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Statistical Properties of Two Asymmetric Stochastic Volatility in Mean Models

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Here we investigate the statistical properties of two normal asymmetric SV models with possibly time varying risk premia. In fact, we investigate two popular autoregressive stochastic volatility specifications. These, although they seem very similar, it turns out, that they possess quite different statistical properties. The derived properties can be employed to develop tests or to check stationarity of various orders, something important for the asymptotic properties of various estimators.

1 Introduction

In economic and financial data there are some well documented statistical facts, the so-called stylized facts. The most important of these is perhaps the volatility clustering, i.e. that, on average, periods of high (low) volatility are followed period with high (low) volatility. However, volatility clustering is also observed in data from physics and specifically from turbulence, called indeterminacy in turbulence terminology (see Barndorff-Nielsen 1997 [9])[1]. This has led to modifications and extensions of the original ARCH (Autoregressive Conditional Heteroskedasticity) model of Engle (1982) [24] and its generalization by Bollerslev (1986) [13], so that now there is a plethora of dynamic heteroskedasticity models (see e.g. Bollerslev, Chou, and Kroner,

[1]For a comparison between the stylized facts of the temporal behavior of asset returns, and differences in velocity of the mean wind direction of a large Reynolds number wind field see Barndorff-Nielsen and Shephard (2001) [10], Mantegna and Stanley (1996) [56], and Mantegna and Stanley (2000) [57].

1992 [?], Bera and Higgins, 1993 [11], Bollerslev, Engle and Nelson 1994 [14], and Francq and Zakoian, 2010 [31] for a book).

Furthermore, economic theory, and specifically financial theory, often postulates specific relationships between first and second conditional moments. For instance, in the stock market context, the first conditional moment of stock market excess returns, say μ_t, is a function of volatility, say σ_t^2, (see e.g. Merton, 1980 [60], and Glosten, Jagannathan and Runkle, 1993 [36]). In fact, rational risk averse investor require higher expected returns during high volatility periods, implying a positive relationship between expected returns and volatility, something which is supported by e.g. French et al. (1987) [32] and Campbell and Hentschel (1992) [16] and Poon and Taylor (1992) [65]. Consequently, Engle, Lilien and Robins (1987) [28] introduced the so called ARCH in Mean model (ARCH-M), which was a first attempt to capture this relationship.

Glosten, Jagannathan and Runkle (1993) [36] and Nelson (1991) [63], among others, give support to a negative relationship between unexpected part of returns volatility. French et al. (1987) [32] interpret it as indirect evidence of a positive correlation between the expected risk premium and ex ante volatility. They suggest that unanticipated large shocks to the return process induce higher expected volatility. If expected volatility and returns are positively related, the current stock price should fall. This is known as the volatility feedback theory (see e.g. Campbell and Hentschel (1992) [16]).

Finally, it has been observed that volatility is higher after the stock market has a fall than after a rise of the same size, meaning that stock returns are negatively correlated with future volatility. This phenomenon was first discussed by Black (1976) [12], who suggested that it could be due to the increase in leverage that occurs when the market of a firm falls. However, it seems that the leverage effect is too small to completely explain this asymmetric response of volatility (see e.g. Christie (1982) [18], Figlewski and Wang (2000) [29], and Schwert (1989) [68], Hasanhodzic and Lo [42], and Bollerslev, T., N. Sizova, and G. Tauchen (2012) [15]). This effect can be accommodated within asymmetric GARCH setup such as the Exponential GARCH of Nelson (1991) [63], the Quadratic GARCH of Sentana (1995) [69] or the model of Glosten, Jagannathan and Runkle (1993) [36]. Hence, and especially in the area of empirical finance, a literature field emerged, where researchers tried to quantify and estimate these relationships using mainly the GARCH-M specification, either symmetric or asymmetric, especially due to its inference tractability (see among others Gonzales-Rivera

1996 [37], Choudhry 1996 [17], Dunne 1999 [23], Arvanitis and Demos 2004 [6] and 2004a [7]).

Statistical properties of various GARCH type models can be found in Milhoj (1985) [62], He and Terasvirta (1999) [43], Karanasos (1999) [50], Rodriguez and Ruiz (2012) [66], Demos (2002) [21], He, Terasvirta and Malmsten (2002) [45], Karanasos and Kim (2003) [51], Tsiotas (2007) [73], etc, whereas for the GARCH-M type models can be found in Anyfantaki and Demos (2011)[4] and (2016) [5], Arvanitis and Demos (2004a) [6] and (2004b) [7]), He and Terasvirta [43]), etc.

All the above mentioned conditional heteroskedastic models are characterized by the fact that the mean error "moves" the next period conditional variance. However, there is another class of conditionally heteroskedastic processes where a second error processes, possibly correlated with the mean error, "drives" the conditional variance, the so-called stochastic variance processes (see e.g. Andersen 1996 [1]). The most popular of the stochastic variance models defines volatility as a logarithmic first-order autoregressive, known as the first-order autoregressive Stochastic Volatility (SV(1)) model. Even thought SV models are considered as competitive alternatives to GARCH ones their application has been limited. One of the reasons is that in the SV setting volatility is not measurable with respect to observable past information. Hence, volatility estimation involves not only filtering but smoothing techniques, as well, making the estimation of the parameters cumbersome (see e.g. Andersen and Benzoni 2009 [2])[2].

Here we investigate the statistical properties of normal asymmetric SV models with possibly time varying risk premia, i.e. the standard deviation could appear as an explanatory variable in the mean equation (SV-M) and the standardized errors are normally distributed. In such a way we model simultaneously the first two moments of the observed process, with errors that can be correlated. In fact, we investigate two popular autoregressive

[2] Classical parameter estimation can be found in e.g., Taylor 1986 [71], and Harvey, Ruiz and Shephard 1994 [40]. Bayesian estimation can be found in e.g., Jacquier et al. (1994) [48] and (2004) [49], Kim, Shephard and Chib (1998) [53], Sandmann and Koopman (1998) [67], Fridman and Harris (1998) [33], and Koopman and Uspensky (2002) [55]. Indirect Inference methods can be found in Gallant and Tauchen (1996) [34], the (Smith (1993) [70] and Gourieroux, Monfort and Renault (1993) [38]), the spectral method of moments in Knight, Satchell (2007) [54], and Yu (2002), the simulated method of moments Duffie and Singleton (1993) [22]and the generalized method of moments Melino and Turnbull (1990) [59], Andersen and Sorensen (1996) [3].

stochastic volatility specifications. These, although they seem very similar, it turns out, that they possess quite different statistical properties. The derived properties can be employed to develop tests, similar to those developed in Horvath, Kokoszka and Zitikis (2006) [47], or to check stationarity of various orders, which are important for the asymptotic properties of various estimators.

Statistical properties of SV models can be found in e.g., Taylor (1994) [72], Harvey and Shephard 1996 [41], Jacquier, Polson and Rossi (1994) [48], Danielsson, J. (1998) [19], Harvey, Ruiz and Shephard (1994) [40] etc. For properties of asymmetric SV models see e.g., Tsiotas, G. (2012) [74], etc., and of long memory SV models see Eric Ghysels, Harvey and Renault[35], Harvey (2007) [39], Perez and Ruiz (2003) [64], etc. Furthermore, Mao, Czellar, Ruiz and Veiga (2020) [58] present the statistical properties of a Generalized Asymmetric SV-M model. The asymmetry in the aforementioned model appears due to the fact that a transformation of the mean error is present in the stochastic conditional variance equation. Here we adopt the classical method of introducing asymmetries by assuming a correlation between the mean and the variance error, as in Jacquier, Polson and Rossi (2004) [49] (JPR hereafter).

In the nest section we present the two SV models and derive their static and dynamic moments. In the final section we compare the properties of the two models and conclude.

2 The Two SVM Models

The normal Autoregressive Stochastic Volatility in Mean models are given by:

$$y_t = c_t + \varepsilon_t^* = c + \lambda\sigma_t + \varepsilon_t\sigma_t \quad where, \tag{2.1}$$

$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_{t-1}\ (SV1)\ and \tag{2.2}$$

$$\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \overset{iid}{\sim} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho\sigma_\eta \\ \rho\sigma_\eta & \sigma_\eta^2 \end{pmatrix}\right).$$

The above model, with $c = \lambda = 0$, has been estimated by quasi maximum likelihood in Harvey and Shephard (1996) [41] and by MCMC in Meyer and Yu (2000) [61] Asai and McAleer (2011) [8] present some properties of the model. A similar model, with $\rho = 0$, has been estimated in Koopman and Uspensky (2002) [55] by simulated maximum likelihood, but they add an autoregressive term in the mean and the employ the conditional variance instead of the standard deviation. Further, with $c = \lambda = 0$ but with non-normal error distribution, in Jacquier, Polson and Ross (2004) [49] by MCMC. However, there is an important difference between the models considered in Jacquier et al. (1994)[48], (2004)[49] (JPR) and Koopman and Uspensky (2002) [55] and the one considered here. Specifically, instead of the above conditional variance specification 2.2 they employ the following one:

$$\ln \sigma_t^2 = \omega + \psi \ln \sigma_{t-1}^2 + \eta_t \ (SV2). \tag{2.3}$$

However, Yu (2002) [76] proved that the partial derivative of future volatility with respect to the error is not necessarily negative when $\rho < 0$, i.e. it could be the case that even if $\rho < 0$ future volatility could decrease with a negative error, claiming the the variance specification in 2.2 is a more "natural" one (see details in Yu 2002 [76]). Let us now explore the properties of these models.

2.1 Properties of the SV1-M Model

First, it is worth noticing that the SV-M model has a relative advantage as compared to GARCH-M type of models. Let us denote by $\sigma_{t|t-1}$ the conditional expectation of the standard deviation on the $\sigma - field$ generated by the observed variables y_t up to time $t-1$, i.e.

$$\sigma_{t|t-1} = E\left(\sigma_t | \sigma\left\{y_{t-1}, y_{t-2}, \ldots, y_1\right\}\right).$$

Then adding and subtracting $\lambda\sigma_{t|t-1}$ to equation we get:

$$y_t = c + \lambda\sigma_{t|t-1} + \lambda\left(\sigma_t - \sigma_{t|t-1}\right) + \varepsilon_t\sigma_t,$$

i.e. the first two terms on the right-hand side of the equation represents the risk premium, implying a positive λ, whereas the third terms represents the volatility feed-back term implying a negative λ (see Koopman and Uspensky (2002) [55], as well). On the other hand, for any GARCH-type specification $\sigma_t = \sigma_{t|t-1}$ and the two effects can not be separated. Of course, one could add to a GARCH-M model an extra term representing the volatility surprise, as in Campbell and Hentschel (1992) [16], i.e. add $\left(\varepsilon_t^2 - 1\right)\sigma_t^2$ a martingale sequence for most GARCH-type models (see Wu 2001 [75], as well). However, in our case $\sigma_{t|t-1}$ is far more complicated.

Let us now investigate the statistical static and dynamic properties of the model and compare them with those of the model employed by JPR and Koopman and Uspensky (2002) [55].

2.1.1 Static Properties

First, it is easy to prove that

$$V\left(\varepsilon_t^*\right) = E\left(\varepsilon_t^2\sigma_t^2\right) = E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

and

$$\kappa_{\varepsilon_t^*} = \frac{3E\left(\sigma_t^4\right)}{\left(V\left(\varepsilon_t\sigma_t\right)\right)^2} = 3\exp\left(\frac{\sigma_\eta^2}{1-\psi^2}\right) > 3.$$

Notice that the kurtosis coefficient of ε_t^* is bigger than 3, i.e. the stochastic volatility increases the kurtosis of the errors, a well known fact of the SV models.

Further

$$E\left(y_t\right) = c + \lambda\exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],$$

$$V\left(y_t\right) = \left\{\left(\lambda^2+1\right)\left(\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]\right) - \lambda^2\right\}\exp\left[\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right],$$

and

$$\frac{V\left(y_t\right)}{V\left(\varepsilon_t^*\right)} = 1 + \lambda^2\left\{1 - \left[-\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]\right\} > 1,$$

i.e. the price of risk parameter λ increases the variance of the observed process independent of the sign of the parameter.

The skewness of y_t is given by

$$sk\left(y_t\right) = \lambda\frac{\lambda^2\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + 3\lambda + 3\left\{\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}{\left\{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]\right\}^{3/2}}$$

In Appendix A is it proven that the skewness coefficient $sk\left(y_t\right)$ can be of either sign.

Now the kurtosis coefficient of y_t is:

$$\kappa\left(y_t\right)=\frac{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{6\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(4\lambda^4+12\lambda^2\right)\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right]}{\left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\lambda^2\right\}^2}$$
$$+3\lambda^2\frac{2\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\lambda^2}{\left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\lambda^2\right\}^2}$$

2.1.2 Dynamic Properties

For the dynamic properties of y_t we get that (see Appendix A):

$$Corr\left(y_t,y_{t-k}\right)=\lambda\frac{\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4(1-\psi^2)}\right]-1\right\}+\frac{1}{2}\rho\phi_\eta\psi^{k-1}\exp\left[\frac{\sigma_\eta^2\psi^k}{4(1-\psi^2)}\right]}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}$$

Notice that when

$$\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4\left(1-\psi^2\right)}\right]-1\right\}>\frac{1}{2}\rho\phi_\eta\psi^{k-1}\exp\left[\frac{\sigma_\eta^2\psi^k}{4\left(1-\psi^2\right)}\right],$$

then $Corr\left(y_t,y_{t-k}\right)$ has the sign of λ, whereas in the opposite case $Corr\left(y_t,y_{t-k}\right)$ has the opposite sign of λ, i.e. $Corr\left(y_t,y_{t-k}\right)$ can be either positive or negative.

In terms of leverage effect we have that

$$Corr\left(\sigma_t^2,\varepsilon_{t-k}^*\right)=\rho\sigma_\eta\psi^{k-1}\frac{\exp\left[\frac{4\psi^k-1}{8(1-\psi^2)}\sigma_\eta^2\right]}{\sqrt{\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)}} \tag{2.4}$$

and has the sign of ρ, i.e. the leverage effect can be satisfied by the model if and only if $\rho<0$, provided that $\psi>0$ something which is very plausible due to volatility clustering (see below).

In terms of dynamic asymmetry, as we call the $Corr\left(y_t^2,y_{t-k}\right)$, notice that ($c=0$ otherwise it is too complicated and it is presented in Appendix A)

$$Corr\left(y_t^2,y_{t-k}\right)=\left(\lambda^2+1\right)\frac{\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2(1-\psi^2)}\right]-1\right\}+\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]}{\sqrt{A\left\{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-\left(\lambda^2+1\right)^2\right\}}},$$

$$A = \left\{ \left(\lambda^2+1\right) \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - \lambda^2 \right\}.$$

and the dynamic asymmetry could be either positive or negative, depending on the numerator being positive or negative, respectively.

Now the volatility clustering is given by (see Appendix A for a proof)

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2 \psi^k}{(1-\psi^2)}\right]-1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}, \tag{2.5}$$

and is the same with the SV2-M specification.

For the dynamic kurtosis, $Cov\left(y_t^2, y_{t-k}^2\right)$, as we call it here, we get:

$$\begin{aligned} Cov\left(y_t^2, y_{t-k}^2\right) &= 4c^2\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}\right) + 2c\lambda\left(2\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}\right) \\ &+\lambda^2\left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + 4c^2\lambda Cov\left(\sigma_t, \sigma_{t-k}\varepsilon_{t-k}\right) + 4c\left(2\lambda^2+1\right) Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) \\ &+2\lambda\left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2c\lambda Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + \left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right), \end{aligned}$$

which is far too complicated. However, under the Efficient Market hypothesis, i.e. for $c=0$, we get

$$\begin{aligned} Corr\left(y_t^2, y_{t-k}^2\right) &= \left(\lambda^2+1\right) \frac{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\} + 2\lambda\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]}{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - \left(\lambda^2+1\right)^2} \\ &+ \frac{\left(\lambda^2+1\right)\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}}{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - \left(\lambda^2+1\right)^2}, \end{aligned}$$

Now if $\rho = c = 0$, we get:

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(\lambda^2+1\right)^2\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}}{\left(\lambda^2+1\right)^2\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right) + \left(4\lambda^2+2\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]},$$

which is positive for any value of λ. It also follows that in this case

$$\frac{Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)}{Corr\left(y_t^2, y_{t-k}^2\right)} = 1 + \frac{\left(4\lambda^2+2\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]}{\left(\lambda^2+1\right)^2\left\{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}} > 1,$$

i.e. $Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) > Corr\left(y_t^2, y_{t-k}^2\right)$.

Now for $\lambda = c = 0$ we get

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1},$$

which positive for any value of ρ and $Corr\left(y_t^2, y_{t-k}^2\right) > Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)$, again.

2.2 Properties of the SV2-M Model

2.2.1 Static Moments

Now, in Appendix B we prove that if instead the conditional variance specification we employ the specification in JPR, equation 2.3, we get

$$E\left(y_t\right) = c + \lambda \exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] + \frac{\rho\sigma_\eta}{2}\exp\left(\frac{1}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

$$\begin{aligned} Var\left(y_t\right) &= \left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]-\lambda^2\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\ &+\rho\sigma_\eta\left\{\left(2\lambda+\rho\sigma_\eta\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]-\lambda-\frac{\rho\sigma_\eta}{4}\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \end{aligned}$$

and

$$\begin{aligned} \kappa\left(\varepsilon_t^*\right) &= \frac{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2} \\ &-3\left(\rho\sigma_\eta\right)^2\frac{\left(3+\left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right)-\frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]+\frac{(\rho\sigma_\eta)^2}{16}}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}. \end{aligned}$$

The complication arises due to the fact that the error term in the mean equation and the conditional variance equation are correlated.

In fact JPR assumes that $c = \lambda = 0$ and consequently we have that

$$E\left(y_t\right) = \frac{\rho\sigma_\eta}{2}\exp\left(\frac{1}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and

$$Var\left(y_t\right) = \left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).$$

Notice that if $\rho < 0$, as it is the case when the leverage effect is satisfied, then $E\left(y_t\right) < 0$, i.e. the unconditional risk premium is negative.

Further if $\rho = 0$ we get

$$E\left(y_t\right) = c + \lambda\exp\left[\frac{\omega}{2\left(1-\psi\right)}+\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$

$$Var\left(y_t\right) = \left\{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]-1\right\}+1\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)$$

The skewness coefficient of the mean error $\varepsilon_t^* = \sigma_t\varepsilon_t$ is given by

$$sk\left(\varepsilon_t^*\right) = \frac{3}{2}\rho\sigma_\eta\frac{\left\{\left(3+\left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)-\left(1+\left(\rho\sigma_\eta\right)^2\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)+\frac{1}{6}\left(\rho\sigma_\eta\right)^2}{\left(\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right)^{3/2}}.$$

It is worth noticing that although the distribution of the standardized errors is normal the skewness of the mean error is non-zero, and if fact it is negative, provided that $\rho < 0$ due to leverage effect.

Now the kurtosis of the mean error is given by

$$\kappa\left(\varepsilon_t^*\right) = \frac{\left[3+24\left(\rho\sigma_\eta\right)^2+16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}$$
$$+3\left(\frac{\rho\sigma_\eta}{2}\right)^2\frac{\left\{2\left(1+\left(\rho\sigma_\eta\right)^2\right)-\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2},$$

and for $\rho = 0$ as in Koopman and Uspensky (2002) [55]

$$\kappa\left(\varepsilon_t^*\right) = 3\exp\left(\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right),$$

as in SV1-M model.

The skewness of y_t is given by

$$
\begin{aligned}
&E\left(y_t - E\left(y_t\right)\right)^3 \\
&= \lambda^3 \left\{ \exp\left[\frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] + 2 \right\} \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
&+ \frac{3}{2}\lambda^2 \rho\sigma_\eta \left\{ 3\exp\left(\frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 5\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) + 2 \right\} \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
&+ 3\lambda \left\{ \begin{array}{c} \left[1 + 2\left(\rho\sigma_\eta\right)^2\right] \left[\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right] \\ + \frac{\rho^2\sigma_\eta^2}{4} \exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) \end{array} \right\} \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
&\qquad + \frac{9}{2}\rho\sigma_\eta \left(1 + \frac{3}{4}\left(\rho\sigma_\eta\right)^2\right) \exp\left(\frac{3\omega}{2}\frac{1}{\left(1-\psi\right)} + \frac{9\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
&+ \frac{\rho\sigma_\eta}{2} \left\{ \left(\rho\sigma_\eta\right)\left(1 + \frac{\rho\sigma_\eta}{2}\right) - 3\left(1 + \left(\rho\sigma_\eta\right)^2\right) \exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).
\end{aligned}
$$

Now for $\lambda = 0$ we get skewness of y_t

$$
\begin{aligned}
sk\left(y_t\right) = & \frac{\rho\sigma_\eta}{2} \frac{3\left(1 + \left(\rho\sigma_\eta\right)^2\right) \left\{ 3\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - 1 \right\} \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)}{\left\{ \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + \left(\rho\sigma_\eta\right)^2 \left\{ \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \frac{1}{4} \right\} \right\}^{3/2}} \\
& + \frac{\left(\rho\sigma_\eta\right)^2}{2} \frac{1 + \frac{\rho\sigma_\eta}{2} - \frac{3}{4}\left(\rho\sigma_\eta\right) \exp\left(\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right)}{\left\{ \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + \left(\rho\sigma_\eta\right)^2 \left\{ \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \frac{1}{4} \right\} \right\}^{3/2}},
\end{aligned}
$$

whereas for $\rho = 0$ we get

$$
sk\left(y_t\right) = \lambda \frac{\lambda^2 \left\{ \exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] + 2 \right\} + 3\left\{ \exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - 1 \right\} \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}{\left\{ \left(\lambda^2 + 1\right) \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \lambda^2 \right\}^{3/2}}
$$

and $sk\left(y_t\right)$ has the sign of λ.

Now, $c=\lambda=0$ the kurtosis coefficient of y_t is:

$$\kappa\left(y_t\right)=3\left(\rho\sigma_\eta\right)^2\frac{\left\{\begin{array}{c}\frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\\-\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\end{array}\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2},$$

$$+\frac{C\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)-3\left(\frac{\rho\sigma_\eta}{2}\right)^4}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}$$

where

$$C=3+6\left(\rho\sigma_\eta 2\right)^2+\left(\rho\sigma_\eta 2\right)^4.$$

Notice that $\kappa\left(y_t\right)<\kappa\left(\varepsilon_t^*\right)$ for any values of ρ.

For $c=\rho=0$ we get

$$\kappa\left(y_t\right)=\lambda^4\frac{\left\{\begin{array}{c}\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-4\right\}\exp\left[\frac{5\sigma_\eta^2}{4(1-\psi^2)}\right]\\+6\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-3\end{array}\right\}}{\left\{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\right\}^2}$$

$$+3\frac{2\lambda^2\left\{\begin{array}{c}\exp\left(\frac{5\sigma_\eta^2}{4(1-\psi^2)}\right)\\-2\exp\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right]+1\end{array}\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]+\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\right\}^2}$$

2.2.2 Dynamic Moments

The 1st order autocorrelation is given by (see Appendix B)

$$Corr\left(y_t,y_{t-1}\right)=\frac{\lambda^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right\}+\frac{1}{2}\lambda\rho\sigma_\eta\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-2\right\}}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+D-\left(\frac{\rho\sigma_\eta}{2}\right)^2}$$

$$+\frac{\left(\frac{\rho\sigma_\eta}{2}\right)\left\{\lambda\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)+\frac{\rho\sigma_\eta}{2}\left[\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi\lambda^2)}\right)-1\right]\right\}}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+D-\left(\frac{\rho\sigma_\eta}{2}\right)^2},$$

where

$$D=\left(1+(\rho\sigma_\eta)^2\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]+\lambda\rho\sigma_\eta\left\{2\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}$$

Now in case of Koopman and Uspensky (2002) [55] we have $\rho=0$ and it follows that

$$Corr\left(y_t,y_{t-1}\right)=\lambda^2\frac{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}.$$

and consequently, $Corr\left(y_t,y_{t-1}\right)$ can be only positive. The same is true in the case of JPR, where we have that $\lambda=0$, i.e.

$$Corr\left(y_t,y_{t-1}\right)=\left(\frac{1}{2}\rho\sigma_\eta\right)^2\frac{(\psi+1)\exp\left(\frac{\psi}{4(1-\psi^2)}\sigma_\eta^2\right)-1}{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]+(\rho\sigma_\eta)^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\frac{1}{4}\right\}}$$

and $Corr\left(y_t,y_{t-1}\right)$ can be only positive.

For the leverage effect we get

$$Corr\left(\sigma_t^2,\sigma_{t-1}\varepsilon_{t-1}\right)=\frac{\rho\sigma_\eta\left(\psi+\frac{1}{2}\right)\exp\left(\frac{\psi\sigma_\eta^2}{2(1-\psi^2)}\right)-\frac{1}{2}}{\sqrt{\left\{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\left\{\left(1+(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\rho\sigma_\eta\frac{1}{2}\right)^2\right\}}},$$

which is negative provided that $\rho<0$, something which also is true for the SV1-M model (see 2.4).

For the SV2-M model the dynamic asymmetry is very complicated for the full model and it is presented in Appendix B. Now for the Koopman and Uspensky (2002) [55] model for $\rho=0$ and $c\neq 0$ we get

$$\begin{aligned}Cov\left(y_t^2,y_{t-1}\right)=\lambda&\left\{\lambda^2\exp\left[\frac{\omega}{2\left(1-\psi\right)}+\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]+2c\lambda+\exp\left(\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)\right\}\\&\times\left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left[\frac{\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right].\end{aligned}$$

It is obvious that depending on the value of c and $Cov\left(y_t^2, y_{t-1}\right)$ could have either the sign or the opposite one of λ. If further, $\rho = c = 0$ then

$$Cov\left(y_t^2, y_{t-1}\right) = \lambda\left\{\lambda^2 \exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] + \exp\left(\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)\right\}$$
$$\times\left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\}\exp\left[\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right].$$

and $Cov\left(y_t^2, y_{t-1}\right)$ can have only the sign of λ.

For $\lambda = 0$ but $\rho \neq 0$ and $c \neq 0$ we get

$$Cov\left(y_t^2, y_{t-1}\right) = c\frac{\left(\rho\sigma_\eta\right)^2}{2}\left\{\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\}\exp\left(\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$
$$+\frac{\rho\sigma_\eta}{2}\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\left\{\left(2\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\}\exp\left(\frac{1}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)\right\}$$
$$\times\exp\left(\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

and $Cov\left(y_t^2, y_{t-1}\right)$ has probably the sign of ρ. If additionally $c = 0$, as in JPR we get

$$Cov\left(y_t^2, y_{t-1}\right) = \frac{\rho\sigma_\eta\left(1+\left(\rho\sigma_\eta\right)^2\right)}{2}\left\{\left(2\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\}$$
$$\times\exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and $Cov\left(y_t^2, y_{t-1}\right)$ has the sign of ρ, negative under the leverage hypothesis.

It is easy to prove that the volatility clustering is given by:

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right] - 1}{\exp\left[\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right] - 1}.$$

The dynamic kurtosis notice that for the SV2-M specification we have that

$$Cov\left(\varepsilon_t\sigma_t, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) \neq 0,\ Cov\left(\varepsilon_t\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) \neq 0\ \ and$$
$$Cov\left(\varepsilon_t^2\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) \neq Cov\left(\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right).$$

Consequently, dynamic kurtosis is very complicated. However, for $\lambda = 0$, as well, as in JPR, we get

$$Corr\left(y_t^2, y_{t-1}^2\right) = \frac{\left(1 + (\rho\sigma_\eta)^2\right)\left\{\left(1 + ((\psi+1)\rho\sigma_\eta)^2\right)\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1 + (\rho\sigma_\eta)^2\right)\right\}}{\left\{\left[3 + 6(2\rho\sigma_\eta)^2 + (2\rho\sigma_\eta)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1 + (\rho\sigma_\eta)^2\right)^2\right\}}.$$

For $c \neq 0$ and $\rho = 0$ as in Koopman and Uspensky (2002) [55]and we get

$$\begin{aligned} Cov\left(y_t^2, y_{t-1}^2\right) = & \frac{4c^2\lambda^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right\}}{\left\{(\lambda^4 + 6\lambda^2 + 3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - (\lambda^4 + 2\lambda^2 + 1)\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{6\sigma_\eta^2}{8(1-\psi^2)}\right)} \\ & + 2c\lambda\frac{\left\{(2\lambda^2+1)\exp\left[\frac{\omega}{2(1-\psi)}\right] + 1\right\}\exp\left(\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right)\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right\}}{\left\{(\lambda^4 + 6\lambda^2 + 3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - (\lambda^4 + 2\lambda^2 + 1)\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{6\sigma_\eta^2}{8(1-\psi^2)}\right)} \\ & + \frac{\lambda^4 + 2\lambda^2 + 1}{\left\{(\lambda^4 + 6\lambda^2 + 3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - (\lambda^4 + 2\lambda^2 + 1)\right\}}\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right\} \end{aligned}$$

For $c = \rho = 0$ we get

$$Corr\left(y_t^2, y_{t-1}^2\right) = \frac{\lambda^4 + 2\lambda^2 + 1}{\left\{(\lambda^4 + 6\lambda^2 + 3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - (\lambda^4 + 2\lambda^2 + 1)\right\}}\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right) - 1\right\}$$

and

$$\begin{aligned} \frac{Corr\left(\sigma_t^2, \sigma_{t-1}^2\right)}{Corr\left(y_t^2, y_{t-1}^2\right)} &= \frac{(\lambda^4 + 6\lambda^2 + 3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - (\lambda^4 + 2\lambda^2 + 1)}{(\lambda^4 + 2\lambda^2 + 1)\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)} \\ &= 1 + \frac{(4\lambda^2 + 2)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)}{(\lambda^4 + 2\lambda^2 + 1)\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1\right)} > 1, \end{aligned}$$

i.e. the volatility clustering in higher than the conditional kurtosis, at least for the first order.

3 Comparisons and Conclusions

The mean and the variance of the observed process under the SV2-M model is richer than the ones of the SV1-M as it involves not only the parameters c and λbut ρ, as well. However, $V(y_t) > V(\varepsilon_t^*)$ under the SV1-M assumption, whereas the same is not necessarily true for the SV2-M model.

It is worth mentioning that the skewness of the mean error, ε_t^*, in the SV2-M model has the sign of ρ, i.e. it is negative under the assumption of leverage effect. On the other hand the skewness of ε_t^* is 0 under the SV1-M. Further, the kurtosis coefficient of ε_t^* for the SV1-M model is higher than 3, however, the one for the SV2-M is higher than the kurtosis coefficient of SV1-M, unless $\rho = 0$, in which case they are equal.

The skewness of the observed process y_t under SV1-M can be either positive or negative and depends on λ only, and not of ρ, apart its dependence on σ_η^2 and ψ. For $\rho = 0$ the skewness of the SV2-M can have only the sign of λ, whereas for $\lambda = 0$ the skewness has only the sign of ρ.

In terms of dynamic moments, the $Corr(y_t, y_{t-k})$ can be of either sign, positive or negative. For the SV2-M model it is difficult to determine the sign of $Corr(y_t, y_{t-1})$. However, for $\rho = 0$ $Corr(y_t, y_{t-1})$ can be only positive for either model, whereas for $\lambda = 0$, $Corr(y_t, y_{t-1})$ can be only positive for the SV2-M model but it is zero for the SV1-M one.

In terms of the leverage effect, $Corr(\sigma_t^2, \varepsilon_{t-1}^*)$, is negative if and only if $\rho < 0$, for either model model, and it is 0 if $\rho = 0$. The dynamic asymmetry is very complicated, for either model, to investigate its sign. However, for $\rho = 0$ the first order dynamic asymmetry $Corr(y_t^2, y_{t-1})$ has the sign of $\lambda$for the SV1-M model wheres as it can be of either sign for the SV2-M one. On the other hand if only $c = 0$ then $Corr(y_t^2, y_{t-1})$ can be of either sign for the SV1-M model whereas its sign is difficult to be determined for the SV2-M. For $\rho = c = 0$, for either model, the sign of $Corr(y_t^2, y_{t-1})$ is the same as the sign of λ. Further, for $\lambda = c = 0$ $Corr(y_t^2, y_{t-1})$ is the same sign as of ρ for either model, whereas for $\lambda = \rho = 0$ the first order dynamic asymmetry is zero. Further, for $\lambda = c = 0$ the first order dynamic kurtosis for the SV1-M model is bigger than the first order volatility clustering

The 2 models have exactly the same first order volatility clustering $Corr(\sigma_t^2, \sigma_{t-1}^2)$. Now the same applies for the dynamic kurtosis, at least for the first order, in the case where $c = \rho = 0$, and in this case the dynamic kurtosis is small than the volatility clustering.

To conclude, both models could satisfy the stylised facts of the financial markets returns. For the models with parameter restrictions the static and dynamic moments of order higher than 2 are very complicated with the ones of the SV2-M model being much more so.

Appendix A

Static Moments

From Demos 2002 [21] we get:

$$A_{k,i}^{(s,d)} = \rho\phi_\eta \left(s\psi_{i+k} + d\psi_i\right) = B_{k,i}^{(s,d)}$$
$$\Gamma_{k,i}^{(s,d)} = \frac{\phi_\eta^2}{2} \left(s\psi_{i+k} + d\psi_i\right)^2 = \Delta_{k,i}^{(s,d)}$$

$$\varpi_k^{(s,d)} = \prod_{i=0}^{\infty} \left[\Phi\left(A_{k,i}^{(s,d)}\right) \exp\left(\Gamma_{k,i}^{(s,d)}\right) + \exp\left(\Delta_{k,i}^{(s,d)}\right) \Phi\left(-B_{k,i}^{(s,d)}\right)\right] \tag{3.1}$$
$$= \prod_{i=0}^{\infty} \left\{\exp\left(\Gamma_{k,i}^{(s,d)}\right) \left[\Phi\left(A_{k,i}^{(s,d)}\right) + \Phi\left(-B_{k,i}^{(s,d)}\right)\right]\right\} = \exp\left(\sum_{i=0}^{\infty} \Gamma_{k,i}^{(s,d)}\right)$$
$$= \exp\left(\frac{\phi_\eta^2}{2} \sum_{i=0}^{\infty} \left(s\psi^{i+k} + d\psi^i\right)^2\right) = \exp\left(\frac{\phi_\eta^2 \left(s\psi^k + d\right)^2}{2} \sum_{i=0}^{\infty} \psi^{2i}\right) = \exp\left(\frac{\phi_\eta^2 \left(s\psi^k + d\right)^2}{2\left(1-\psi^2\right)}\right)$$

$$\Lambda_{(k-1)}^{s} = \prod_{i=0}^{k-1} \left[\Phi\left(A_{0,i}^{(0,s)}\right) exp\left(\Gamma_{0,i}^{(0,s)}\right) + exp\left(\Delta_{0,i}^{(0,s)}\right) \Phi\left(-B_{0,i}^{(0,s)}\right)\right]$$
$$= \prod_{i=0}^{k-1} \exp\left(\Gamma_{0,i}^{(0,s)}\right) = \exp\sum_{i=0}^{k-1} \left(\frac{\phi_\eta^2}{2} s^2 \psi_i^2\right)$$

By employing the above formulae we get:

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\right) = \exp\left[(s+d)\frac{\omega}{1-\psi}\right] \varpi_k^{(s,d)} \Lambda_{(k-1)}^{s}$$
$$= \exp\left[\frac{(s+d)\,\omega}{1-\psi} + \frac{\sigma_\eta^2 \left(s\psi^k + d\right)^2}{2\left(1-\psi^2\right)} + \frac{\sigma_\eta^2 s^2}{2} \sum_{i=0}^{k-1} \psi^{2i}\right]$$
$$= \exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2 \left(2sd\psi^k + d^2 + s^2\right)}{2\left(1-\psi^2\right)}\right]$$

Now from Demos 2002 [21], again,

$$\begin{aligned}D_{k-1}^{s} &= A_{0,k-1}^{(0,s)}\Phi\left(A_{0,k-1}^{(0,s)}\right)\exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)+B_{0,k-1}^{(0,s)}\exp\left(\Delta_{0,k-1}^{(0,s)}\right)\Phi\left(-B_{0,k-1}^{(0,s)}\right)\\ &= A_{0,k-1}^{(0,s)}\exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)=s\rho\phi_{\eta}\psi^{k-1}\exp\left(\frac{\phi_{\eta}^{2}}{2}s^{2}\psi^{2k-2}\right)\end{aligned}$$

$$\begin{aligned}Y_{k-1}^{s} &= \Phi\left(A_{0,k-1}^{(0,s)}\right)\exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)+\exp\left(\Delta_{0,k-1}^{(0,s)}\right)\Phi\left(-B_{0,k-1}^{(0,s)}\right)\\ &= \exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)=\exp\left(\frac{\phi_{\eta}^{2}}{2}s^{2}\psi^{2k-2}\right)\end{aligned}$$

$$\begin{aligned}F_{k-1}^{s} &= \left(A_{0,k-1}^{(0,s)}\right)^{2}\Phi\left(A_{0,k-1}^{(0,s)}\right)\exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)+\left(B_{0,k-1}^{(0,s)}\right)^{2}\exp\left(\Delta_{0,k-1}^{(0,s)}\right)\Phi\left(-B_{0,k-1}^{(0,s)}\right)\\ &= \left(A_{0,k-1}^{(0,s)}\right)^{2}\exp\left(\mathit{\Gamma}_{0,k-1}^{(0,s)}\right)=s^{2}\rho^{2}\phi_{\eta}^{2}\psi^{2k-2}\exp\left(\frac{\phi_{\eta}^{2}}{2}s^{2}\psi^{2k-2}\right)\end{aligned}$$

$$\begin{aligned}E\left(\sigma_{t}^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}\right) &= E\left(\sigma_{t}^{2s}\sigma_{t-k}^{2d}\right)D_{k-1}^{s}\left(Y_{k-1}^{s}\right)^{-1}\quad and\\ E\left(\sigma_{t}^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}^{2}\right) &= E\left(\sigma_{t}^{2s}\sigma_{t-k}^{2d}\right)\left[1+F_{k-1}^{s}\left(Y_{k-1}^{s}\right)^{-1}\right]\end{aligned}$$

Hence employing the above formulae we get

$$\begin{aligned}E\left(\sigma_{t}\sigma_{t-k}\varepsilon_{t-k}\right) &= \frac{1}{2}\rho\sigma_{\eta}\psi^{k-1}\exp\left[\frac{\omega}{1-\psi}+\frac{\sigma_{\eta}^{2}\left(\psi^{k}+1\right)}{4\left(1-\psi^{2}\right)}\right]\\ E\left(\sigma_{t}^{2}\sigma_{t-k}\varepsilon_{t-k}\right) &= \rho\sigma_{\eta}\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{1-\psi}+\frac{\sigma_{\eta}^{2}\left(4\psi^{k}+5\right)}{8\left(1-\psi^{2}\right)}\right]\\ E\left(\sigma_{t}\sigma_{t-k}^{2}\varepsilon_{t-k}\right) &= \frac{1}{2}\rho\sigma_{\eta}\psi^{k-1}\exp\left[\frac{3}{2}\frac{\omega}{1-\psi}+\frac{\sigma_{\eta}^{2}\left(4\psi^{k}+5\right)}{8\left(1-\psi^{2}\right)}\right]=\frac{1}{2}E\left(\sigma_{t}^{2}\sigma_{t-k}\varepsilon_{t-k}\right)\\ E\left(\sigma_{t}^{2}\sigma_{t-k}^{2}\varepsilon_{t-k}\right) &= \rho\sigma_{\eta}\psi^{k-1}\exp\left[2\frac{\omega}{1-\psi}+\frac{\sigma_{\eta}^{2}\left(\psi^{k}+1\right)}{\left(1-\psi^{2}\right)}\right]\end{aligned}$$

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}\right) = \exp\left(\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\phi_\eta^2\left(s\psi^k+d\right)^2}{2\left(1-\psi^2\right)}\right)\exp\left(\sum_{i=0}^{k-1}\frac{\phi_\eta^2}{2}s^2\psi_i^2\right)$$

$$\times s\rho\phi_\eta\psi^{k-1}\frac{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}$$

$$= s\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k+d^2+s^2\right)}{2\left(1-\psi^2\right)}\right]$$

and

$$E\left(\sigma_t^{2s}\sigma_{t-k}^{2d}\varepsilon_{t-k}^2\right) = \exp\left(\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\phi_\eta^2\left(s\psi^k+d\right)^2}{2\left(1-\psi^2\right)}\right)\exp\left(\sum_{i=0}^{k-1}\frac{\phi_\eta^2}{2}s^2\psi_i^2\right)$$

$$\times\left[1+s^2\rho^2\phi_\eta^2\psi^{2k-2}\frac{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}{\exp\left(\frac{\phi_\eta^2}{2}s^2\psi^{2k-2}\right)}\right]$$

$$= \left(1+s^2\rho^2\phi_\eta^2\psi^{2k-2}\right)\exp\left[\frac{(s+d)\,\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(2sd\psi^k+d^2+s^2\right)}{2\left(1-\psi^2\right)}\right]$$

Now

$$E\left(\sigma_t\right) = \exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],\;\; E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

$$E\left(\sigma_t^3\right) = \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{9\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],\;\; \textit{and}\;\; E\left(\sigma_t^4\right) = \exp\left[2\frac{\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right].$$

Hence it follows that

$$V\left(\varepsilon_t^*\right) = E\left(\varepsilon_t^2\sigma_t^2\right) = E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

and

$$\kappa\left(\varepsilon_t^*\right) = \frac{3E\left(\sigma_t^4\right)}{\left(V\left(\varepsilon_t\sigma_t\right)\right)^2} = 3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] > 3$$

which is equal to $\kappa\left(\varepsilon_t^*\right)$ for the SV2-M model.

Further

$$E\left(y_t\right) = c+\lambda E\left(\sigma_t\right) = c+\lambda\exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$

$$Var\left(y_t\right) = E\left(y_t - E\left(y_t\right)\right)^2 = E\left(\left(\lambda+\varepsilon_t\right)\sigma_t - \lambda E\left(\sigma_t\right)\right)^2$$
$$= \left(\lambda^2+1\right)E\left(\sigma_t^2\right) - 2\lambda^2 E^2\left(\sigma_t\right) + \lambda^2 E^2\left(\sigma_t\right)$$
$$= \left(\lambda^2+1\right)\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] - \lambda^2\exp\left[\frac{2\omega}{2\left(1-\psi\right)} + \frac{2\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$

$$E\left(y_t^2\right) = E\left(c + \left(\lambda+\varepsilon_t\right)\sigma_t\right)^2 = E\left(c^2 + 2c\left(\lambda+\varepsilon_t\right)\sigma_t + \left(\lambda+\varepsilon_t\right)^2\sigma_t^2\right)$$
$$= c^2 + 2\lambda c E\left(\sigma_t\right) + \left(\lambda^2+1\right)E\left(\sigma_t^2\right)$$
$$= c^2 + 2c\lambda\exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] + \left(\lambda^2+1\right)\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

$$V\left(y_t^2\right) = E\left(y_t^2 - E\left(y_t^2\right)\right)^2 =$$
$$= E\left\{\left[2c\left(\lambda+\varepsilon_t\right)\sigma_t + \left(\lambda+\varepsilon_t\right)^2\sigma_t^2\right] - \left[2c\lambda E\left(\sigma_t\right) + \left(\lambda^2+1\right)E\left(\sigma_t^2\right)\right]\right\}^2$$
$$= \left(\lambda^4+6\lambda^2+3\right)E\left(\sigma_t^4\right) - \left(\lambda^2+1\right)^2E^2\left(\sigma_t^2\right) + 4c^2\left(\lambda^2+1\right)^2E\left(\sigma_t^2\right) - 4c^2\lambda^2E^2\left(\sigma_t\right)$$
$$+4c\lambda\left(\lambda^2+3\right)E\left(\sigma_t^3\right) - 4c\lambda\left(\lambda^2+1\right)E\left(\sigma_t^2\right)E\left(\sigma_t\right)$$
$$= \left\{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - \left(\lambda^2+1\right)^2\right\}\exp\left[2\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$+4c^2\left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - \lambda^2\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$
$$+4c\lambda\left\{\left(\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] - \left(\lambda^2+1\right)\right\}\exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$
$$= \left[\begin{array}{c} \left\{(\lambda^4+6\lambda^2+3)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - (\lambda^2+1)^2\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] \\ +4c^2\left\{(\lambda^2+1)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \lambda^2\right\} \\ +4c\lambda\left\{(\lambda^2+3)\exp\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right] - (\lambda^2+1)\right\}\exp\left[\frac{\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right] \end{array}\right]$$
$$\times\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right].$$

Further

$$E\left(y_t^3\right) = E\left[c + (\lambda + \varepsilon_t)\,\sigma_t\right]^3$$
$$= c^3 + E\left[3c^2(\lambda+\varepsilon_t)\,\sigma_t + 3c(\lambda+\varepsilon_t)^2\sigma_t^2 + (\lambda+\varepsilon_t)^3\sigma_t^3\right]$$
$$= c^3 + 3c^2\lambda E(\sigma_t) + 3c\left(\lambda^2+1\right)E\left[\sigma_t^2\right] + \lambda\left(\lambda^2+3\right)E\left(\sigma_t^3\right)$$
$$= c^3 + 3c^2\lambda\exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right] + 3c\left(\lambda^2+1\right)\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]$$
$$+\lambda\left(\lambda^2+3\right)\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{9\sigma_\eta^2}{8(1-\psi^2)}\right],$$

and it follows that

$$E\left(y_t - E(y_t)\right)^3 = E\left((\lambda+\varepsilon_t)\,\sigma_t - \lambda E(\sigma_t)\right)^3$$
$$= E\left((\lambda+\varepsilon_t)^3\sigma_t^3 - 3(\lambda+\varepsilon_t)^2\sigma_t^2\lambda E(\sigma_t) + 3(\lambda+\varepsilon_t)\,\sigma_t\lambda E^2(\sigma_t) - \lambda^3E^3(\sigma_t)\right)$$
$$= \lambda^3E\left(\sigma_t^3\right) + 3\lambda E\left(\sigma_t^3\right) - 3\lambda\left(\lambda^2+1\right)E\left(\sigma_t^2\right)E(\sigma_t) + \lambda^2(3-\lambda)\,E^3(\sigma_t)$$

$$sk(y_t) = \frac{E\left(y_t - E(y_t)\right)^3}{\left[V(y_t)\right]^{3/2}}$$
$$= \lambda\frac{\lambda^2\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + 3\lambda + 3\left\{\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right] - 1\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}{\left\{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]\right\}^{3/2}}.$$

The denominator is positive. For the numerator

$$\lambda^2\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\}$$
$$+3\lambda + 3\left(\exp\left[\frac{\sigma_\eta^2}{3(1-\psi^2)}\right] - 1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]$$

if $\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1 < 0$, as it is usually the case, then the determinant is positive, i.e.

$$9 - 12\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\}$$
$$\times\left(\exp\left[\frac{\sigma_\eta^2}{3(1-\psi^2)}\right] - 1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] > 0$$

and we have two roots with opposite signs and bigger, in absolute value, the positive one. Hence the skewness coefficient $sk\left(y_t\right)$ can be of either sign.

Now

$$E\left(y_t - E\left(y_t\right)\right)^4 = E\left(\left(\lambda+\varepsilon_t\right)\sigma_t - \lambda E\left(\sigma_t\right)\right)^4$$
$$= E\left(\left(\lambda+\varepsilon_t\right)^4\sigma_t^4 - 4\left(\lambda+\varepsilon_t\right)^3\sigma_t^3\lambda E\left(\sigma_t\right) + 6\left(\lambda+\varepsilon_t\right)^2\sigma_t^2\lambda^2E^2\left(\sigma_t\right)\right)$$
$$+E\left(-4\left(\lambda+\varepsilon_t\right)\sigma_t\lambda^3E^3\left(\sigma_t\right)+\lambda^4E^4\left(\sigma_t\right)\right)$$
$$= \left(\lambda^4+6\lambda^2+3\right)E\left(\sigma_t^4\right) - \left(4\lambda^4+12\lambda^2\right)E\left(\sigma_t^3\right)E\left(\sigma_t\right)$$
$$+6\lambda^2\left(\lambda^2+1\right)E\left(\sigma_t^2\right)E^2\left(\sigma_t\right) - 3\lambda^4E^4\left(\sigma_t\right)$$

and it follows

$$\kappa\left(y_t\right) = \frac{E\left(y_t-E\left(y_t\right)\right)^4}{V^2\left(y_t\right)} = \frac{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{6\sigma_\eta^2}{4(1-\psi^2)}\right]}{\left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\lambda^2\right\}^2}$$
$$+\lambda^2\frac{6\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-4\left(\lambda^2+3\right)\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right]-3\lambda^2}{\left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\lambda^2\right\}^2},$$

which is equal to $\kappa\left(\varepsilon_t^*\right)$ for $\lambda = 0$.

Dynamic Moments

For the dynamic properties of y_t we get that:

$$Cov\left(y_t, y_{t-k}\right) = E\left(y_ty_{t-k}\right) - E^2\left(y_t\right) = E\left(\left(c+\left(\lambda+\varepsilon_t\right)\sigma_t\right)\left(c+\left(\lambda+\varepsilon_{t-k}\right)\sigma_{t-k}\right)\right) - E^2\left(y_t\right)$$
$$= c^2 + c\lambda E\left(\sigma_{t-k}\right) + c\lambda E\left(\sigma_t\right) + \lambda^2E\left(\sigma_t\sigma_{t-k}\right) + E\left(\left(\lambda\sigma_t+\varepsilon_t\sigma_t\right)\varepsilon_{t-k}\sigma_{t-k}\right) - \left(c+\lambda E\left(\sigma_t\right)\right)^2$$
$$= \lambda^2E\left(\sigma_t\sigma_{t-k}\right) + \lambda E\left(\sigma_t\sigma_{t-k}\varepsilon_{t-k}\right) - \lambda^2E^2\left(\sigma_t\right)$$
$$= \lambda^2\exp\left[\frac{\omega}{1-\psi}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right] + \frac{\lambda}{2}\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\omega}{1-\psi}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right]$$
$$-\lambda^2\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$

and

$$Corr\left(y_t, y_{t-k}\right) = \lambda\frac{\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4(1-\psi^2)}\right]-1\right\}+\frac{1}{2}\rho\phi_\eta\psi^{k-1}\exp\left[\frac{\sigma_\eta^2\psi^k}{4(1-\psi^2)}\right]}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]}$$

Leverage

$$Cov\left(\sigma_t^2, \varepsilon_{t-k}^*\right) = E\left(\sigma_t^2 \varepsilon_{t-k}^*\right) = E\left(\sigma_t^2 \sigma_{t-k} \varepsilon_{t-k}\right)$$
$$= \rho\sigma_\eta \psi^{k-1} \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right]$$

and it follows that

$$Corr\left(\sigma_t^2, \varepsilon_{t-k}^*\right) = \frac{\rho\sigma_\eta \psi^{k-1} \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8(1-\psi^2)}\right]}{\sqrt{V\left(\sigma_t^2\right) V\left(\varepsilon_{t-k}^*\right)}}$$
$$= \frac{\rho\sigma_\eta \psi^{k-1} \exp\left[\frac{4\psi^k-1}{8(1-\psi^2)}\sigma_\eta^2\right]}{\sqrt{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}}.$$

Dynamic Asymmetry

$$Cov\left(y_t^2, y_{t-k}\right) = E\left(\left[y_t^2 - E\left(y_t^2\right)\right]\left[y_{t-k} - E\left(y_{t-k}\right)\right]\right)$$
$$= E\left\{\left[2c\left(\lambda+\varepsilon_t\right)\sigma_t + \left(\lambda^2 + 2\lambda\varepsilon_t + \varepsilon_t^2\right)\sigma_t^2 - 2\lambda c E\left(\sigma_t\right) - \left(\lambda^2+1\right)E\left(\sigma_t^2\right)\right]\left(\lambda+\varepsilon_{t-k}\right)\sigma_{t-k}\right\}$$
$$-\lambda E\left\{\left[2c\left(\lambda+\varepsilon_t\right)\sigma_t + \left(\lambda^2 + 2\lambda\varepsilon_t + \varepsilon_t^2\right)\sigma_t^2 - 2\lambda c E\left(\sigma_t\right) - \left(\lambda^2+1\right)E\left(\sigma_t^2\right)\right]E\left(\sigma_{t-k}\right)\right\}$$
$$= 2c\lambda^2 E\left(\sigma_t\sigma_{t-k}\right) + 2c\lambda E\left(\sigma_t\sigma_{t-k}\varepsilon_{t-k}\right) + \lambda\left(\lambda^2+1\right)E\left(\sigma_t^2\sigma_{t-k}\right)$$
$$+\left(\lambda^2+1\right)E\left(\sigma_t^2\sigma_{t-k}\varepsilon_{t-k}\right) - \lambda\left(\lambda^2+1\right)E\left[\sigma_t^2\right]E\left(\sigma_{t-k}\right) - 2c\lambda^2 E\left[\sigma_t\right]E\left(\sigma_{t-k}\right)$$
$$= 2c\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}\right) + 2c\lambda E\left(\sigma_t\sigma_{t-k}\varepsilon_{t-k}\right) + \lambda\left(\lambda^2+1\right)Cov\left(\sigma_t^2, \sigma_{t-k}\right)$$
$$+\left(\lambda^2+1\right)E\left(\sigma_t^2\sigma_{t-k}\varepsilon_{t-k}\right)$$
$$= \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$
$$\times\left[\begin{array}{c} c\lambda\left\{\left[2\lambda + \rho\sigma_\eta\psi^{k-1}\right]\exp\left[\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right] - 2\lambda\right\} \\ +\left(\lambda^2+1\right)\left\{\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2(1-\psi^2)}\right]-1\right\} + \rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]\right\}\exp\left[\frac{\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]\end{array}\right]$$

and

$$Corr\left(y_t^2, y_{t-k}\right) = \frac{\left[\begin{array}{c} c\lambda\left\{\left[2\lambda+\rho\sigma_\eta\psi^{k-1}\right]\exp\left[\frac{\psi^k\sigma_\eta^2}{4(1-\psi^2)}\right]-2\lambda\right\} \\ +\left(\lambda^2+1\right)\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\omega}{2(1-\psi)}+\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right] \\ +\left(\lambda^2+1\right)\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]\exp\left[\frac{\omega}{2(1-\psi)}+\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]\end{array}\right]}{\sqrt{A\left[\begin{array}{c}\left\{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-\left(\lambda^2+1\right)^2\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] \\ +4c^2A+4c\lambda\left(\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right]\exp\left[\frac{\omega}{2(1-\psi)}+\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right] \\ -4c\lambda\left(\lambda^2+1\right)\exp\left[\frac{\omega}{2(1-\psi)}+\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]\end{array}\right]}},$$

where

$$A = \left\{\left(\lambda^2+1\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]-\lambda^2\right\}.$$

For $c = 0$ we get

$$Corr\left(y_t^2, y_{t-k}\right) = \left(\lambda^2+1\right)\frac{\lambda\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2(1-\psi^2)}\right]-1\right\}+\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\psi^k\sigma_\eta^2}{2(1-\psi^2)}\right]}{\sqrt{A\left\{\left(\lambda^4+6\lambda^2+3\right)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-\left(\lambda^2+1\right)^2\right\}}}.$$

Volatility Clustering

$$\begin{aligned} Cov\left(\sigma_t, \sigma_{t-k}\right) &= E\left(\sigma_t\sigma_{t-k}\right)-E\left(\sigma_t\right)E\left(\sigma_{t-k}\right) \\ &= \exp\left[\frac{\omega}{1-\psi}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right]-\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] \\ &= \left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]\end{aligned}$$

$$Cov\left(\sigma_t^2, \sigma_{t-k}\right) = \left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{3\omega}{2\left(1-\psi\right)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$

$$Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) = \left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{2\omega}{\left(1-\psi\right)}+\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right]$$

and it follows that

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2 \psi^k}{(1-\psi^2)}\right] - 1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right] - 1}.$$

Now

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) &= Cov\left(2c\left(\lambda+\varepsilon_t\right)\sigma_t + \left(\lambda+\varepsilon_t\right)^2\sigma_t^2, 2c\left(\lambda+\varepsilon_{t-k}\right)\sigma_{t-k} + \left(\lambda+\varepsilon_{t-k}\right)^2\sigma_{t-k}^2\right) \\
&= Cov\left(2c\left(\lambda+\varepsilon_t\right)\sigma_t, 2c\left(\lambda+\varepsilon_{t-k}\right)\sigma_{t-k} + \left(\lambda^2+2\lambda\varepsilon_{t-k}+\varepsilon_{t-k}^2\right)\sigma_{t-k}^2\right) \\
&+ Cov\left(\left(\lambda^2+2\lambda\varepsilon_t+\varepsilon_t^2\right)\sigma_t^2, 2c\left(\lambda+\varepsilon_{t-k}\right)\sigma_{t-k} + \left(\lambda^2+2\lambda\varepsilon_{t-k}+\varepsilon_{t-k}^2\right)\sigma_{t-k}^2\right) \\
&= 4c^2\lambda Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}\right) + 4c^2 Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}\varepsilon_{t-k}\right) + 4c^2\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}\right) \\
&+ 4c^2\lambda Cov\left(\sigma_t, \sigma_{t-k}\varepsilon_{t-k}\right) + 2c\lambda^3 Cov\left(\sigma_t, \sigma_{t-k}^2\right) + 2c\lambda^2 Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}^2\right) \\
&+ 4c\lambda Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 4c\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2c\lambda Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) \\
&+ 2c Cov\left(\varepsilon_t\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + 2c\lambda^3 Cov\left(\sigma_t^2, \sigma_{t-k}\right) + 2c\lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) \\
&+ 4c\lambda^2 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}\right) + 4c\lambda Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) \\
&+ 2c\lambda Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}\right) + 2c Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) \\
&+ \lambda^4 Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + 2\lambda^3 Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + \lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) \\
&+ 2\lambda^3 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\right) + 4\lambda^2 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2\lambda Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) \\
&+ \lambda^2 Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\right) + 2\lambda Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right).
\end{aligned}$$

Now as

$$\begin{gathered}
Cov\left(\varepsilon_t\sigma_t, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = 0,\ Cov\left(\varepsilon_t\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = 0 \ \ and \\
Cov\left(\varepsilon_t^2\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right) = Cov\left(\sigma_t^2, f\left(\varepsilon_{t-k}, \sigma_{t-k}\right)\right)
\end{gathered}$$

and it follows that

$$\begin{aligned}
Cov\left(y_t^2, y_{t-k}^2\right) &= 4c^2\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}\right) + 2c\lambda^3 Cov\left(\sigma_t, \sigma_{t-k}^2\right) + 2c\lambda\left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}\right) \\
&+ \lambda^4 Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + \lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + 4c^2\lambda Cov\left(\sigma_t, \sigma_{t-k}\varepsilon_{t-k}\right) \\
&+ 4c\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2c\left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) + 2\lambda\left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) \\
&+ 2c\lambda Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + \left(\lambda^2+1\right) Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right).
\end{aligned}$$

Now

$$Cov\left(\sigma_t,\sigma_{t-k}\right)=E\left(\sigma_t\sigma_{t-k}\right)-E\left(\sigma_t\right)E\left(\sigma_{t-k}\right)$$
$$=\exp\left[\frac{\omega}{1-\psi}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right]-\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$
$$=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{4\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$

$$Cov\left(\sigma_t^2,\sigma_{t-k}\right)=Cov\left(\sigma_t,\sigma_{t-k}^2\right)=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{2\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$
$$Cov\left(\sigma_t^2,\sigma_{t-k}^2\right)=\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right]$$

$$Cov\left(\sigma_t,\sigma_{t-k}\varepsilon_{t-k}\right)=\frac{1}{2}\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{4\left(1-\psi^2\right)}\right]$$
$$Cov\left(\sigma_t^2,\sigma_{t-k}\varepsilon_{t-k}\right)=\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right]=2Cov\left(\sigma_t,\sigma_{t-k}^2\varepsilon_{t-k}\right)$$
$$Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}\right)=\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{\left(1-\psi^2\right)}\right]$$
$$Cov\left(\sigma_t,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)=\left\{\left(1+\frac{1}{4}\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{2\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$
$$Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)=\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{\left(1-\psi^2\right)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right].$$

$$Cov\left(\sigma_t,\sigma_{t-k}^2\varepsilon_{t-k}\right)=E\left(\sigma_t\sigma_{t-k}^2\varepsilon_{t-k}\right)=\frac{1}{2}\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{\sigma_\eta^2\left(4\psi^k+5\right)}{8\left(1-\psi^2\right)}\right]$$

Hence we get

$$Cov\left(y_t^2,y_{t-k}^2\right)=4c^2\lambda^2Cov\left(\sigma_t,\sigma_{t-k}\right)+2c\left(2\lambda^2+1\right)\left[\lambda Cov\left(\sigma_t^2,\sigma_{t-k}\right)+2Cov\left(\sigma_t,\sigma_{t-k}^2\varepsilon_{t-k}\right)\right]$$
$$+\lambda^2\left(\lambda^2+1\right)Cov\left(\sigma_t^2,\sigma_{t-k}^2\right)+4c^2\lambda Cov\left(\sigma_t,\sigma_{t-k}\varepsilon_{t-k}\right)$$
$$+\left(\lambda^2+1\right)\left[2\lambda Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}\right)+Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)\right]+2c\lambda Cov\left(\sigma_t,\sigma_{t-k}^2\varepsilon_{t-k}^2\right).$$

It follows that $Corr\left(y_t^2, y_{t-k}^2\right)$ is far too complicated. However, under the Efficient Market Hypothesis $c = 0$ and we get

$$Cov\left(y_t^2, y_{t-k}^2\right) = \left(\lambda^2+1\right)\left[\lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + 2\lambda Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right)\right]$$
$$= \lambda^2\left(\lambda^2+1\right)\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$+2\lambda\left(\lambda^2+1\right)\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2\left(\psi^k+1\right)}{(1-\psi^2)}\right]$$
$$+\left(\lambda^2+1\right)\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right],$$

and

$$Corr\left(y_t^2, y_{t-k}^2\right) = \left(\lambda^2+1\right)\frac{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}+2\lambda\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]}{(\lambda^4+6\lambda^2+3)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-\left(\lambda^2+1\right)^2}$$
$$+\frac{(\lambda^2+1)\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}}{(\lambda^4+6\lambda^2+3)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-\left(\lambda^2+1\right)^2},$$

Now if $\rho = c = 0$, we get:

$$Corr\left(y_t^2, y_{t-k}^2\right) = \left(\lambda^2+1\right)^2\frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{\left\{(\lambda^2+1)^2\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)+(4\lambda^2+2)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]\right\}},$$

and

$$\frac{Corr\left(\sigma_t^2, \sigma_{t-k}^2\right)}{Corr\left(y_t^2, y_{t-k}^2\right)} = 1+\frac{(4\lambda^2+2)\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]}{(\lambda^2+1)^2\left\{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}} > 1,$$

i.e. $Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) > Corr\left(y_t^2, y_{t-k}^2\right)$. Further, for $\lambda = \rho = 0$ then

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}$$

Now for $\lambda = 0$ but $c \neq 0$ we get

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{2c\rho\sigma_\eta\psi^{k-1}\exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2\left(4\psi^k+1\right)}{8(1-\psi^2)}\right]}{\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]+4c^2}$$
$$+\frac{\left\{\left(1+\rho^2\sigma_\eta^2\psi^{2k-2}\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]}{\left\{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right\}\exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]+4c^2}.$$

and for $c = \lambda = 0$ we have

$$Corr\left(y_t^2, y_{t-k}^2\right) = \frac{\left(\rho^2\sigma_\eta^2\psi^{2k-2}+1\right)\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{3\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1} > Corr\left(\sigma_t^2, \sigma_{t-k}^2\right).$$

Appendix B

The SV2-M is given by

$$y_t = c + (\lambda + \varepsilon_t)\,\sigma_t,$$
$$\ln\sigma_t^2 = \omega + \psi\ln\sigma_{t-1}^2 + \eta_t = \frac{\omega}{1-\psi} + \sum_{i=0}^{\infty}\psi^i\eta_{t-i}.$$

First notice that if

$$x \sim N\left(A, \left(1-\rho^2\right)\sigma_\eta^2\right)$$

then as

$$\int (x-A)\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\{x-A\}^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx = 0$$

we get that

$$\int x\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\{x-A\}^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$
$$= A\int \frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\{x-A\}^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx = A.$$

Further,

$$E\left[\exp\left(B\eta_t\right)\right] = \int e^{Bx}\frac{1}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{x^2}{2\sigma_\eta^2}\right]dx = \int\frac{1}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{x^2-2B\sigma_\eta^2x}{2\sigma_\eta^2}\right]dx$$
$$= \int\frac{1}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{\left(x^2-B\sigma_\eta^2\right)-B^2\sigma_\eta^4}{2\sigma_\eta^2}\right]dx$$
$$= \exp\left[\frac{B^2\sigma_\eta^4}{2\sigma_\eta^2}\right]\int\frac{1}{\sqrt{2\pi\sigma_\eta^2}}\exp\left[-\frac{\left(x^2-B\sigma_\eta^2\right)}{2\sigma_\eta^2}\right]dx = \exp\left[\frac{B^2\sigma_\eta^2}{2}\right].$$

Notice that

$$E\left(\varepsilon_t\exp\left[B\varepsilon_t\right]\right) = \int xe^{Bx}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x)^2}{2}\right]dx = \int x\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-B)^2-B^2}{2}\right]dx$$
$$= \exp\left[\frac{B^2}{2}\right]\int x\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x-B)^2}{2}\right]dx = B\exp\left(\frac{B^2}{2}\right).$$

Further

$$E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right] = \int e^{Ax}\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\left(x-\rho\sigma_\eta\varepsilon_t\right)^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$

$$= \int\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\left(x-\rho\sigma_\eta\varepsilon_t\right)^2-2A\left(1-\rho^2\right)\sigma_\eta^2 x}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$

$$= \int\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{x^2-2x\left(\rho\sigma_\eta\varepsilon_t+A\left(1-\rho^2\right)\sigma_\eta^2\right)+\left(\rho\sigma_\eta\varepsilon_t\right)^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$

$$= \int\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\exp\left[-\frac{\begin{array}{c}\left[x-\left(\rho\sigma_\eta\varepsilon_t-A\left(1-\rho^2\right)\sigma_\eta^2\right)\right]^2\\ -\left(2\rho\sigma_\eta\varepsilon_t A\left(1-\rho^2\right)\sigma_\eta^2+A^2\left(1-\rho^2\right)^2\sigma_\eta^4\right)\end{array}}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$

$$= \frac{\exp\left[\frac{\left(2\rho\sigma_\eta\varepsilon_t A+A^2\left(1-\rho^2\right)\sigma_\eta^2\right)}{2}\right]}{\sqrt{2\pi\left(1-\rho^2\right)\sigma_\eta^2}}\int\exp\left[-\frac{\left[x-\left(\rho\sigma_\eta\varepsilon_t-A\left(1-\rho^2\right)\sigma_\eta^2\right)\right]^2}{2\left(1-\rho^2\right)\sigma_\eta^2}\right]dx$$

$$= \exp\left[\frac{\left(2\rho\sigma_\eta\varepsilon_t A+A^2\left(1-\rho^2\right)\sigma_\eta^2\right)}{2}\right] = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]\exp\left[\rho\sigma_\eta A\varepsilon_t\right],$$

hence

$$E\left[\varepsilon_t\exp\left(A\eta_t\right)\right] = E\left(\varepsilon_t E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right]\right) = E\left(\varepsilon_t\exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]\exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right)$$

$$= \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]E\left(\varepsilon_t\exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]\rho\sigma_\eta A\exp\left(\frac{\left(\rho\sigma_\eta A\right)^2}{2}\right)$$

$$= \rho\sigma_\eta A\exp\left[\frac{A^2\sigma_\eta^2}{2}\right]. \tag{3.2}$$

Also

$$E\left[\varepsilon_t^2\exp\left[D\varepsilon_t\right]\right] = \int x^2 e^{Dx}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2}{2}\right]dx = \int x^2\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{x^2-2Dx+D^2-D^2}{2}\right]dx$$

$$= \exp\left[\frac{D^2}{2}\right]\int x^2\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{\left(x-D\right)^2}{2}\right]dx = \left(1+D^2\right)\exp\left[\frac{D^2}{2}\right]. \tag{3.3}$$

Further,

$$E\left(\varepsilon_t^2 \exp\left(A\eta_t\right)\right) = E\left(\varepsilon_t^2 E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right] E\left(\varepsilon_t^2 \exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right)$$
$$= \left(1+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right].$$

Additionally,

$$E\left(\varepsilon_t^3 \exp\left(A\eta_t\right)\right) = E\left(\varepsilon_t^3 E\left[\exp\left(A\eta_t\right)|\varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right] E\left(\varepsilon_t^3 \exp\left[\rho\sigma_\eta A\varepsilon_t\right]\right)$$
$$= \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right]\int x^3 \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(x)^2 - 2\rho\sigma_\eta A x}{2}\right]dx$$
$$= \exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\frac{1}{\sqrt{2\pi}}\int\left(x-\left(\rho\sigma_\eta A\right)\right)^3\exp\left[-\frac{\left(x-\rho\sigma_\eta A\right)^2}{2}\right]dx$$
$$+\exp\left[\frac{A^2\sigma_\eta^2}{2}\right]\frac{1}{\sqrt{2\pi}}\int\left[3\left(\rho\sigma_\eta A\right)x^2 - 3\left(\rho\sigma_\eta A\right)^2 x + \left(\rho\sigma_\eta A\right)^3\right]\exp\left[-\frac{\left(x-\rho\sigma_\eta A\right)^2}{2}\right]dx$$
$$= \left(\rho\sigma_\eta A\right)\left(3+\left(\rho\sigma_\eta A\right)^2\right)\exp\left[\frac{A^2\sigma_\eta^2}{2}\right]$$

as

$$x^3 = (x-B)^3 + 3Bx^2 - 3B^2x + B^3.$$

Now as

$$\ln\sigma_t^2 = \frac{\omega}{1-\psi} + \sum_{i=0}^{\infty}\psi^i\eta_{t-i}$$

then

$$E\left[\exp\left(C\ln\sigma_t^2\right)\right] = E\left[\exp\left(C\frac{\omega}{1-\psi} + C\sum_{i=0}^{\infty}\psi^i\eta_{t-i}\right)\right]$$
$$= \exp\left(C\frac{\omega}{1-\psi}\right)E\left[\exp\left(C\sum_{i=0}^{\infty}\psi^i\eta_{t-i}\right)\right] = \exp\left(C\frac{\omega}{1-\psi}\right)\prod_{i=0}^{\infty}E\left[\exp\left(C\psi^i\eta_{t-i}\right)\right]$$
$$= \exp\left(C\frac{\omega}{1-\psi}\right)\prod_{i=0}^{\infty}\exp\left[\frac{C^2\psi^{2i}\sigma_\eta^2}{2}\right] = \exp\left(C\frac{\omega}{1-\psi} + \frac{C^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right). \quad (3.4)$$

It follows that

$$E(\sigma_t) = \exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right], \;\; E\left(\sigma_t^2\right) = \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right]$$
$$E\left(\sigma_t^3\right) = \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{9\sigma_\eta^2}{8(1-\psi^2)}\right], \;\; \textit{and} \;\; E\left(\sigma_t^4\right) = \exp\left[2\frac{\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right]$$

as for the SV1-M model.

Static Moments

First,

$$E\left(\sigma_t^B \varepsilon_t^A\right) = E\left(\exp\left(\frac{B\omega}{2} + \frac{B}{2}\psi \ln \sigma_{t-1}^2 + \frac{B}{2}\eta_t\right)\varepsilon_t^A\right)$$
$$= E\left(\exp\left(\frac{B}{2}\eta_t\right)\varepsilon_t^A\right)\exp\left(\frac{B\omega}{2}\frac{1}{(1-\psi)} + \frac{(B\psi)^2\sigma_\eta^2}{8(1-\psi^2)}\right) \qquad (3.5)$$

and employing equation 3.2 we get

$$E(\sigma_t \varepsilon_t) = \frac{\rho\sigma_\eta}{2}\exp\left(\frac{1}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right).$$

It follows that

$$E(y_t) = E(c + \lambda\sigma_t + \varepsilon_t\sigma_t) = c + \lambda E(\sigma_t) + E(\varepsilon_t\sigma_t)$$
$$= c + \lambda\exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right] + \frac{\rho\sigma_\eta}{2}\exp\left(\frac{1}{2}\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right).$$

Now to find $Var(y_t)$ we need $E(\sigma_t^2\varepsilon_t)$ and $E(\varepsilon_t^2\sigma_t^2)$. Hence employing equations 3.5, 3.2 and 3.3 we get

$$E\left(\sigma_t^2\varepsilon_t\right) = \rho\sigma_\eta\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right)$$
$$E\left(\varepsilon_t^2\sigma_t^2\right) = \left(1 + (\rho\sigma_\eta)^2\right)\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2(1-\psi^2)}\right),$$

and it follows that

$$V(\varepsilon_t^*) = V(\varepsilon_t\sigma_t) = \left\{\left(1 + (\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4(1-\psi^2)}\right).$$

$$
\begin{aligned}
Var\left(y_t\right) = E\left(y_t^2\right) - E^2\left(y_t\right) &= \lambda^2 E\left(\sigma_t^2\right) + 2\lambda E\left(\sigma_t^2 \varepsilon_t\right) + E\left(\varepsilon_t^2 \sigma_t^2\right) \\
&- \left(\lambda^2 E^2\left(\sigma_t\right) + 2\lambda E\left(\sigma_t\right) E\left(\varepsilon_t \sigma_t\right) + E^2\left(\varepsilon_t \sigma_t\right)\right) \\
&= \lambda^2 \left\{ \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - 1 \right\} \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] \\
&+ \lambda \rho \sigma_\eta \left\{ 2\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1 \right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
&+ \left\{ \left(1 + \left(\rho\sigma_\eta\right)^2\right) \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2 \right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).
\end{aligned}
\tag{3.6}
$$

Hence,

$$
\begin{aligned}
Var\left(y_t\right) = E\left(y_t^2\right) - E^2\left(y_t\right) &= \lambda^2 E\left(\sigma_t^2\right) + 2\lambda E\left(\sigma_t^2 \varepsilon_t\right) + E\left(\varepsilon_t^2 \sigma_t^2\right) \\
&- \left(\lambda^2 E^2\left(\sigma_t\right) + 2\lambda E\left(\sigma_t\right) E\left(\varepsilon_t \sigma_t\right) + E^2\left(\varepsilon_t \sigma_t\right)\right) \\
&= \lambda^2 \left\{ \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - 1 \right\} \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] \\
&+ \lambda \rho \sigma_\eta \left\{ 2\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1 \right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
&+ \left\{ \left(1 + \left(\rho\sigma_\eta\right)^2\right) \exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2 \right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).
\end{aligned}
\tag{3.7}
$$

Notice that

$$
\begin{aligned}
E\left(\varepsilon_t^4 \exp\left(A\eta_t\right)\right) &= E\left(\varepsilon_t^4 E\left[\exp\left(A\eta_t\right) | \varepsilon_t\right]\right) = \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right] E\left(\varepsilon_t^4 \exp\left[\rho\sigma_\eta A \varepsilon_t\right]\right) \\
&= \exp\left[\frac{A^2\left(1-\rho^2\right)\sigma_\eta^2}{2}\right] \int x^4 e^{\rho\sigma_\eta A x} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(x)^2}{2}\right] dx \\
&= \exp\left[\frac{A^2\sigma_\eta^2}{2}\right] \int x^4 \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\left(x-\rho\sigma_\eta A\right)^2}{2}\right] dx \\
&= \exp\left[\frac{A^2\sigma_\eta^2}{2}\right] \left[3 + 6\left(\rho\sigma_\eta A\right)^2 + \left(\rho\sigma_\eta A\right)^4\right]
\end{aligned}
$$

as

$$x^4 = (x-B)^4 + 4Bx^3 - 6B^2x^2 + 4B^3x - B^4 \ \ and$$
$$x^3 = (x-B)^3 + 3Bx^2 - 3B^2x + B^3.$$

Employing now equation 3.5 we get:

$$E\left(\sigma_t^4\varepsilon_t^4\right) = E\left(\exp\left(2\omega + 2\psi\ln\sigma_{t-1}^2 + 2\eta_t\right)\varepsilon_t^4\right) = \exp\left(2\omega\right)E\left(\exp\left(2\eta_t\right)\varepsilon_t^4\right)E\left(\exp\left(2\psi\ln\sigma_{t-1}^2\right)\right)$$
$$= \left[3 + 6\left(\rho\sigma_\eta 2\right)^2 + \left(\rho\sigma_\eta 2\right)^4\right]\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right)$$

Notice that

$$E\left(\sigma_t\varepsilon_t - E\left(\sigma_t\varepsilon_t\right)\right)^3 = E\left(\sigma_t^3\varepsilon_t^3 - 3\sigma_t^2\varepsilon_t^2E\left(\sigma_t\varepsilon_t\right) + 3\sigma_t\varepsilon_tE^2\left(\sigma_t\varepsilon_t\right) - E^3\left(\sigma_t\varepsilon_t\right)\right)$$
$$= E\left(\sigma_t^3\varepsilon_t^3\right) - 3E\left(\sigma_t^2\varepsilon_t^2\right)E\left(\sigma_t\varepsilon_t\right) + 2E^3\left(\sigma_t\varepsilon_t\right)$$
$$= \frac{3}{2}\rho\sigma_\eta\left\{\left\{\left(3 + \left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - \left(1 + \left(\rho\sigma_\eta\right)^2\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) + \frac{1}{6}\left(\rho\sigma_\eta\right)^2\right\}$$

and the skewness coefficient is given by

$$sk\left(\varepsilon_t^*\right) = \frac{3}{2}\rho\sigma_\eta\frac{\left\{\left(3 + \left(\frac{3}{2}\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - \left(1 + \left(\rho\sigma_\eta\right)^2\right)\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + \frac{1}{6}\left(\rho\sigma_\eta\right)^2}{\left(\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right)^{3/2}}.$$

Now the kurtosis of the mean error $\varepsilon_t^* = \varepsilon_t\sigma_t$ is given by

$$\kappa\left(\varepsilon_t^*\right) = \frac{E\left(\varepsilon_t\sigma_t - E\left(\varepsilon_t\sigma_t\right)\right)^4}{Var^2\left(\varepsilon_t\sigma_t\right)}$$
$$= \frac{E\left(\varepsilon_t^4\sigma_t^4 - 4\varepsilon_t^3\sigma_t^3E\left(\varepsilon_t\sigma_t\right) + 6\varepsilon_t^2\sigma_t^2E^2\left(\varepsilon_t\sigma_t\right) - 4\varepsilon_t\sigma_tE^3\left(\varepsilon_t\sigma_t\right) + E^4\left(\varepsilon_t\sigma_t\right)\right)}{\left(E\left(\varepsilon_t^2\sigma_t^2\right) - E^2\left(\varepsilon_t\sigma_t\right)\right)^2}$$
$$= \frac{\left[3 + 24\left(\rho\sigma_\eta\right)^2 + 16\left(\rho\sigma_\eta\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right) - 3\left(\rho\sigma_\eta\right)^2\left(3 + \left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right)}{\left\{\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}$$
$$+\frac{6\left(\frac{\rho\sigma_\eta}{2}\right)^2\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\left(\frac{\rho\sigma_\eta}{2}\right)^4}{\left\{\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}.$$

The asymmetry is very complicated for the full model. Hence we assume that $c = 0$ and as

$$E\left(\sigma_t^3\varepsilon_t\right) = \frac{3}{2}\rho\sigma_\eta \exp\left(\frac{3\omega}{2}\frac{1}{(1-\psi)} + \frac{9\sigma_\eta^2}{8(1-\psi^2)}\right) \quad and$$

$$E\left(\varepsilon_t^2\sigma_t^3\right) = \left(1 + \left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{3\omega}{2}\frac{1}{(1-\psi)} + \frac{9\sigma_\eta^2}{8(1-\psi^2)}\right)$$

we have

$$E\left(y_t - E\left(y_t\right)\right)^3 = E\left(\left(\lambda\sigma_t + \varepsilon_t\sigma_t\right) - \left(\lambda E\left(\sigma_t\right) + E\left[\varepsilon_t\sigma_t\right]\right)\right)^3$$

$$= \lambda^3 E\left(\sigma_t^3\right) + 2\lambda^3 E^3\left(\sigma_t\right) - 3\lambda^3 E\left(\sigma_t\right)E\left(\sigma_t^2\right) + 3\lambda^2 E\left(\sigma_t^3\varepsilon_t\right) - 6\lambda^2 E\left(\sigma_t\right)E\left(\sigma_t^2\varepsilon_t\right)$$
$$-3\lambda^2 E\left[\varepsilon_t\sigma_t\right]E\left(\sigma_t^2\right) + 6\lambda^2 E\left(\varepsilon_t\sigma_t\right)E^2\left(\sigma_t\right) - 6\lambda E\left[\varepsilon_t\sigma_t\right]E\left(\sigma_t^2\varepsilon_t\right) + 6\lambda E^2\left(\varepsilon_t\sigma_t\right)E\left(\sigma_t\right)$$
$$+3\lambda E\left(\varepsilon_t^2\sigma_t^3\right) - 3\lambda E\left(\sigma_t\right)E\left(\varepsilon_t^2\sigma_t^2\right) + E\left(\varepsilon_t^3\sigma_t^3\right) + 2E^3\left[\varepsilon_t\sigma_t\right] - 3E\left[\varepsilon_t\sigma_t\right]E\left(\varepsilon_t^2\sigma_t^2\right)$$
$$= \lambda^3\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] + 2\right\}\exp\left[\frac{3\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]$$
$$+\frac{3}{2}\lambda^2\rho\sigma_\eta\left\{3\exp\left(\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right) - 5\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) + 2\right\}\exp\left[\frac{3\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]$$
$$+3\lambda\left\{\begin{array}{c}\left[1 + 2\left(\rho\sigma_\eta\right)^2\right]\left[\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right] \\ +\frac{\rho^2\sigma_\eta^2}{4}\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\end{array}\right\}\exp\left[\frac{3\omega}{2(1-\psi)} + \frac{5\sigma_\eta^2}{8(1-\psi^2)}\right]$$
$$+\frac{9}{2}\rho\sigma_\eta\left(1 + \frac{3}{4}\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{3\omega}{2}\frac{1}{(1-\psi)} + \frac{9\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$+\frac{\rho\sigma_\eta}{2}\left\{\left(\rho\sigma_\eta\right)\left(1 + \frac{\rho\sigma_\eta}{2}\right) - 3\left(1 + \left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right)$$

Now for $\rho = 0$, as in Koopman and Uspensky (2002)

$$E\left(y_t - E\left(y_t\right)\right)^3 = \lambda\left[\begin{array}{c}\lambda^2\left\{\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] + 2\right\} \\ +3\left\{\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right) - 1\right\}\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]\end{array}\right]\exp\left[\frac{3\omega}{2(1-\psi)} + \frac{3\sigma_\eta^2}{8(1-\psi^2)}\right]$$

Notice that the function $f(x) = \exp(3x) + 2 - 3\exp(x)$ is strictly increasing in $\{0, \infty)$ and it follows that $\exp\left[\frac{3\sigma_\eta^2}{4(1-\psi^2)}\right] + 2 - 3\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] > 0$. Hence, the skewness of y_t has the sign of λ.

On the other hand, for $\lambda = 0$ as in JPR in we get

$$E\left(y_t - E\left(y_t\right)\right)^3 = 3\frac{\rho\sigma_\eta}{2}\left\{ \frac{(\rho\sigma_\eta)^2}{3} + D\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right) \right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

where

$$D = \left[3 + \left(\frac{3\rho\sigma_\eta}{2}\right)^2\right]\exp\left(\frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - \left(1 + \left(\rho\sigma_\eta\right)^2\right),$$

and the skewness of y_t has the sign of ρ, i.e. negative under the assumption that the leverage effect is satisfied (see below).

The kurtosis is very complicated for the full model. Hence we assume that $c = 0$and we have

$$\begin{aligned}
E\left(y_t - E\left(y_t\right)\right)^4 &= E\left(\left(\lambda\sigma_t + \varepsilon_t\sigma_t\right) - \left(\lambda E\left(\sigma_t\right) + E\left[\varepsilon_t\sigma_t\right]\right)\right)^4 \\
&= \lambda^4 E\left(\sigma_t^4\right) - 4\lambda^4 E\left(\sigma_t\right)E\left(\sigma_t^3\right) - 3\lambda^4 E^4\left(\sigma_t\right) + 6\lambda^4 E\left(\sigma_t^2\right)E^2\left(\sigma_t\right) \\
&\quad + 4\lambda^3 E\left(\sigma_t^4\varepsilon_t\right) - 12\lambda^3 E\left(\sigma_t\right)E\left(\sigma_t^3\varepsilon_t\right) + 12\lambda^3 E\left(\sigma_t^2\varepsilon_t\right)E^2\left(\sigma_t\right) \\
&\quad - 4\lambda^3 E\left[\varepsilon_t\sigma_t\right]E\left(\sigma_t^3\right) - 12\lambda^3 E^3\left(\sigma_t\right)E\left[\varepsilon_t\sigma_t\right] + 12\lambda^3 E\left(\sigma_t^2\right)E\left(\sigma_t\right)E\left[\varepsilon_t\sigma_t\right] \\
&\quad + 6\lambda^2 E\left(\varepsilon_t^2\sigma_t^2\right)E^2\left(\sigma_t\right) - 12\lambda^2 E\left(\sigma_t\right)E\left(\varepsilon_t^2\sigma_t^3\right) - 12\lambda^2 E\left[\varepsilon_t\sigma_t\right]E\left(\sigma_t^3\varepsilon_t\right) \\
&\quad + 6\lambda^2 E\left(\varepsilon_t^2\sigma_t^4\right) + 24\lambda^2 E\left(\sigma_t^2\varepsilon_t\right)E\left(\sigma_t\right)E\left[\varepsilon_t\sigma_t\right] - 18\lambda^2 E^2\left(\sigma_t\right)E^2\left[\varepsilon_t\sigma_t\right] \\
&\quad + 6\lambda^2 E\left(\sigma_t^2\right)E^2\left[\varepsilon_t\sigma_t\right] + 12\lambda E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\sigma_t\right)E\left[\varepsilon_t\sigma_t\right] + 4\lambda E\left(\varepsilon_t^3\sigma_t^4\right) - 4\lambda E\left(\sigma_t\right)E\left(\varepsilon_t^3\sigma_t^3\right) \\
&\quad - 12\lambda E\left(\sigma_t\right)E^3\left[\varepsilon_t\sigma_t\right] + 12\lambda E\left(\sigma_t^2\varepsilon_t\right)E^2\left[\varepsilon_t\sigma_t\right] - 12\lambda E\left[\varepsilon_t\sigma_t\right]E\left(\varepsilon_t^2\sigma_t^3\right) \\
&\quad + 6E\left(\varepsilon_t^2\sigma_t^2\right)E^2\left[\varepsilon_t\sigma_t\right] - 4E\left[\varepsilon_t\sigma_t\right]E\left(\varepsilon_t^3\sigma_t^3\right) + E\left(\varepsilon_t^4\sigma_t^4\right) - 3E^4\left[\varepsilon_t\sigma_t\right].
\end{aligned}$$

Further

$$\begin{aligned}
Var\left(y_t^2\right) &= \lambda^4 E\left(\sigma_t^4\right) + 4\lambda^3 E\left(\sigma_t^4\varepsilon_t\right) + 6\lambda^2 E\left(\sigma_t^4\varepsilon_t^2\right) + 4\lambda E\left(\sigma_t^4\varepsilon_t^3\right) + E\left(\sigma_t^4\varepsilon_t^4\right) \\
&\quad - 4\lambda^3 E\left(\sigma_t^2\right)E\left(\sigma_t^2\varepsilon_t\right) - 2\lambda^2 E\left(\sigma_t^2\right)E\left(\sigma_t^2\varepsilon_t^2\right) - 4\lambda E\left(\sigma_t^2\varepsilon_t\right)E\left(\sigma_t^2\varepsilon_t^2\right) \\
&\quad - \lambda^4 E^2\left(\sigma_t^2\right) - 4\lambda^2 E^2\left(\sigma_t^2\varepsilon_t\right) - E^2\left(\sigma_t^2\varepsilon_t^2\right)
\end{aligned}$$

For $\lambda = 0$ as in JPR, and as

$$E\left(\sigma_t^3\varepsilon_t^3\right) = \frac{3}{2}\rho\sigma_\eta\left(3 + \left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{9\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

we get

$$E\left(y_t - E\left(y_t\right)\right)^4 = 6E\left(\varepsilon_t^2\sigma_t^2\right)E^2\left[\varepsilon_t\sigma_t\right] - 4E\left[\varepsilon_t\sigma_t\right]E\left(\varepsilon_t^3\sigma_t^3\right) + E\left(\varepsilon_t^4\sigma_t^4\right) - 3E^4\left[\varepsilon_t\sigma_t\right]$$

$$= 3\left(\rho\sigma_\eta\right)^2\left\{\begin{array}{c}\frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\\ -\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\end{array}\right\}\exp\left(2\frac{\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

$$+\left\{\left[3+6\left(\rho\sigma_\eta 2\right)^2 + \left(\rho\sigma_\eta 2\right)^4\right]\exp\left(\frac{3\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 3\left(\frac{\rho\sigma_\eta}{2}\right)^4\right\}\exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right).$$

and

$$Var\left(y_t^2\right) = E\left(\sigma_t^4\varepsilon_t^4\right) - E^2\left(\sigma_t^2\varepsilon_t^2\right)$$

$$= \left\{\left[3+6\left(\rho\sigma_\eta 2\right)^2 + \left(\rho\sigma_\eta 2\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right) - \left(1+\left(\rho\sigma_\eta\right)^2\right)^2\right\}\exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right)$$

and the kurtosis coefficient is given by

$$\kappa\left(y_t\right) = \frac{3\left(\rho\sigma_\eta\right)^2\left\{\begin{array}{c}\frac{1}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\\ -\left(3+\left(\rho\sigma_\eta\frac{3}{2}\right)^2\right)\exp\left(\frac{\sigma_\eta^2}{2(1-\psi^2)}\right)\end{array}\right\}\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2},$$

$$+\frac{C\exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right) - 3\left(\frac{\rho\sigma_\eta}{2}\right)^4}{\left\{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - \left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}^2}$$

where

$$C = 3 + 6\left(\rho\sigma_\eta 2\right)^2 + \left(\rho\sigma_\eta 2\right)^4,$$

For $\rho = 0$ as in Koopman and Uspensky (2002), and taking into account that

$$E\left(\sigma_t^4\varepsilon_t\right) = 2\rho\sigma_\eta\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right),$$

$$E\left(\sigma_t^4\varepsilon_t^2\right) = \left(1+\left(2\rho\sigma_\eta\right)^2\right)\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right),\ and$$

$$E\left(\sigma_t^4\varepsilon_t^3\right) = 2\rho\sigma_\eta\left(3+\left(2\rho\sigma_\eta\right)^2\right)\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right),$$

we get

$$E\left(y_t - E\left(y_t\right)\right)^4 = \lambda^4 \left\{ \begin{array}{c} \left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 4\right\} \exp\left[\frac{5\sigma_\eta^2}{4(1-\psi^2)}\right] \\ +6\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 3 \end{array} \right\} \exp\left[\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right]$$

$$+6\lambda^2 \left\{\exp\left(\frac{5\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 2\exp\left[\frac{\sigma_\eta^2}{2\left(1-\psi^2\right)}\right] + 1\right\} \exp\left[\frac{2\omega}{(1-\psi)} + \frac{3\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]$$

$$+3\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right).$$

and

$$Var\left(y_t^2\right) = \left(\lambda^4 + 6\lambda^2 + 3\right)\exp\left(\frac{2\omega}{(1-\psi)} + \frac{2\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$-\left(2\lambda^2 + \lambda^4 + 1\right)\exp\left(\frac{2\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{(1-\psi^2)}\right)$$

and the kurtosis coefficient is

$$\kappa\left(y_t\right) = \frac{\lambda^4 \left\{ \left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 4\right\} \exp\left[\frac{5\sigma_\eta^2}{4(1-\psi^2)}\right] + 6\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 3 \right\}}{\left\{\lambda^2 \left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\right\}^2}$$
$$+3\frac{2\lambda^2 \left\{ \begin{array}{c} \exp\left(\frac{5\sigma_\eta^2}{4(1-\psi^2)}\right) \\ -2\exp\left[\frac{\sigma_\eta^2}{2(1-\psi^2)}\right] + 1 \end{array} \right\} \exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] + \exp\left(\frac{3\sigma_\eta^2}{2(1-\psi^2)}\right)}{\left\{\lambda^2 \left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right] - 1\right\} + \exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)\right\}^2}$$

Autocorrelation

First, notice that

$$E\left(\varepsilon_t^A\sigma_t^B\varepsilon_{t-1}^C\sigma_{t-1}^D\right) = \exp\left(\frac{\omega}{2}\frac{D+B}{(1-\psi)} + \frac{(B\psi+D)^2\psi^2\sigma_\eta^2}{8(1-\psi^2)}\right)$$
$$\times E\left(\varepsilon_t^A \exp\left(\frac{B}{2}\eta_t\right)\right) E\left(\exp\left(\frac{1}{2}(B\psi+D)\eta_{t-1}\right)\varepsilon_{t-1}^C\right),$$
$$E\left(\varepsilon_t^A\sigma_t^B\sigma_{t-1}^D\right) = E\left(\varepsilon_t^A \exp\left(\frac{B}{2}\eta_t\right)\right)\exp\left((B+D)\frac{\omega}{2(1-\psi)} + \frac{(B\psi+D)^2\sigma_\eta^2}{8(1-\psi^2)}\right),$$
$$E\left(\sigma_t^B\sigma_{t-1}^D\varepsilon_{t-1}^C\right) = E\left(\exp\left(\frac{B}{2}\omega + \frac{B}{2}\psi\ln\sigma_{t-1}^2 + \frac{B}{2}\eta_t\right)\varepsilon_{t-1}^C\sigma_{t-1}^D\right),$$
$$= \exp\left((D+B)\frac{\omega}{2(1-\psi)} + \frac{B^2\psi^4 + 2BD\psi^3 + D^2(\psi^2) + B^2 - \psi^2B^2}{8(1-\psi^2)}\sigma_\eta^2\right)$$
$$\times E\left(\exp\left(\frac{B\psi+D}{2}\eta_{t-1}\right)\varepsilon_{t-1}^C\right),$$
$$E\left(\sigma_t^B\sigma_{t-1}^A\right) = E\left(\exp\left(\frac{B\omega}{2} + \frac{B}{2}\psi\ln\sigma_{t-1}^2 + \frac{B}{2}\eta_t\right)\sigma_{t-1}^A\right)$$
$$= \exp\left((B+A)\frac{\omega}{2(1-\psi)} + \frac{2AB\psi + A^2 + B^2}{8(1-\psi^2)}\sigma_\eta^2\right).$$

It follows that

$$E(\sigma_t\sigma_{t-1}) = \exp\left(\frac{\omega}{(1-\psi)} + \frac{\psi+1}{4(1-\psi^2)}\sigma_\eta^2\right)$$
$$E(\varepsilon_t\sigma_t\sigma_{t-1}) = \frac{1}{2}\rho\sigma_\eta\exp\left(\frac{\omega}{(1-\psi)} + \frac{\psi+1}{4(1-\psi^2)}\sigma_\eta^2\right)$$
$$E(\sigma_t\varepsilon_{t-1}\sigma_{t-1}) = \frac{\psi+1}{2}\rho\sigma_\eta\exp\left(\frac{\omega}{(1-\psi)} + \frac{\psi+1}{4(1-\psi^2)}\sigma_\eta^2\right)$$
$$E(\varepsilon_t\sigma_t\varepsilon_{t-1}\sigma_{t-1}) = \frac{\psi+1}{4}(\rho\sigma_\eta)^2\exp\left(\frac{\omega}{(1-\psi)} + \frac{\psi+1}{4(1-\psi^2)}\sigma_\eta^2\right)$$

The first order autocorrelation is given by

$$
\begin{aligned}
Cov\left(y_t, y_{t-1}\right) &= E\left(\left(c+\left(\lambda+\varepsilon_t\right)\sigma_t\right)\left(c+\left(\lambda+\varepsilon_{t-1}\right)\sigma_{t-1}\right)\right) - E^2\left(y_t\right) \\
&= \lambda^2 E\left(\sigma_t\sigma_{t-1}\right) + \lambda E\left(\varepsilon_t\sigma_t\sigma_{t-1}\right) + \lambda E\left(\sigma_t\varepsilon_{t-1}\sigma_{t-1}\right) \\
&+ E\left(\varepsilon_t\sigma_t\varepsilon_{t-1}\sigma_{t-1}\right) - \left(\lambda^2 E^2\left(\sigma_t\right) + E^2\left(\varepsilon_t\sigma_t\right) + 2\lambda E\left(\sigma_t\right)E\left(\varepsilon_t\sigma_t\right)\right) \\
&= \lambda^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] \\
&+ \left(\frac{\rho\sigma_\eta}{2}\right)\left\{\begin{array}{c}\lambda\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right) \\ +\frac{\rho\sigma_\eta}{2}\left[\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right]\end{array}\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
&+ \frac{1}{2}\lambda\rho\sigma_\eta\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-2\right\}\exp\left[\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]
\end{aligned}
$$

and employing equation 3.7 we get

$$
\begin{aligned}
Corr\left(y_t, y_{t-1}\right) &= \frac{\lambda^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right\}+\frac{1}{2}\lambda\rho\sigma_\eta\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-2\right\}}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+G-\left(\frac{\rho\sigma_\eta}{2}\right)^2} \\
&+\frac{\left(\frac{\rho\sigma_\eta}{2}\right)\left\{\lambda\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)+\frac{\rho\sigma_\eta}{2}\left[\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi\lambda^2)}\right)-1\right]\right\}}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+G-\left(\frac{\rho\sigma_\eta}{2}\right)^2},
\end{aligned}
$$

where

$$
G = \left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]+\lambda\rho\sigma_\eta\left\{2\exp\left(\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}.
$$

For $\lambda = 0$ as in JPR

$$
Corr\left(y_t, y_{t-1}\right) = \frac{\left(\frac{\rho\sigma_\eta}{2}\right)^2\left[\left(\psi+1\right)\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right]}{\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-\left(\frac{\rho\sigma_\eta}{2}\right)^2} > 0
$$

for any value of ρ.

For $\rho = 0$ as in Koopman and Uspensky (2002)

$$
Corr\left(y_t, y_{t-1}\right) = \frac{\lambda^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4(1-\psi^2)}\right)-1\right\}}{\lambda^2\left\{\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]-1\right\}+\exp\left[\frac{\sigma_\eta^2}{4(1-\psi^2)}\right]} > 0
$$

for any value of λ.

Leverage $Cov\left(\sigma_t^2, \sigma_{t-1}\varepsilon_{t-1}\right)$

Now

$$
\begin{aligned}
E\left(\sigma_t^2\sigma_{t-1}\varepsilon_{t-1}\right) &= E\left(\exp\left[\ln\sigma_t^2\right]\exp\left[\frac{1}{2}\ln\sigma_{t-1}^2\right]\varepsilon_{t-1}\right) \\
&= \exp(\omega)\,E\left(\exp[\eta_t]\exp\left[\left(\psi+\frac{1}{2}\right)\ln\sigma_{t-1}^2\right]\varepsilon_{t-1}\right) \\
&= \exp(\omega)\,E\left(\exp[\eta_t]\exp\left[\left(\psi+\frac{1}{2}\right)\left[\omega+\psi\ln\sigma_{t-2}^2+\eta_{t-1}\right]\right]\varepsilon_{t-1}\right) \\
&= \exp\left[\omega+\left(\psi+\frac{1}{2}\right)\omega\right]E\left(\exp[\eta_t]\right)E\left(\exp\left[\left(\psi+\frac{1}{2}\right)\psi\ln\sigma_{t-2}^2\right]\right) \\
&\quad \times E\left(\exp\left[\left(\psi+\frac{1}{2}\right)\eta_{t-1}\right]\varepsilon_{t-1}\right) \\
&= \exp\left[\omega+\left(\psi+\frac{1}{2}\right)\omega+\frac{\sigma_\eta^2}{2}\right]\exp\left(\left(\psi+\frac{1}{2}\right)\psi\frac{\omega}{(1-\psi)}+\frac{\left(\psi+\frac{1}{2}\right)^2\psi^2\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) \\
&\quad \times\left(\psi+\frac{1}{2}\right)\rho\sigma_\eta\exp\left[\frac{\left(\psi+\frac{1}{2}\right)^2\sigma_\eta^2}{2}\right] \\
&= \left(\psi+\frac{1}{2}\right)\rho\sigma_\eta\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{4\psi+5}{8\left(1-\psi^2\right)}\sigma_\eta^2\right]
\end{aligned}
$$

Hence

$$
\begin{aligned}
Cov\left(\sigma_t^2,\sigma_{t-1}\varepsilon_{t-1}\right) &= \left(\psi+\frac{1}{2}\right)\rho\sigma_\eta\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{4\psi+5}{8\left(1-\psi^2\right)}\sigma_\eta^2\right] \\
&\quad -\frac{\rho\sigma_\eta}{2}\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
&= \rho\sigma_\eta\left\{\left(\psi+\frac{1}{2}\right)\exp\left[\frac{4\psi}{8\left(1-\psi^2\right)}\sigma_\eta^2\right]-\frac{1}{2}\right\}\exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right].
\end{aligned}
$$

and

$$
Corr\left(\sigma_t^2,\sigma_{t-1}\varepsilon_{t-1}\right)=\frac{\rho\sigma_\eta\left\{\left(\psi+\frac{1}{2}\right)\exp\left[\frac{4\psi}{8(1-\psi^2)}\sigma_\eta^2\right]-\frac{1}{2}\right\}}{\sqrt{\left[\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right]\left\{\left(1+(\rho\sigma_\eta)^2\right)\exp\left(\frac{\sigma_\eta^2}{4(1-\psi^2)}\right)-\left(\frac{\rho\sigma_\eta}{2}\right)^2\right\}}},
$$

which has the sign of ρ, provide that $\psi > 0$ due to volatility clustering. Consequently, the leverage effect is satisfied if and only if $\rho < 0$.

Dynamic Asymmetry $Cov\left(y_t^2, y_{t-1}\right)$

Now

$$Cov\left(y_t^2, y_{t-1}\right) = Cov\left(c^2 + 2c\left(\lambda + \varepsilon_t\right)\sigma_t + \left(\lambda + \varepsilon_t\right)^2 \sigma_t^2, c + \left(\lambda + \varepsilon_{t-1}\right)\sigma_{t-1}\right)$$
$$= 2c\lambda^2 Cov\left(\sigma_t, \sigma_{t-1}\right) + \lambda^3 Cov\left(\sigma_t^2, \sigma_{t-1}\right) + 2c\lambda Cov\left(\sigma_t, \varepsilon_{t-1}\sigma_{t-1}\right) + \lambda Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-1}\right)$$
$$+2\lambda^2 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-1}\right) + 2cCov\left(\varepsilon_t\sigma_t, \varepsilon_{t-1}\sigma_{t-1}\right) + 2c\lambda Cov\left(\varepsilon_t\sigma_t, \sigma_{t-1}\right)$$
$$+\lambda^2 Cov\left(\sigma_t^2, \varepsilon_{t-1}\sigma_{t-1}\right) + 2\lambda Cov\left(\varepsilon_t\sigma_t^2, \varepsilon_{t-1}\sigma_{t-1}\right) + Cov\left(\varepsilon_t^2\sigma_t^2, \varepsilon_{t-1}\sigma_{t-1}\right).$$

First, we need

$$Cov\left(\sigma_t^A, \sigma_{t-1}\right) = E\left(\sigma_t^A\sigma_{t-1}\right) - E\left(\sigma_t^A\right)E\left(\sigma_{t-1}\right)$$
$$= \left\{\exp\left(\frac{A\psi}{4\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\}\exp\left(\frac{A+1}{2}\frac{\omega}{1-\psi} + \frac{\left(A^2+1\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).$$

Second,

$$Cov\left(\varepsilon_t^C\sigma_t^A, \sigma_{t-1}\right) = E\left(\varepsilon_t^C\sigma_t^A\sigma_{t-1}\right) - E\left(\varepsilon_t^C\sigma_t^A\right)E\left(\sigma_{t-1}\right)$$
$$= E\left(\varepsilon_t^C\exp\left(\frac{A}{2}\eta_t\right)\right)\left\{\exp\left(\frac{A\psi}{4\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\}\exp\left(\frac{(A+1)}{2}\frac{\omega}{(1-\psi)} + \frac{(A\psi)^2+1}{8\left(1-\psi^2\right)}\sigma_\eta^2\right).$$

Hence we have that for $C = 1$ $A = 1$

$$Cov\left(\varepsilon_t\sigma_t, \sigma_{t-1}\right) = \frac{1}{2}\rho\sigma_\eta\left\{\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right),$$

for $C = 1$ $A = 2$

$$Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-1}\right) = \rho\sigma_\eta\left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

and $C = 2$ $A = 1$

$$Cov\left(\varepsilon_t^2\sigma_t, \sigma_{t-1}\right) = \left(1 + \left(\frac{1}{2}\rho\sigma_\eta\right)^2\right)\left\{\exp\left(\frac{\psi}{4\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$

and $C = 2$ $A = 2$

$$Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-1}\right) = \left(1 + \left(\rho\sigma_\eta\right)^2\right)\left\{\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\}\exp\left(\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).$$

Further,

$$Cov\left(\sigma_t, \varepsilon_{t-1}\sigma_{t-1}\right) = E\left(\sigma_t\varepsilon_{t-1}\sigma_{t-1}\right) - E\left(\sigma_t\right)E\left(\varepsilon_{t-1}\sigma_{t-1}\right)$$
$$= \frac{\rho\sigma_\eta}{2}\left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right).$$

Finally,

$$Cov\left(\varepsilon_t\sigma_t, \varepsilon_{t-1}\sigma_{t-1}\right) = E\left(\varepsilon_t\sigma_t\varepsilon_{t-1}\sigma_{t-1}\right) - E\left(\varepsilon_t\sigma_t\right)Cov\left(\varepsilon_{t-1}\sigma_{t-1}\right)$$
$$= \left(\frac{\rho\sigma_\eta}{2}\right)^2\left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right),$$

and as

$$E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-1}\sigma_{t-1}\right) = \rho\sigma_\eta\frac{(2\psi+1)}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(3\frac{\omega}{2(1-\psi)}+\frac{4\psi+5}{8\left(1-\psi^2\right)}\sigma_\eta^2\right)$$

we get

$$Cov\left(\varepsilon_t^2\sigma_t^2, \varepsilon_{t-1}\sigma_{t-1}\right) = E\left(\varepsilon_t^2\sigma_t^2\varepsilon_{t-1}\sigma_{t-1}\right) - E\left(\varepsilon_t^2\sigma_t^2\right)E\left(\varepsilon_{t-1}\sigma_{t-1}\right)$$
$$= \frac{\rho\sigma_\eta}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

and

$$Cov\left(\varepsilon_t\sigma_t^2, \varepsilon_{t-1}\sigma_{t-1}\right) = E\left(\varepsilon_t\sigma_t^2\varepsilon_{t-1}\sigma_{t-1}\right) - E\left(\varepsilon_t\sigma_t^2\right)E\left(\varepsilon_{t-1}\sigma_{t-1}\right)$$
$$= \frac{\left(\rho\sigma_\eta\right)^2}{2}\left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).$$

Also

$$Cov\left(\sigma_t^2\sigma_{t-1}\right) = \left[\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right)-1\right]\exp\left[\frac{3\omega}{2(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right]$$

Hence

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-1}\right) = {} & 2c\lambda^2 \left\{\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\} \exp\left[\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right] \\
& + \lambda^3 \left[\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right] \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
& + 2c\lambda\frac{\rho\sigma_\eta}{2} \left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
& + \lambda\left(1 + (\rho\sigma_\eta)^2\right) \left\{\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& + 2\lambda^2\rho\sigma_\eta \left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& + 2c\left(\frac{\rho\sigma_\eta}{2}\right)^2 \left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
& + c\lambda\rho\sigma_\eta \left\{\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{\omega}{(1-\psi)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right) \\
& + \lambda^2\rho\sigma_\eta \left\{\left(\psi + \frac{1}{2}\right)\exp\left[\frac{4\psi}{8\left(1-\psi^2\right)}\sigma_\eta^2\right] - \frac{1}{2}\right\} \exp\left[\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
& + 2\lambda\frac{(\rho\sigma_\eta)^2}{2} \left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& + \frac{\rho\sigma_\eta}{2}\left(1 + (\rho\sigma_\eta)^2\right) \left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).
\end{aligned}
$$

Now for $c = 0$ (Efficient Market) we get

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-1}\right) = {} & \lambda^3 \left[\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right] \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
& +2\lambda^2\rho\sigma_\eta \left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +\lambda^2\rho\sigma_\eta \left\{\left(\psi + \frac{1}{2}\right) \exp\left[\frac{4\psi}{8\left(1-\psi^2\right)}\sigma_\eta^2\right] - \frac{1}{2}\right\} \exp\left[\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] \\
& +\lambda\left(1 + \left(\rho\sigma_\eta\right)^2\right) \left\{\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +\lambda\left(\rho\sigma_\eta\right)^2 \left\{\left(2\psi + 1\right)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
& +\frac{\rho\sigma_\eta}{2}\left(1 + \left(\rho\sigma_\eta\right)^2\right) \left\{\left(2\psi + 1\right)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right).
\end{aligned}
$$

which is very complicated.

On the other hand for $\rho = 0$ and $c \neq 0$ as in Koopman and Uspensky (2002)

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-1}\right) = {} & \lambda \left\{\begin{array}{c} \lambda^2 \exp\left[\frac{\omega}{2(1-\psi)} + \frac{\sigma_\eta^2}{8(1-\psi^2)}\right] \\ +2c\lambda + \exp\left(\frac{\sigma_\eta^2}{8(1-\psi^2)}\right) \end{array}\right\} \left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \\
& \times \exp\left[\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right].
\end{aligned}
$$

It is obvious that depending on the value of c and $Cov\left(y_t^2, y_{t-1}\right)$ could have either the sign or the opposite one of λ. If further, $\rho = c = 0$ then

$$
\begin{aligned}
Cov\left(y_t^2, y_{t-1}\right) = {} & \lambda \left\{\lambda^2 \exp\left[\frac{\omega}{2\left(1-\psi\right)} + \frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] + \exp\left(\frac{\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)\right\} \\
& \times \left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left[\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right]
\end{aligned}
$$

and $Cov\left(y_t^2, y_{t-1}\right)$ can have only the sign of λ.

For $\lambda = 0$ but $\rho \neq 0$ and $c \neq 0$ we get

$$Cov\left(y_t^2, y_{t-1}\right) = \frac{c\left(\rho\sigma_\eta\right)^2}{2}\left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right)$$
$$+\frac{\rho\sigma_\eta}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and $Cov\left(y_t^2, y_{t-1}\right)$ has probably the sign of ρ. If additionally $c = 0$, as in JPR we get

$$Cov\left(y_t^2, y_{t-1}\right) = \frac{\rho\sigma_\eta}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\left\{(2\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}$$
$$\times\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right)$$

and $Cov\left(y_t^2, y_{t-1}\right)$ has the sign of ρ, negative under the leverage hypothesis.

Volatility Clustering

It is easy to prove that

$$Corr\left(\sigma_t^2, \sigma_{t-k}^2\right) = \frac{\exp\left[\frac{\sigma_\eta^2\psi^k}{(1-\psi^2)}\right]-1}{\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1}.$$

dynamic kurtosis

In terms of dynamic kurtosis notice that

$$Cov\left(y_t^2, y_{t-k}^2\right) = 4c^2\lambda Cov\left(\sigma_t, \sigma_{t-k}\varepsilon_{t-k}\right) + 4c^2 Cov\left(\sigma_t\varepsilon_t, \sigma_{t-k}\varepsilon_{t-k}\right) + 4c^2\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}\right)$$
$$+4c\lambda^2 Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2c\lambda^3 Cov\left(\sigma_t, \sigma_{t-k}^2\right) + 2c\lambda^2 Cov\left(\sigma_t\varepsilon_t, \sigma_{t-k}^2\right)$$
$$+4c\lambda Cov\left(\sigma_t\varepsilon_t, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 4c^2\lambda Cov\left(\sigma_t\varepsilon_t, \sigma_{t-k}\right) + 2c\lambda Cov\left(\sigma_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right)$$
$$+2c Cov\left(\sigma_t\varepsilon_t, \sigma_{t-k}^2\varepsilon_{t-k}^2\right) + 2c\lambda^3 Cov\left(\sigma_t^2, \sigma_{t-k}\right) + 2c\lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) + 4c\lambda^2 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}\right)$$
$$+4c\lambda Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right) + 2c\lambda Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}\right) + 2c Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}\varepsilon_{t-k}\right)$$
$$+\lambda^4 Cov\left(\sigma_t^2, \sigma_{t-k}^2\right) + 2\lambda^3 Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + \lambda^2 Cov\left(\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right)$$
$$+2\lambda^3 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\right) + 4\lambda^2 Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + 2\lambda Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right)$$
$$+\lambda^2 Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\right) + 2\lambda Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}\right) + Cov\left(\varepsilon_t^2\sigma_t^2, \sigma_{t-k}^2\varepsilon_{t-k}^2\right).$$

However, for the SV2-M model we have that

$$Cov\left(\varepsilon_t\sigma_t, f\left(\varepsilon_{t-k},\sigma_{t-k}\right)\right)\neq 0,\ Cov\left(\varepsilon_t\sigma_t^2, f\left(\varepsilon_{t-k},\sigma_{t-k}\right)\right)\neq 0\ \ and$$
$$Cov\left(\varepsilon_t^2\sigma_t^2, f\left(\varepsilon_{t-k},\sigma_{t-k}\right)\right)\neq Cov\left(\sigma_t^2, f\left(\varepsilon_{t-k},\sigma_{t-k}\right)\right).$$

Now under the Efficient Market Hypothesis, i.e. $c=0$, and for $k=1$, we get

$$\begin{aligned}Cov\left(y_t^2,y_{t-k}^2\right) &= \lambda^4 Cov\left(\sigma_t^2,\sigma_{t-k}^2\right)+2\lambda^3 Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}\right)+\lambda^2 Cov\left(\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)\\ &+2\lambda^3 Cov\left(\varepsilon_t\sigma_t^2,\sigma_{t-k}^2\right)+4\lambda^2 Cov\left(\varepsilon_t\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}\right)+2\lambda Cov\left(\varepsilon_t\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)\\ &+\lambda^2 Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-k}^2\right)+2\lambda Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}\right)+Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-k}^2\varepsilon_{t-k}^2\right)\end{aligned}$$

and

$$\begin{aligned}Var\left(y_t^2\right) &= \lambda^4 E\left(\sigma_t^4\right)+4\lambda^3 E\left(\sigma_t^4\varepsilon_t\right)+6\lambda^2 E\left(\sigma_t^4\varepsilon_t^2\right)+4\lambda E\left(\sigma_t^4\varepsilon_t^3\right)+E\left(\sigma_t^4\varepsilon_t^4\right)\\ &-4\lambda^3 E\left(\sigma_t^2\right)E\left(\sigma_t^2\varepsilon_t\right)-2\lambda^2 E\left(\sigma_t^2\right)E\left(\sigma_t^2\varepsilon_t^2\right)-4\lambda E\left(\sigma_t^2\varepsilon_t\right)E\left(\sigma_t^2\varepsilon_t^2\right)\\ &-\lambda^4 E^2\left(\sigma_t^2\right)-4\lambda^2 E^2\left(\sigma_t^2\varepsilon_t\right)-E^2\left(\sigma_t^2\varepsilon_t^2\right).\end{aligned}$$

Further for $\lambda=0$, as well, as in JPR, we get

$$\begin{aligned}&Cov\left(y_t^2,y_{t-1}^2\right)=Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-1}^2\varepsilon_{t-1}^2\right)\\ &=\left(1+(\rho\sigma_\eta)^2\right)\left\{\left(1+\left((\psi+1)\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)\right\}\\ &\times\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right)\end{aligned}$$

and as

$$\begin{aligned}Var\left(y_t^2\right) &= \left\{\left[3+6\left(2\rho\sigma_\eta\right)^2+\left(2\rho\sigma_\eta\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)^2\right\}\\ &\times\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right)\end{aligned}$$

$$Corr\left(y_t^2,y_{t-1}^2\right)=\frac{\left(1+(\rho\sigma_\eta)^2\right)\left\{\left(1+\left((\psi+1)\rho\sigma_\eta\right)^2\right)\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)\right\}}{\left\{\left[3+6\left(2\rho\sigma_\eta\right)^2+\left(2\rho\sigma_\eta\right)^4\right]\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(1+(\rho\sigma_\eta)^2\right)^2\right\}}.$$

For $c \neq 0$ and $\rho = 0$ as in Koopman and Usp, first notice that

$$Cov\left(\sigma_t, \sigma_{t-1}\right) = \left\{\exp\left(\frac{\psi}{4\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(\frac{\omega}{1-\psi} + \frac{\sigma_\eta^2}{4\left(1-\psi^2\right)}\right),$$

$$Cov\left(\sigma_t, \sigma_{t-1}^2\right) = \left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],$$

$$Cov\left(\sigma_t^2, \sigma_{t-1}^2\right) = \left\{\exp\left(\frac{\psi\sigma_\eta^2}{\left(1-\psi^2\right)}\right) - 1\right\} \exp\left[\frac{2\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right],$$

$$Cov\left(\sigma_t, \sigma_{t-1}^2\varepsilon_{t-1}\right) = \rho\sigma_\eta \left\{\frac{\psi+2}{2}\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right) - 1\right\} \exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],$$

$$Cov\left(\sigma_t^2, \sigma_{t-1}\varepsilon_{t-1}\right) = \frac{\rho\sigma_\eta}{2}\left\{\left(2\psi+1\right)\exp\left(\frac{\left(\psi+2\right)\psi}{4\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

$$Cov\left(\sigma_t, \sigma_{t-1}^2\varepsilon_{t-1}^2\right) = \left(1 + \left(\frac{\psi+2}{2}\rho\sigma_\eta\right)^2\right)\exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{\left(4\psi+5\right)\sigma_\eta^2}{8\left(1-\psi^2\right)}\right] - \left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left[\frac{3\omega}{2\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right],$$

$$Cov\left(\sigma_t\varepsilon_t, \sigma_{t-1}^2\right) = \frac{\rho\sigma_\eta}{2}\left\{\exp\left(\frac{\psi}{2\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(\frac{3}{2}\frac{\omega}{\left(1-\psi\right)} + \frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),$$

$$Cov\left(\sigma_t^2\varepsilon_t, \sigma_{t-1}^2\right) = \rho\sigma_\eta\left\{\exp\left(\frac{\psi}{\left(1-\psi^2\right)}\sigma_\eta^2\right) - 1\right\} \exp\left(2\frac{\omega}{\left(1-\psi\right)} + \frac{\sigma_\eta^2}{\left(1-\psi^2\right)}\right),$$

$$
\begin{aligned}
Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-1}^2\varepsilon_{t-1}\right) &= \left(\rho\sigma_\eta\right)^2\left\{(\psi+1)\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\psi}{(1-\psi^2)}\sigma_\eta^2\right)-1\right\} \\
&\quad\times\exp\left(2\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right), \\
Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-1}^2\varepsilon_{t-1}^2\right) &= \rho\sigma_\eta\left(1+\left(\rho\sigma_\eta\left(\psi+1\right)\right)^2\right)\exp\left(\frac{2\omega}{(1-\psi)}+\frac{(\psi+1)\,\sigma_\eta^2}{(1-\psi^2)}\right) \\
&\quad-\rho\sigma_\eta\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right), \\
Cov\left(\varepsilon_t\sigma_t, \sigma_{t-1}^2\varepsilon_{t-1}\right) &= \frac{\left(\rho\sigma_\eta\right)^2}{2}\left\{\frac{1}{2}\left(\psi+2\right)\exp\left(\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\}\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
Cov\left(\varepsilon_t\sigma_t, \sigma_{t-1}^2\varepsilon_{t-1}^2\right) &= \frac{\rho\sigma_\eta}{2}\left(1+\left(\frac{1}{2}\left(\psi+2\right)\rho\sigma_\eta\right)^2\right)\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{(4\psi+5)\,\sigma_\eta^2}{8\left(1-\psi^2\right)}\right) \\
&\quad-\frac{\rho\sigma_\eta}{2}\left(1+\left(\rho\sigma_\eta\right)^2\right)\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right), \\
Cov\left(\varepsilon_t\sigma_t^2, \sigma_{t-1}\varepsilon_{t-1}\right) &= \frac{\left(\rho\sigma_\eta\right)^2}{2}\left\{(2\psi+1)\exp\left(\frac{\omega}{2}\frac{1+2}{(1-\psi)}+\frac{\psi\sigma_\eta^2}{2\left(1-\psi^2\right)}\right)-1\right\} \\
&\quad\times\exp\left(\frac{3}{2}\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\left(1-\psi^2\right)}\right),
\end{aligned}
$$

$$Cov\left(\sigma_t^2,\sigma_{t-1}^2\varepsilon_{t-1}\right)=\rho\sigma_\eta\left\{(\psi+1)\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}\exp\left(2\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right),$$

$$Cov\left(\sigma_t^2,\sigma_{t-1}^2\varepsilon_{t-1}^2\right)=\left(1+\left(\rho\sigma_\eta\left(\psi+1\right)\right)^2\right)\exp\left[2\frac{\omega}{(1-\psi)}+\frac{(\psi+1)\,\sigma_\eta^2}{(1-\psi^2)}\right]$$
$$-\left(1+(\rho\sigma_\eta)^2\right)\exp\left[2\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right],$$

$$Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-1}^2\right)=\left(1+(\rho\sigma_\eta)^2\right)\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}\exp\left(2\frac{\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right),$$

$$Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-1}^2\varepsilon_{t-1}\right)=\rho\sigma_\eta\left(1+(\rho\sigma_\eta)^2\right)\left((\psi+1)\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\psi+1}{(1-\psi^2)}\sigma_\eta^2\right)-1\right)$$
$$\times\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right),$$

$$Cov\left(\varepsilon_t^2\sigma_t^2,\sigma_{t-1}^2\varepsilon_{t-1}^2\right)=\left(1+(\rho\sigma_\eta)^2\right)\left(1+\left((\psi+1)\,\rho\sigma_\eta\right)^2\right)\exp\left(\frac{2\omega}{(1-\psi)}+\frac{(\psi+1)\,\sigma_\eta^2}{(1-\psi^2)}\right)$$
$$-\left(1+(\rho\sigma_\eta)^2\right)^2\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right),$$

and we get

$$Cov\left(y_t^2,y_{t-1}^2\right)=4c^2\lambda^2\left\{\exp\left(\frac{\psi}{4\,(1-\psi^2)}\sigma_\eta^2\right)-1\right\}\exp\left(\frac{\omega}{1-\psi}+\frac{\sigma_\eta^2}{4\,(1-\psi^2)}\right)$$
$$+2c\lambda\left\{\left(2\lambda^2+1\right)\exp\left[\frac{\omega}{2\,(1-\psi)}\right]+1\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{5\sigma_\eta^2}{8\,(1-\psi^2)}\right)\left\{\exp\left(\frac{\psi\sigma_\eta^2}{2\,(1-\psi^2)}\right)-1\right\}$$
$$+\lambda^4\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}\exp\left[\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right]+$$
$$+\left(2\lambda^2+1\right)\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right).$$

and

$$Var\left(y_t^2\right)=\left\{\left(\lambda^4+6\lambda^2+3\right)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-\left(\lambda^4+2\lambda^2+1\right)\right\}\exp\left(\frac{2\omega}{(1-\psi)}+\frac{\sigma_\eta^2}{(1-\psi^2)}\right)$$

and

$$Cov\left(y_t^2, y_{t-1}^2\right) = 2c\lambda \frac{\left\{2c\lambda + \left\{(2\lambda^2+1)\exp\left[\frac{\omega}{2(1-\psi)}\right]+1\right\}\exp\left(\frac{3\sigma_\eta^2}{8(1-\psi^2)}\right)\right\}\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}}{\left\{(\lambda^4+6\lambda^2+3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-(\lambda^4+2\lambda^2+1)\right\}\exp\left(\frac{\omega}{(1-\psi)}+\frac{6\sigma_\eta^2}{8(1-\psi^2)}\right)}$$
$$+\frac{\lambda^4+2\lambda^2+1}{\left\{(\lambda^4+6\lambda^2+3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-(\lambda^4+2\lambda^2+1)\right\}}\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}$$

For $c=\rho=0$ we get

$$Corr\left(y_t^2, y_{t-1}^2\right) = \frac{(\lambda^2+1)^2\left\{\exp\left(\frac{\psi\sigma_\eta^2}{(1-\psi^2)}\right)-1\right\}}{\left\{(\lambda^4+6\lambda^2+3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-(\lambda^4+2\lambda^2+1)\right\}}$$

$$\frac{Corr\left(\sigma_t^2, \sigma_{t-1}^2\right)}{Corr\left(y_t^2, y_{t-1}^2\right)} = \frac{(\lambda^4+6\lambda^2+3)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)-(\lambda^4+2\lambda^2+1)}{(\lambda^4+2\lambda^2+1)\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)}$$
$$= 1 + \frac{(4\lambda^2+2)\exp\left(\frac{\sigma_\eta^2}{(1-\psi^2)}\right)}{(\lambda^4+2\lambda^2+1)\left(\exp\left[\frac{\sigma_\eta^2}{(1-\psi^2)}\right]-1\right)} > 1,$$

the volatility clustering in higher than the conditional kurtosis, at least for the first order.

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