(0)}, h^{(0)})$. However, this is not possible and we choose the less ambitious task of minimizing the deviation of X from 0 at the minimum possible cost. This is equivalent to the robust optimal control problem[6]

$$\begin{gathered} \min_u \max_v \bar{J}(u, v), \\ \text{subject to (14)} \end{gathered} \tag{15}$$

where

$$\bar{J}(u, v) := \mathbb{E}\left[\frac{1}{2}\int_0^\infty e^{-rt}\left(X^{tr}(t)\mathsf{P}X(t) + u^{tr}(t)\mathsf{Q}u(t) - v^{tr}(t)\mathsf{R}v(t)\right)dt\right].$$

The solution to problem (15) guarantees that we get as close as possible to the desired state, at the worst possible deviation from our ideal model (11). The matrices $\mathsf{P} \in \mathbb{R}^{2N\times 2N}$, $\mathsf{Q} \in \mathbb{R}^{N\times N}$ and $\mathsf{R} \in \mathbb{R}^{N\times N}$ are positive definite and invertible and without loss of generality can be considered to be copies of the identity matrix, i.e.

$$\mathsf{P} = \begin{pmatrix} pI & 0 \\ 0 & \bar{p}I \end{pmatrix}, \ \mathsf{Q} = qI, \ \mathsf{R} = \theta I,$$

where I is the $N \times N$ identity matrix. For this particular case the objective functional becomes

$$\bar{J}(u, v) = E\left[\frac{1}{2}\int_0^\infty e^{-rt}\sum_n \left(p(x_n^{(1)}(t))^2 + \bar{p}(V_n^{(1)}(t))^2 + q(h_n^{(1)}(t))^2 + \theta(v_n^{(1)}(t))^2\right)dt\right]. \tag{16}$$

In objective (16) the coefficients $(p, \bar{p}, q, \theta)$ reflect the relative importance attache by the regulator to deviations from the optimal deterministic path, with r expressing the cost or being robust. Without loss of generality and to simplify the expressions we may choose $p = \bar{p}$. The parameter θ in (16) should be interpreted as the parameter associated with the global entropic constraint. If we are dealing with local entropic constraints matrix R should be defined as:

[6] It is obvious that upon setting $\hat{J} = -\bar{J}$ the $\min_u \max_v \bar{J}$ problem becomes equivalent to the $\max_u \min_v \hat{J}$ problem, which is in a form similar to the robust control problem (9), where now the profit functional is a replaced by the negative of a loss functional quantifying costs of deviation from a target.

$$\mathsf{R} = \begin{pmatrix} \theta_1 & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \theta_n \end{pmatrix}$$

This a more complicated case which can be dealt with methods appropriate for the solution of the general linear quadratic robust control problem presented in Brock et al. (2012). Because of its relative simplicity functional (16) allows us to use the Fourier space solution of the problem as we will see in the next section.

Note that this problem is different from the problem treated in Magill (1977a), Magill (1977b) where a linear quadratic approximation of a nonlinear stochastic optimal control problem is proposed. Here instead, we propose an exact linear quadratic procedure, which minimizes the tracking error from the optimal solution of a nonlinear idealized deterministic profit maximization problem. Our approach differs in spirit, however, correspond to a realistic situation. Most policy is designed upon ideal and simplified models (as for instance model (11)). It is important for the policy maker to have guidelines concerning the necessary corrections needed when the true state of the system deviates from the ideal state (as for instance under model (13)), so as to correct her policy in order to minimize deviations from the target. However, the generalization of Magill's procedure to a robust control problem is of interest in its own right, and will be treated elsewhere.

3 Robust stabilization of the benchmark solution

Problem (15) can now be treated using the Hamilton-Jacobi-Belman-Isaacs equation. This is expressed in terms of the generator $\mathscr{L}$ of the Ornstein-Uhlenbeck process (14), defined through its action on a twice continuously differentiable function $\mathscr{V} : \mathbb{R}^{2N} \to \mathbb{R}$

$$\mathscr{L}\mathscr{V} = (\mathbb{A}X + \mathbb{B}u + \mathbb{S}v) D_X \mathscr{V} + \frac{1}{2} \mathbb{S}\mathbb{S}^{tr} D_X^2 \mathscr{V}$$

where $D_X \mathscr{V}$ is the gradient of $\mathscr{V}$ with respect to the coordinates of the vector X and $D_X^2 \mathscr{V}$ is the Hessian matrix of the function $\mathscr{V}$ with respect to the coordinates of the vector X. The above are shorthands for the relevant expressions in coordinate form, e.g.,

$$D_X \mathscr{V} = \left(\frac{\partial \mathscr{V}}{\partial x_1^{(1)}}, \cdots, \frac{\partial \mathscr{V}}{\partial x_N^{(1)}}, \frac{\partial \mathscr{V}}{\partial V_1^{(1)}}, \cdots, \frac{\partial \mathscr{V}}{\partial V_N^{(1)}} \right)^{tr},$$

and similarly for the Hessian. Using the generator we may define the Hamiltonian function

$$H(\mathscr{V}; X, u, v) = \mathscr{L}\mathscr{V} + \langle \mathsf{P}X, X \rangle + \langle \mathsf{Q}u, u \rangle - \theta \, \langle \mathsf{R}v, v \rangle$$

where by $\langle \cdot, \cdot \rangle$ we denote the inner product in $\mathbb{R}^{2N}$. The value function $\mathscr{V}$ is the solution of the Hamilton-Jacobi-Belman-Isaacs (HJBI) equation

$$r\mathscr{V} + \min_u \max_v H(\mathscr{V}; X, u, v) = 0$$

(since by the saddle point theorem we may interchange the order of the $\min_u$ and $\max_v$ operations). The optimal policy is then related to the solution of the optimization problem for the Hamiltonian function. The HJBI equation is a fully nonlinear PDE, but on account of the linear quadratic nature of the system it can be solved in terms of the matrix Ricatti equation. Adapting the general results of Brock et al. (2012) to the model under consideration we find that the optimal correction policy is given by

$$u = -\mathsf{Q}^{-1}\mathbb{B}^{tr}\mathsf{H}X, \tag{17}$$

where $\mathsf{H} \in \mathbb{R}^{2N\times 2N}$ is the symmetric solution of the matrix Ricatti equation

$$\mathsf{H}\mathbb{A} + \mathbb{A}^{tr}\mathsf{H} - \mathsf{H}\mathsf{E}^s\mathsf{H} - r\mathsf{H} + \mathsf{P} = 0$$

and

$$\mathsf{E}^s = \frac{1}{2}(\mathsf{E} + \mathsf{E}^{tr}),$$
$$\mathsf{E} = \mathbb{B}\mathsf{Q}^{-1}\mathbb{B}^{tr} - \frac{1}{\theta}\mathbb{S}\mathsf{R}^{-1}\mathbb{S}^{tr}.$$

Once the matrix H computed, in principle numerically, the correction $u = h^{(1)}$ needed to modify the benchmark control procedure $h^{(0)}$ so as to keep the true state of the system as close as possible to the benchmark optimal state $(x^{(0)}, V^{(0)})$ is given by the feedback rule (17). This rule is very easy to apply as it only requires the manager to monitor the current value of the state $X(t)$, i.e. the current deviations $(x^{(1)}(t), V^{(1)}(t))$ of the true state of the system from the benchmark optimal state $(x^{(0)}, V^{(0)})$. We remark that our approach through the Ricatti equation does not necessarily require the benchmark state to be time independent not spatially homogeneous. However, even it the benchmark state enjoys both these properties, the deviations from this state, $X(t)$ will not necessarily satisfy them; it is in general expected to be both time varying and is expected to display spatial patterns. Furthermore, the optimal state X (i.e. the optimal deviations from the benchmark state once the optimal correction policy $u = h^{(1)}$ is adopted) is given by the solution of the Ornstein-Uhlenbeck equation

$$dX = \left(\mathbb{A} - \mathbb{B}\mathsf{Q}^{-1}\mathbb{B}^{tr}\mathsf{H} + \frac{1}{\theta}\mathbb{S}\mathsf{R}^{-1}\mathbb{S}^{tr}\mathsf{H}\right)X dt + \mathbb{S} dw.$$

The matrix Ricatti equation can be treated through a multitude of analytic or numerical methods leading to either interesting qualitative features of its solution, or to accurate computations, therefore the above analysis provides a general an computationally feasible approach to the problem of correcting the benchmark optimal strategy in order to lead the realistic system to the desired state. Here, in order to provide some qualitative results with the less possible technicalities involved, we treat the simple, yet realistic case where the operators related to the matrices $\mathbb{A}$, $\mathbb{B}$,$\mathbb{S}$

are translation invariant, the fishery is situated on a ring (i.e. periodic boundary conditions $x_1(t) = x_N(t)$ and $V_1(t) = V_N(t)$ for all t are imposed) and the loss functional related to the deviations of the system from the benchmark model is given in the form (16). We note that operators such as the discrete Laplacian often employed in models concerning the transport of biomass enjoy the translation invariant property. Furthermore, for this particular approach we have to assume that the benchmark state is spatially invariant, while the analysis is simplified considerably if it is also a steady state.

When all the above assumptions are satisfied, we may treat the robust control problem (15) with the choice of objective functional as in (16) by using the discrete Fourier transform. Importantly, the problem decouples[7] in Fourier space, a fact that allows us to obtain closed form solutions in terms of the Fourier transform of X.

For a vector $x = \{x_n\} = (x_1, \cdots, x_N)$ defined on the spatial domain $\mathbb{Z}_N$, we may define a vector $\hat{x} = \{\hat{x}_k\} = (\hat{x}_1, \cdots, \hat{x}_N)$, by

$$\hat{x}_k := \sum_{n=1}^{N} x_n \exp\left(-i2\pi k \frac{n-1}{N}\right), \; k \in \mathbb{Z}_N.$$

The k coordinates of the vector $\hat{x}$ are considered as taking values in a dual space, often called the Pontryagin dual space or simply Fourier space, which in this simple case coincides with $\mathbb{Z}_N$. The discrete Fourier transform of $X = (x, V)$ where x and V are vectors defined on the spatial domain $\mathbb{Z}_N$ is defined by $\hat{X} = (\hat{x}, \hat{V})$. The discrete Fourier transform has very interesting properties, one of which is very important in the simplification of problem (15) with the choice of objective functional as in (16). The Fourier transform turns a convolution to a product, in the sense that the Fourier transform of $\mathsf{A}x$ is equal to $\hat{\mathsf{A}}\hat{x}$ as long as A is translation invariant, i.e. commutes for all m with the translation operators T_m defined by $(T_m x)_n = x_{n-m}$ where of course periodicity is taken into account. Matrices such as those corresponding to the discrete Laplacian have this form. This property leads to a decoupled set on equations for the state variables, where treated in Fourier space. Furthermore, by the special form of the objective functional (16), the use of the Parceval identity allows us to rewrite the objective functional in essentially identical form but now interpreted in Fourier space. This leads to a decoupling of the full problem into N scalar problems which are amenable to full analytic consideration.

Denoting by $\hat{X}$ the Fourier transform of X it can be shown (see Brock et al. (2012)) that the optimal state is the solution of the Ornstein-Uhlenbeck equation

$$d\hat{X}_k = R_k \hat{X}_k + \hat{\sigma}_k dw_k, \; k \in \mathbb{Z}_N$$

where $\hat{\sigma}_k$ is a constant whose exact expression is not needed for what follows,

$$R_k := \hat{a}_k - \frac{\hat{b}_k^2 M_{2,k}}{2q} + \frac{\hat{c}_k^2 M_{2,k}}{2\theta}.$$

[7] Essentially turning the matrix Ricatti equation to a set of scalar, uncoupled Ricatti equations, amenable to analytic solution.

and $M_{2,k}$ is the solution of

$$\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right) M_{2,k}^2 + (2\,\hat{a}_k - r)\,M_{2,k} + 2p = 0. \tag{18}$$

The terms $\hat{a}_k$, $\hat{b}_k$ and $\hat{c}_k$ are related to the Fourier transform of the matrices $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{S}$. Furthermore, the optimal controls are given by the feedback laws

$$\hat{u}_k = -\frac{\hat{b}_k M_{2,k}}{2q}\hat{x}_k, \qquad \hat{v}_k = \frac{\hat{c}_k M_{2,k}}{2\theta}\hat{x}_k.$$

4 Hot spot formation

In this section we study the validity and the qualitative behavior of the controlled system (15). We will call the qualitative changes of the behavior of the system **hot spots**. In the present context, hot spots will correspond to important deviations of the stabilization procedure presented in the previous section, that will have as consequence important quantitative and qualitative deviations of the true controlled system from the desired ideal benchmark model, no matter what the decision maker does in order to correct her policy by proper adjustment procedures. We may thus consider hot spots as possible important failures of the adjustment procedure, which may have important consequences on the true state of the controlled fishery.

We will define three types of hot spots:

- **Hot spot of type I:** This is a breakdown of the solution procedure, i.e., a set of parameters where a solution to the above problem does not exist.
- **Hot spot of type II:** This corresponds to the case where the solution exists but may lead to spatial pattern formation, i.e., to spatial instability similar to the Turing instability.
- **Hot spot of type III:** This corresponds to the case where the cost of robustness becomes more that what is offering us, i.e., where the relative cost of robustness may become very large.

In what follows, for simplicity, we discuss the formation of hot spots under the assumption that the tracking problem (15) with the choice of objective functional as in (16) is translation invariant (which requires certain symmetry conditions). The results are stated in terms of a number of propositions, providing relevant parameter values for the formation of the different types of hotspots, the proofs of which may be found in Brock et al. (2012). However, similar results hold for the general case of non-translation invariant systems, by the full treatment of the matrix Ricatti equation (see Brock et al. (2012)).

4.1 Hot spots of type I

The breakdown of the solution procedure can be seen quite easily by the following simple argument. The value function assumes a simple quadratic form, as long as the algebraic quadratic equation

$$\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right) M_{2,k}^2 + (2\,\hat{a}_k - r)\,M_{2,k} + 2p = 0. \tag{19}$$

admits real valued solutions, at least one of which is positive. The positivity of the real root is needed since, by general considerations in optimal control, the value function must be convex. If the above algebraic quadratic equation does not admit at least one positive real valued solution this is an indication of breakdown of the existence of a solution to the robust control problem which will be called a hot spot of Type I.

Proposition 1 (Type I hot spot creation:). *Hot spots of Type I may be created in one of the following two cases:*

(I_A)Either,

$$(2\hat{a}_k - r)^2 < 8\,p\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right), \tag{20}$$

(I_B)Or,

$$(2\hat{a}_k - r)^2 > 8\,p\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right),\ \left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right) > 0,\ 2\hat{a}_k - r > 0. \tag{21}$$

Hot spots of this type may arise either due to low values of θ, or due to high values of q or low values of r. For example, they may arise either if

$$\theta < \frac{p\hat{c}_k^2}{\left(\hat{a}_k - \frac{r}{2}\right)^2 + \frac{p}{q}\hat{b}_k^2},\ \ k \in \mathbb{Z}_N.$$

or if

$$\theta > \frac{p\hat{c}_k^2}{\left(\hat{a}_k - \frac{r}{2}\right)^2 + \frac{p}{q}\hat{b}_k^2},\ \ \frac{q}{\theta} > \frac{\hat{b}_k^2}{\hat{c}_k^2},\ \ r < 2\hat{a}_k,\ \ k \in \mathbb{Z}_N.$$

In particular hot spots are expected to occur in the limit as $\theta \to 0$ while they are not expected to occur in the limit as $\theta \to \infty$.

As mentioned above, a hot spot of Type I represents breakdown of the solvability of the optimal control problem. We argue that this represents some sort of loss of convexity of the problem thus leading to non existence of solution. To illustrate this point more clearly let us take the limit as $\theta \to 0$ which corresponds to hot spot formation. For such values of θ, the particular ansatz employed for the solution breaks down and in fact as $\theta \to 0$ we expect $M_{2,k} \to 0$ so that the quadratic term in the value function will disappear. This leads to loss of strict concavity of the functional, which may be seen as follows: The functional contains a contribution from $\hat{v}_k$ through the dependence of $\hat{x}_k$ on $\hat{v}_k$ which contributes a quadratic term of positive sign in $\hat{v}_k$. The robustness term, which is proportional to $-\theta$ contributes a quadratic term of negative sign in $\hat{v}_k$. For large enough values of θ the latter term dominates in the functional and guarantees the strict concavity, therefore, leading to a well defined maximization problem. In the limit of small θ the former term dominates and thus turn the functional into a convex functional leading to problems with respect to the maximization problem over $\{\hat{v}_k\}$. We call this breakdown of concavity in v, which lead to loss of convexity of the value function in x, for small values of θ a hot spot of type I. When this happens, there is a duality gap, since the assumptions of the min-max theorem do not hold. In terms or regulatory objectives this means that concerns about model misspecification make regulation impossible.

The effect of the parameters of the fishery model employed on the formation of hot spots, can quantified by the results of Proposition 1 through the dependence of the Fourier transformed operators $\mathbb{A}$,$\mathbb{B}$, $\mathbb{S}$ on the model parameters. For instance, if prices $\mathscr{P}$ increase, whereas the rest of the parameters remain fixed, then $\hat{b}_k$ will increase with respect to the other parameters $\hat{a}_k$ and $\hat{c}_k$. This will result to a decrease of the right hand side of e.g., equation (20) thus leading to a suppression of such a hot spot. Due to the large number of parameters of the model, extreme care should be taken when interpreting qualitatively the above conditions. However, having chosen a particular model and having estimated some of the parameters, the decision maker may investigate numerically the above analytic conditions and provide parameter regimes for creation or suppression of the various type of hot spots. Since our major interest here is the formulation of a general methodology, rather than a detailed treatment of a particular model, we provide two simplified examples that allow us to provide a qualitative understanding of hot spot formation as a result of the various interacting "forces" that influence the system and comment upon their relative importance. These examples are porivided here, for lack of space, to hot spots of type I only, but they can be extended to the study of the other hot spots as well.

The following examples show some interesting limiting situations, in terms of simplifications of the operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{S}$:

Example 1. Assume that $\mathbb{A}$ is the discrete Laplacian whereas $\mathbb{B}$ and $\mathbb{S}$ are copies of the identity operator. This corresponds to the case that there is diffusive coupling in the state equation but controls as well as the uncertainty have purely localized effects. A quick calculation shows that in this case $a_k = \alpha\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)$ where α is the diffusion coefficient whereas $b_k = \beta$ and $c_k = \gamma$ for every $k \in \mathbb{Z}_N$ where β and γ is a measure for the control and the uncertainty respectively. In this particular case, the quadratic equation becomes

$$\left(\frac{\gamma^2}{2\theta}-\frac{\beta^2}{2q}\right)M_{2,k}^2+\left(2\,\alpha\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)-r\right)M_{2,k}+2p=0.$$

which must have a real valued solution for every k. There will not exist real valued solutions if

$$\Delta:=\left(2\,\alpha\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)-r\right)^2-8\,p\left(\frac{\gamma^2}{2\theta}-\frac{\beta^2}{2q}\right)<0$$

or equivalently after some algebra

$$\left(\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)^2-\frac{r}{2\alpha}\right)<\frac{p}{\alpha^2}\left(\frac{\gamma^2}{\theta}-\frac{\beta^2}{q}\right).$$

This is the condition for generation of a hot spot of Type I in this particular example. If this condition holds for some $k\in\mathbb{Z}_N$, this particular k is a candidate for such a hot spot. We may spot directly that this cannot hold for any $k\in\mathbb{Z}_N$ if the right hand side of this inequality is negative, i.e., when $\theta>\theta_{cr}:=q\frac{\gamma^2}{\beta^2}$, therefore hot spots of this type will never occur for large enough values of θ. The critical value of θ for the formation of such hot spots will depend on the relative magnitude of uncertainty over control . For $\theta<\theta_{cr}$ then a hot spot of Type I may occur for the modes k such that

$$\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)^2\leqslant\frac{r}{2\alpha}+\rho$$

or equivalently for k such that

$$\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)^2\leqslant\left(\frac{r}{2\alpha}+\rho\right)^{\frac{1}{2}}$$

where $\rho^2=\frac{p}{\alpha^2}\left(\frac{\gamma^2}{\theta}-\frac{\beta^2}{q}\right)$.

Example 2. The opposite case is when $\mathbb{A}$ is again the discrete Laplacian while $\mathbb{B}$ and $\mathbb{S}$ are multiples of matrices containing 1 in the diagonal and the same entry ν in every other position. This means that the controls as well as the uncertainty has a globalized effect to all lattice points, in the sense that the controls even at remote lattice sites have an effect at each lattice point. Then $\hat{b}_k=\beta\,\delta_{k,0}$, $\hat{c}_k=\gamma\delta_{k,0}$, i.e., the Fourier transform is fully localized and is a delta function. Then, for $k=0$ the quadratic equation becomes

$$\left(\frac{\gamma^2}{2\theta}-\frac{\beta^2}{2q}\right)M_{2,0}^2-(6\alpha-r)M_{2,0}+2p=0$$

while for $k\neq0$ the quadratic term vanishes yielding

$$-\left(2\alpha\left(1+2\cos\left(\frac{2\pi k}{N}\right)\right)-r\right)M_{2,0}+2p=0$$

4.2 Hot spots of type II

We now consider the spatial behavior of the optimal path, as given by the Itō stochastic differential equation

$$d\hat{x}^*_k = R_k \hat{x}^*_k dt + \hat{c}_k d\hat{w}_k$$

The optimal path is a random field, thus leading to random patterns in space, some of which may be short lived and generated simply by the fluctuations of the Wiener process. We thus look for the spatial behavior of the mean field as describable by the expectation $\hat{X}_k := \mathbb{E}_Q[\hat{x}^*_k]$. By standard linear theory $\hat{X}_k(t) = \hat{X}_k(0)\exp(R_k t)$ and this means that for the modes $k \in \mathbb{Z}_N$ such that $R_k \geqslant 0$ we have temporal growth and these modes will dominate the long term temporal behavior. On the contrary modes k such that $R_k < 0$ decay as $t \to \infty$ therefore such modes correspond to (short term) transient temporal behavior, not likely to be observable in the long term temporal behavior. The above discussion implies that the long time asymptotic of the solution in Fourier space will be given by

$$\hat{X}_k(t) \simeq \begin{cases} \hat{x}_k(0)\,\exp(R_k t),\, k \in \mathscr{P} := \{k \in \mathbb{Z}_N \,:\, R_k \geqslant 0\} \\ 0 \qquad\qquad\qquad \text{otherwise} \end{cases}$$

To see what this pattern will look like in real space, we simply need to invert the Fourier transform, thus obtaining a spatial pattern of the form

$$X_n(t) := \mathbb{E}_Q[x_n(t)] = \sum_{k\in\mathscr{P}} \hat{x}_k(0)\,\exp(R_k t)\,\cos\left(2\pi\frac{k}{N}n\right). \qquad (22)$$

The above discussion therefore leads us to a very important conclusion, which is of importance to economic theory of spatially interconnected systems:

> If as an effect of the robust optimal control procedure exerted on the system there exist modes $k \in \mathbb{Z}_N$ such that $R_k > 0$, then this will lead to spatial pattern formation which will create spatial patterns of the form (22). As we will see there are cases what such patterns will not exist in the uncontrolled system and will appear as an effect of the control procedure. We will call such patterns an **optimal robustness induced spatial instability** or hot spot of Type II.

The economic significance of this result should be stressed. We show the emergence of a spatial pattern formation instability, which can be triggered by the optimal control procedures exerted on the system; in other words emergence of spatial clustering and agglomerations in the fishery (as observed in the spatial distribution of the biomass and the number of vessels) caused by uncertainty aversion and robust control. This observation can further be extended in the case of nonlinear dynamics, in the weakly nonlinear case. When the dynamics are nonlinear in the state the emergence of hot spots of Type II and optimal robustness induced spatial instability should be linked to the spatial instability of a spatially uniform steady state corresponding to the linear quadratic approximation of a nonlinear system. This instability which can be thought as pattern formation precursor will induce the emergence of spatial clustering. As time progresses and the linearized solution (22) grows beyond a certain critical value (in terms of a relevant norm) then the deviation from the homogeneous steady state is so large that the linearized dynamics are no longer a valid approximation. Then the nonlinear dynamics will take over and as an effect of that some of the exponentially growing modes could be balanced thus leading to more complicated stable patterns. At any rate even in the nonlinear case the mechanism described here will be a Turing type pattern formation mechanism explaining the onset of spatial patterns in the fishery.

The next proposition identifies which modes can lead to hot spot of Type II formation (optimal robustness induced spatial instability) and in this way through equation (22) identifies possible spatial patterns that can emerge in the fishery.

Proposition 2 (Pattern formation for the primal problem). *There exist pattern formation behavior for the primal problem if there exist modes k such that $R_k > 0$, i.e., if there exist modes k such that*

$$\frac{1}{2}\left(r-\sqrt{r^2+8\,p\left(\frac{\hat{c}_k^2}{2\theta}-\frac{\hat{b}_k^2}{2q}\right)}\right)\leqslant \hat{a}_k \leqslant \frac{1}{2}\left(r+\sqrt{r^2+8\,p\left(\frac{\hat{c}_k^2}{2\theta}-\frac{\hat{b}_k^2}{2q}\right)}\right),$$
$$r^2+8\,p\left(\frac{\hat{c}_k^2}{2\theta}-\frac{\hat{b}_k^2}{2q}\right)\geqslant 0. \tag{23}$$

It is interesting to see what is the behavior of the system as a function of parameters with respect to pattern formation and the qualitative behavior of the optimal path.

Note that this pattern formation behavior is in full accordance with the fact that our state equation is the optimal path for the linear quadratic control problem. Since it solves this problem it is guaranteed that $I := \mathbb{E}_Q[\int_0^\infty e^{-rt}\hat{x}_k^2(t)dt]$ is finite[8] therefore $\hat{x}_k(t)$ can at most grow as $e^{\frac{r}{2}t}$, otherwise the quantity I would be infinite. This is verified explicitly by the observation that $R_k \leqslant \frac{r}{2}$ for every $k \in \mathbb{Z}_N$. Therefore, all possible patterns may at most exhibit growth rates less or equal to $r/2$. In the limit as $r \to 0$ i.e. in the limit of small discount rates pattern formation is becoming increasingly difficult in the linear quadratic model since growing patterns will be suppressed by the control procedures.

Proposition 3 (Stabilizing or destabilizing effects of control). *The robust control procedure may either have a stabilizing or destabilizing effect with respect to pattern formation. in the sense that it may either stabilize an unstable mode of the uncontrolled system or on the contrary facilitate the onset of instabilities.*

[8] This is in fact equivalent to the assertion that the optimal path satisfies temporal transversality conditions at infinity.

In particular,

(i) If $\frac{q}{\theta} < \frac{\hat{b}_k^2}{\hat{c}_k^2}$ *then the robust control procedure has a stabilizing effect*

(ii)If $\frac{q}{\theta} > \frac{\hat{b}_k^2}{\hat{c}_k^2}$ *then the robust control procedure has a destabilizing effect*

Case (ii) suggests robust control caused pattern formation, in the sense that we obtain a growing mode leading to a pattern which would not have appeared in the uncontrolled system.

As seen by Proposition 3 in the $\theta \to \infty$ limit, the control has a stabilizing effect on unstable modes of the uncontrolled system. Similarly, by Proposition 3 in the $\theta \to 0$ limit, the robust control has a destabilizing effect on modes of the uncontrolled system which are "marginal" to be stable i.e. with $\hat{\alpha}_k$ negative but close to zero.

In closing this discussion we wish to ponder upon some similarities and differences of Type II hot spots with the occurence of the celebrated Turing instability; Formation of hot spots of type II is similar to Turing instability leading to pattern formation but with a very important difference! In contrast to Turing instability which is observed in an uncontrolled forward Cauchy problem, this instability is created in an optimally controlled problem in the infinite horizon. This has important consequences and repercussions both from the conceptual as well as from the practical point of view. On the conceptual level, a controlled system is related to a system that somehow its final state (at $t \to \infty$ in our case) is predescribed. Therefore, our result is an "extension" of Turing instability in a forward-backward system and not just to a forward Cauchy problem, as is the case for the Turing instability. On the practical point of view, the optimal control nature of the problem we study here induces serious constraints on the growth rate of the allowed patterns which has a strict upper bound is related only to the discount factor of the model and not on the operator A. This is not the case for the standard Turing pattern formation mechanism, in which the growth rate upper bound is simply related to the spectrum of the operator A.

4.3 Hot spots of type III: The cost of robustness

The value function is of the form $V_k = \frac{M_{2,k}}{2}\hat{x}_k^2 + \frac{\hat{c}_k^2 M_{2,k}}{2r}$. This gives us the total cost of the minimum possible deviation from the desired goal and it is made up from contributions by three terms:

- the term proportional to p in the cost functional which corresponds to the cost related to the deviation from the desired target,
- the term proportional to q in the cost functional which corresponds to the cost related to the cost of the control u needed to drive the system to the desired target and
- the term proportional to θ in the cost functional which corresponds to the cost of robustness (which is the cost incurred by the regulator because she wants to be robust when she has concerns about the misspecification of the model).

The value functions depends on all these three contributions and this may be clearly seen since $M_{2,k}$ is in fact a function of the parameters p, q, θ.

An interesting question is which is the relevant importance of each of these contributions in the overall value function. Does one term dominates over the others or not?

A simple answer to this question will be given by the elasticity of the value function with respect to these parameters, i.e., by the calculation of the quantities $\frac{1}{V}\frac{\partial V}{\partial p}$, $\frac{1}{V}\frac{\partial V}{\partial q}$ and $\frac{1}{V}\frac{\partial V}{\partial \theta}$. It is easily seen that these elasticities are independent of $\hat{x}_k$ and reduce to $\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial p}$, $\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial q}$ and $\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta}$, respectively. Whenever one of these quantities tends to infinity, that means that the contribution of the relevant procedure dominates the control problem[9]

In particular whenever $\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta} \to \infty$, then we say that the cost of robustness becomes more expensive than what it offers, and we will call that a hot spot of type III. This quantity can be calculated directly from the solution of the quadratic equation (18) through straightforward but tedious algebraic manipulations, which we choose not to reproduce here.

However, an illustrative partial case, which allows some insight on the nature of hot spots of type III is the following:

Differentiating (18) with respect to θ yields

$$-\frac{\hat{c}_k^2}{2\theta^2}M_{2,k}^2 + 2\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right)M_{2,k}\frac{\partial M_{2,k}}{\partial \theta} + (2\,\hat{a}_k - r)\frac{\partial M_{2,k}}{\partial \theta} = 0.$$

Dividing by $M_{2,k}^2$ we obtain

$$-\frac{\hat{c}_k^2}{2\theta^2} + 2\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right)\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta} + (2\,\hat{a}_k - r)\frac{1}{M_{2,k}^2}\frac{\partial M_{2,k}}{\partial \theta} = 0.$$

Let us now take the particular case where $2\hat{a}_k = r$, so that

$$\frac{1}{M_{2,k}}\frac{\partial M_{2,k}}{\partial \theta} = \frac{\hat{c}_k^2}{4\theta\left(\frac{\hat{c}_k^2}{2\theta} - \frac{\hat{b}_k^2}{2q}\right)}$$

which becomes infinite for values of θ such that $\theta \to \frac{q\hat{c}_k^2}{\hat{b}_k^2}$. The general case $2\hat{a}_k \neq r$ may present similar phenomena.

[9] This interpretation arises from observation that close to a point (p_0, q_0, θ_0) the value function behaves as

$$V_k \simeq \left.\frac{\partial V_k}{\partial p}\right|_{p=p_0}(p-p_0) + \left.\frac{\partial V_k}{\partial q}\right|_{q=q_0}(q-q_0) + \left.\frac{\partial V_k}{\partial \theta}\right|_{\theta=\theta_0}(\theta-\theta_0).$$

5 Concluding Remarks

In this paper we studied the optimal management of a commercial fishery which is distributed over a finite spatial domain, is characterized by stock, congestion and productivity externalities, and the fishery manager has concerns about model misspecification.

We solve this problem as a robust control linear quadratic distributed parameter model. The linear quadratic approximation is formulated as a tracking problem where stochastic dynamics indicating model uncertainty are linearized around a deterministic optimal path, and the control process aims at keeping the system close to the optimal path. Harvesting rules are obtained as robust tracking rules which can be used by the manager to set policy such as quotas on each site of the spatial domain.

An important result of our paper is the identification of spatial hot spots, which are sites of special interest emerging from the interactions between concerns about model uncertainty, spatial interactions and the structure of the fishery. In such hot spots optimal robust regulation may be impossible and the inability to regulate is extended to the whole domain (type I hot spot); regulation may lead to spatial non-homogeneity in the harvesting rules, implying spatially differentiated quotas (type II hot spots); or misspecification concerns may lead to very costly regulation, indicating excessive cost of robust regulation (type III hot spot).

These results although qualitative in nature provide insights to regulation of a commercial fishery under model uncertainty and under explicit spatial interactions. Future research may include solution of a linear quadratic approximation in the sense of Magill approximations, instead of solution of the optimal tracking problem, or attempts to characterize the solution of the full nonlinear problem.

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