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ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Spatial Growth: The Distribution of Capital across Locations when Saving Rates are Exogenous

Anastasios Xepapadeas

Athanasios Yannacopoulos

Andreas Ioannidis

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Spatial Growth: The Distribution of Capital across Locations when Saving Rates are Exogenous*

A. Xepapadeas, A. Yannacopoulos and A. Ioannidis
Athens University of Economics and Business

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Abstract

Economic growth has traditionally been analyzed in the temporal domain, while the spatial dimension is captured by cross-country income differences. Data suggest great inequality in income per capita across countries, with a slight but noticeable increase over time (Acemoglu 2009). Seeking to explore the mechanism underlying the temporal evolution of the cross sectional distribution of economies, we develop a spatial growth model where saving rates are exogenous. Capital movements across locations are governed by having capital moving towards locations of relatively higher marginal productivity, with a velocity determined by the existing stock of capital and its marginal productivity. This mechanism leads to a capital accumulation equation augmented by a nonlinear diffusion term, which characterizes spatial movements. Our results suggest that under diminishing returns, the growth process leads to a stable spatially non-homogenous distribution for per capita capital and income in the long run. AK production functions and increasing returns lead to strong persistent and increasing concentration of capital in a very few locations. Insufficient savings may lead to the emergence of poverty cores where capital stock is depleted in some locations.

Keywords: Economic growth, space, capital flows, nonlinear diffusion, Solow model, steady-state distributions, stability.

JEL Classification: O4, R1, C6

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1 Introduction

In formal growth models, economic growth has traditionally been analyzed in the temporal domain with the main focus of analysis being the development of models capable of explaining stylized facts, expressed in terms of the temporal evolution of key variables such as output or capital per capita or the capital-labor ratio. A central issue, however, is cross-country income differences which exemplify the spatial dimension of the problem. Acemoglu [1], Chapter 1, using data on GDP per capita and per worker since 1960, points out that there is great inequality in income per capita and income per worker across countries, which has been slowly but noticeably increasing over time. The geographical or spatial dimension is also taken into account in the context of convergence. Data suggest that there is no unconditional convergence during the post-war period (e.g. Acemoglu [1]). However, the results from Barro and Sala-i-Martin [2] suggest that conditional convergence takes place with poor countries growing faster than rich ones in terms of per capita GDP within a group that shares similar characteristics. Conditional convergence even at different steady states may not, however, adequately describe the evolution of the spatial distribution of per capita GDP across countries. Quah [3, 4, 5] points out that convergence concerns poor economies catching up with rich ones and that what one wants to know is what happens to the entire cross sectional distribution of economies, not whether a single economy is tending towards its own, individual steady state.

A cross sectional distribution of real GDP per capita in eleven regions of the world from 1980 to 2011 is presented in Figure 1.[1]

[Figure 1. Spatial distribution of GDP per capita, by world regions]

Figure 1 suggests that in the thirty year period covered by the data, the distribution of per capita GDP seems to be becoming less uniform and that the evolution of the spatial distribution could be regarded as an approximately symmetric distribution around an increasing peak that corresponds to North America. Figure 2 depicts a similar distribution for high income countries.[2]

[1]The World Bank data base was used for GDP per capita (PPP constant 2005 international $). The regions, according to the World Bank classification, are: Arab World, ARB; Caribbean small states, CSS; East Asia & Pacific (developing only), EAP; European Union, EUU; Europe & Central Asia (developing only), ECA; Latin America & Caribbean (developing only), LAC; Middle East & North Africa (developing only), MNA; North America, NAC; Pacific Island Small states, PSS; South Asia, SAS; Sub-Saharan Africa (developing only), SSA.

For better visualization, in Figures 1 and 2 the region or country with the highest average GDP per capita for the sample period was placed in the middle and the rest of the regions on either side in order of descending average GDP per capita.

[2]The group of high income countries includes Australia, Austria, Belgium, Canada, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hungary, Iceland,

[Figure 2. Spatial distribution of GDP per capita, developed countries]

As a measure of the evolution of the regional or spatial inhomogeneity of GDP per capita, the quantity

$$D_t = \sum_{i \neq j} \left(\frac{y_{it} - y_{jt}}{\bar{y}} \right)^2, \quad j = 1, \ldots N, \quad t = 1950, \ldots 2007$$

can be used, where y_{it}, y_{jt} denotes per capita GDP in countries i, j at time t for a sample of countries $i, j = 1, ..., N$, and $\bar{y}$ denotes the overall average (over all countries) per capita GDP. This quantity can be regarded as a measure of spatial inhomogeneity of GDP per capita, in the sense that an increasing D_t over time means that the spatial distribution of GDP becomes more spatially heterogenous or "less flat" relative to space.[3] Thus an increasing D_t over time indicates that the dispersal of per capita GDP across the countries of the sample increased during the sample period. The inhomogeneity measure D_t, along with the corresponding linear trend, is presented in Figures 3 and 4 for the regions shown in Figures 1 and 2.

[Figure 3. Regional inhomogeneity measure]
[Figure 4. Inhomogeneity measure, developed countries]

The evolution of the inhomogeneity measure and the associated linear trend suggest that the overall dispersal is increasing somewhat both at the world regional level and within the group of high income countries.

These empirical observations, although broad in nature, indicate that the spatial distribution of GDP per capita does not tend to become more uniform with the passage of time. The spatial distribution is characterized by a sharp peak which increases with time and is located in North America and Norway/USA in Figures 1 and 2 respectively, which are the regions or countries with the highest GDP per capita within the group. These "stylized facts" seem therefore to support the idea that in a geographical context, the growth process induces an approximately bell-shaped distribution with a rather sharp peak. This distribution does not seem to become flatter with the passage of time or, to put it differently, does not seem to converge to a geographical homogeneous state for countries grouped in the traditional way according to their level of economic development. Countries that start with lower per capita income in the region may grow faster than high income counties, which is consistent with β convergence arguments, but this growth does not seem to result in a spatially flatter distribution in the long run.

Ireland, Israel, Italy, Japan, Korean Rep., Luxembourg, Netherlands, New Zealand, Norway, Poland, Portugal, Slovak Republic, Slovenia, Spain, Sweden, Switzerland, United Kingdom, United States. The World Bank classification is used. Some outliers countries have been omitted from the graph.

[3] This measure can be related to a discretized version of a Sobolev norm.

In this context the purpose of this paper is to develop a spatial model of economic growth and by doing so to explore mechanisms that could generate, through economic forces, persistent nonuniform spatial distributions of per capita capital and GDP across locations, and determine the temporal evolution of the spatial distributions. To put it differently, we explore how traditional neoclassical growth theory can be extended to a spatial growth theory which would provide models capable of approximating the spatial distributions observed in Figures 1 or 2.

Economic geography and economic growth has been discussed in the so-called second generation of new economic geography models, but not in a formal growth context (e.g., Martin and Ottaviano [6], Baldwin et al. [7], [8], Baldwin and Martin [9], Fujita and Mori [10], Desmet and Rossi-Hansberg [11], [12]). Models of optimal development over space and time, which could be regarded as a suitable vehicle for studying economic growth in a geographical context, were developed in the 1970s by Isard and Liossatos (e.g., [13], [14], [15]) and Carlson et al. [16]. Dynamic spatial economic models were developed in the context of economic growth and resource management mainly during the 2000s (e.g., Brito [17], Camacho and Zou [18], Boucekkine et al. [19], [20] [21], Brock and Xepapadeas [22], [23], Brock et al. [24], [25]). The main feature of current spatial growth models is that the spatial movements of the stock of capital across locations are modeled through a trade balance approach with respect to a closed region where capital flows are such that capital is received from the left of the region and flows away to the right of the region. This leads to a model of classic local diffusion with a constant diffusion coefficient. Modeling capital movements in this way means that capital stock moves from locations of high concentration to locations of low concentration. This property, although consistent with diminishing returns to capital (since high concentration implies low marginal productivity and vice versa), seems not to be compatible with empirical findings. As indicated in the context of the Lucas paradox ([26], [27]), although diminishing returns suggest that capital will flow from locations of high concentration to locations of low concentration, this does not happen in reality.

In the present paper we contribute to the ongoing research on spatiotemporal dynamics and spatial growth by developing a model in which the basic mechanism underlying the movements of capital across space is the quest for locations where the marginal productivity of capital is relatively higher than the productivity at the location of origin, without imposing the constraint that capital moves from locations of high concentration to locations of low concentration. By assuming that capital flows towards locations of high returns, which is a plausible assumption underlying capital flows, with velocity depending on endogenous factors such as the existing stock of capital or the size of profitability, our model implies that the spatiotemporal evolution of capital is governed by a nonlinear diffusion equation. In this case the "diffusion coefficient" is not constant but depends on the capital stock and the rate of change of marginal productivity of capital (the second derivative of the production function). This approach for modeling capital flows essentially differs from the classic diffusion models used in the existing literature which are based on the trade balance (e.g., Carlson et al. [16], Brito [17], Camacho and Zou [18], Boucekkine et al. [19], [20], [21]), and describe the spatiotemporal evolution of capital by a parabolic partial differential equation (PDE). Our contribution is that by using the plausible mechanism that capital moves towards locations of higher productivity, and not a mechanism where capital moves necessarily from higher to lower concentration locations, we obtain - using standard neoclassical growth assumptions - spatial distributions which are characterized by large and persistent concentration gradients that can be compatible with existing observations. Furthermore, we are not confronted with the Lucas paradox which appears in models with the trade balance mechanism which essentially imposes the constraint that capital moves from locations of high concentration to locations of low concentration even though such behavior seems not to be supported by empirical findings. Our approach, which is based on the notion that capital moves to locations of relatively higher productivity, but not necessarily from locations of high concentration to locations of low concentration, and leads to nonlinear diffusion, does not face this difficulty.

By considering a distance metric concept based on economic distance, we develop local models of capital diffusion, where the spatiotemporal evolution of capital in pursuit of higher returns is governed by a nonlinear PDE which incorporates the velocity by which capital flows across regions. We use this analytical framework to extend the standard Solow model in a geographical context. The spatial Solow model with a mechanism underlying capital flows which leads to nonlinear diffusion, generates solutions in which spatially nonhomogeneous distributions of per capita capital and income across locations persist over time. In certain cases locations may end up at a steady state in poverty cores with capital stock approaching zero. Our results about persistent spatial heterogeneity and non-smoothing of spatial differences do not require increasing returns and are obtained under standard diminishing returns to capital.

2 Capital Flows and Distance Metrics

The existing literature in spatial growth models the spatiotemporal evolution of capital by a parabolic PDE with a constant diffusion coefficient.[4] This implies that capital should flow from rich countries to poor countries given diminishing returns, which is not, however, what is observed in reality.

Lucas ([26], [27]) explains the paradox in terms of misspecification of the production technology where important factors such as human capital,

[4] See for example, [16], [17], [18], [19], [20], [21].

and capital market imperfections including costs of international trade (e.g. transportation costs, capital controls, tariffs and other trade costs), information asymmetries or sovereign risk are omitted.[5] These factors are closely related to spatial heterogeneities across the spatial domain of interest. The importance of the Lucas paradox for models of economic geography and growth, which are modeled using classic diffusion and are essentially based on the heat equation, is that it suggests that these models might not be an adequate representation of reality, since they are based on the assumption that capital moves for high abundance to low abundance locations. In the present paper we model capital flows in space based on the notion that capital moves to location of relatively higher productivity, but not necessarily from locations of high concentration to locations of low concentration. The notion of capital which we employ is a "mechanistic" kind that cannot move very fast, like financial capital can, to areas of high marginal productivity because of adjustment costs and other potential institutional barriers.

A second issue that a spatial growth model should address is the topology of the space where capital flows take place and the definition of an appropriate distance metric. The most common metric of the distance between two spatial points (say countries) where capital flows take place is geographical distance, as measured for example by the distance between capital cities. Conley and Ligon [30] suggest that a more appropriate metric for measuring distances associated with economic activities is that of the economic distance - the economic metric - reflected by transportation costs. They use United Parcel Service (UPS) distance as a proxy for transportation cost associated with physical capital, while airfare distance is used as a proxy for transportation cost associated with human capital. It turns out that the distance between countries might be very different depending on whether the geographic or the economic metric is used. For example while the geographical distance between Australia and Egypt is smaller than the distance between Australia-UK and Australia-USA, the corresponding economic distance both in terms of UPS and airfare distance between Australia and Egypt is larger than the distance between Australia-UK and Australia-USA.

The choice of the distance metric is important for modeling purposes since it provides a basis for choosing between a local model of capital diffusion, or non-local model of capital flows which will incorporate long range effects. If an economic metric is adopted, a local model might be regarded as adequate. This is because it is reasonable to assume that capital, given the restrictions imposed by technology and institutions, will flow among sites which are close in terms of the economic metric, since this would imply less frictions, with the flow directed towards sites where returns grow faster. On the other hand if the geographic metric is used, then a non-local model

[5] See also Razin and Sadka [28], Alfaro et al.[29].

seems to be the most appropriate, since in this case the geographical distance might not be a good proxy for frictions associated with capital flows. In this case capital will flow again towards sites where returns grow faster, but these locations might not be close to each other in terms of the geographical metric, which means that a non-local model of spatial interactions is required.

3 Modeling the Spatiotemporal Evolution of Capital

Following the previous discussion, we develop a local model that enables us to study the spatiotemporal evolution of capital in the context of an economic metric. Since each element of the economic space can be mapped to one and only one element of the geographical space, any spatial distribution defined in economic space can be transformed to a corresponding distribution in the geographical space. This equivalence allows us to work with local models defined in economic space. In these models the movement of capital to sites where returns are higher can be defined in a more tractable way through local transport operators, an approach which is not appropriate when capital flows are defined in geographical space.

In what follows, the "spatial" variable z can be considered as describing a point in a generalized notion of space. To avoid unnecessary complications we assume that economic (or physical) space may be embedded in a sufficiently high dimensional Euclidean space.[6] We will thus allow z to take values in $U \subset \mathbb{R}^d$ where d is the dimension of space.

Following standard neoclassical growth theory, we assume that the aggregate output is produced at a location (site or spatial point) z and time t according to a production function $Y(t,z) = F(K(t,z), L(t,z), A(t,z))$, where K, L represent capital stock and labor input respectively and A represents technology. We assume, unless stated differently, that with respect to K and L, for fixed A, the production function is twice continuously differentiable, strictly concave, and homogeneous of degree one. Furthermore, in order to simplify, we assume that there is no population growth, i.e., $L(t,z) = L(z)$, and no technical change, i.e., $A(t,z) = A(z)$. Thus $L(z)$ and $A(z)$ reflect spatial heterogeneities related to population and a productivity factor that may reflect positive spatial externalities associated with location z. Under these assumptions, the production function can be written in per worker terms as

$$y(t,z) = f(k(t,z), A(z)), \; y(t,z) = \frac{Y(t,z)}{L(z)}, \; k(t,z) = \frac{K(t,z)}{L(z)}. \qquad (1)$$

[6] This does not mean that the space itself is Euclidean. For example a circle is a one-dimensional nonlinear manifold which is embedded in a plane, which is a two-dimensional Euclidean space.

Given the production function (1), the state of the system is described by the per worker capital stock function $k : [0, \infty) \times U \to \mathbb{R}$. The state of the system can also be described by a vector valued function $\check{k} : [0, T] \to \mathbb{X}$, where $\mathbb{X}$ is a function space which accounts for the spatial behavior of the state variable. Then $\check{k}(t) = \omega \in \mathbb{X}$, where $\omega : U \to \mathbb{R}$ is a function of z such that $(\check{k}(t))(z) = \omega(x) = k(t, z)$.[7]

The law of motion of per capita capital at site z and time t, $(t, z) \in [0, \infty) \times U$ is determined by two fundamental terms:

- A term characterizing capital mobility across space, i.e., the net flow of capital in site z from all possible points in U which will be denoted abstractly as $T(t, z)$.
- A term characterizing the local net accumulation of capital at site z.

3.1 A simple discrete model of capital flows

To make the derivation of the spatiotemporal evolution clear, we start with a family of discrete space models defined in the economic space and go to the continuous limit. We assume that space is a discrete lattice, which without loss of generality is assumed to be one dimensional. Therefore $U = \mathbb{Z}$ is the integer lattice and we may consider the function $\check{k}$ as a function $\check{k} : [0, T] \to \mathbb{R}^{\mathbb{Z}}$, i.e., for every t the state of the system $\check{k}(t) = (k_i(t))_{i \in \mathbb{Z}}$, which means that it is expressed by a real valued sequence. We will then use $k_i(t)$ in lieu of $k(t, z)$ at $z = i$, $i \in \mathbb{Z}$, to denote the capital stock and $f(k_i(t), A_i)$ and the production function at site i at time t. A similar notation $m_i(t) := f' = \frac{df}{dk_i}$ is used for the marginal productivity of capital. We assume in the context of the economic space that each site i is connected only with two neighboring - from an economic point of view - sites, $i - 1$ and $i + 1$, and that capital may flow from i to either one of these sites, or vice versa.

Our basic assumption is that capital may flow from i to j (where $j = i \pm 1$) at time t only if $m_i(t) < m_j(t)$. Then, a fraction of capital stock $\phi(t, i \to j)$ will move from site i to j, so that the total capital stock that moves from i to j is $\phi(t, i \to j)k(t, i)$. We assume that

$$\phi(t, i \to j) = \begin{cases} \psi(m(t, j) - m(t, i)) & , \ m(t, j) \geq m(t, i), \\ 0 & , \ m(t, j) < m(t, i) \end{cases} \tag{2}$$

where $\psi : \mathbb{R} \to \mathbb{R}_+$ is a positive function. We assume here for simplicity that $\psi(\omega) = \lambda \omega$ where $\lambda > 0$ is assumed small enough so that $\psi(m(t, j) -$

[7] The choice of function space $\mathbb{X}$ depends on the spatial behavior of capital, for instance we may consider $\mathbb{X} = C(U)$, the space of continuous functions on U, if the spatial behavior of the capital distribution can be modeled by a continuous function, or $\mathbb{X} = L^p(U)$ if the spatial behavior of the capital distribution can be modeled in terms of the element of a Lebesgue space. When working in terms of $\check{k}$, the capital flow is modeled in terms of a (typically) infinite-dimensional dynamical system on the function space $\mathbb{X}$.

$m(t,i)) \in [0,1]$. In terms of the production function, the same relation can be written as:

$$\phi(t, i \to j) = \begin{cases} \psi\left(f'(k(t,j)) - f'(k(t,i))\right) & , \ f'(k(t,j)) \geq f'(k(t,i)), \\ 0 & , \ f'(k(t,j)) < f'(k(t,i)), \end{cases} \tag{3}$$

where to simplify notation we omit the term $A(z)$ from the production function.

In a similar fashion, capital may migrate from site j to site i (where in the present simple model $j = i \pm 1$). The fraction of capital stock $\phi(t, j \to i)$ that moves from j to i depends on the difference $m(t,i) - m(t,j)$, and the total capital stock which is transported from j to i will be equal to $\phi(t, j \to i)k(t,j)$. The fraction $\phi(t, j \to i)$ is given by expression (2) or (3) with the roles of i and j interchanged.

The net capital flow at time t and site i is given by

$$\begin{aligned} T(t,i) = & \underbrace{\phi(t, i-1 \to i)k(t,i-1) + \phi(t, i+1 \to i)k(t,i+1)}_{Inflow} \\ & - \underbrace{\left(\phi(t, i \to i-1) + \phi(t, i \to i+1)\right)k(t,i)}_{Outflow}. \end{aligned}$$

Let $c(t,z)$ denote consumption per unit of labor at time t and site z, where consumption is modelled by a vector valued function $\check{c} : [0, \infty) \to \mathbb{Y}$, and let δ denote the time invariant exponential depreciation rate of capital; then local capital formation is determined by $\Phi(t,z) = f(k(t,z) - c(t,z) - \delta k(t,z)$. If we assume that in each site a constant fraction $s(z)$ of income is saved, then $\Phi(t,z) = s(z) f(k(t,z), A(z)) - \delta k(t,z)$ with $c(t,z) = (1 - s(z)) f(k(t,z), A(z))$, and the local net accumulation is defined in the context of the Solow model. Denoting by $\Phi(k_i)$ net capital formation at site i, in terms of the discrete lattice, the total balance equation at site i will be

$$\frac{dk_i(t)}{dt} = k_i^{'}(t) = \Phi(k_i(t)) + T(t,i), \ \ i \in \mathbb{Z}. \tag{4}$$

This is a set of differential equations, the solution of which will provide the spatiotemporal evolution of capital on the lattice. It is useful for the purpose of modeling to investigate the continuous space limit of this discrete model. This is the limit when $z = i\delta z$ and we allow $\delta z \to 0$. Furthermore, to ease notation we abandon the notation $\check{k}$ and use k invariably. The next proposition provides the continuous limit of the discrete model proposed above.

Proposition 1 *Assume that capital flows to locations of relatively higher marginal productivity, according to the transport law (2). The continuous* *limit of the lattice transport equation (4) describing the spatiotemporal evolution of capital stock is given by the nonlinear parabolic PDE,*

$$\frac{\partial k}{\partial t} = \Phi(k) - \alpha \frac{\partial}{\partial z}\left(kf''(k)\frac{\partial k}{\partial z}\right), \tag{5}$$

and the transport operator determining capital flows across locations is given by:

$$\mathfrak{T}k\,(t,z) = T\,(t, k\,(t,z)) =$$
$$-\frac{\partial}{\partial z}\left(k(f''(k)\frac{\partial k}{\partial z}\right) = -\frac{\partial}{\partial z}\left(D(k)\frac{\partial k}{\partial z}\right).$$

For proof, see Appendix 1.

The transport operator $\mathfrak{T}$ is a nonlinear diffusion operator, with a state dependent diffusion coefficient $D(k) = f''(k)k$, meaning that the flow of capital from or to a location depends on the spatial rate of change of the marginal productivity of capital and the stock of capital in the location.[8] The growth equation (5) is thus a nonlinear diffusion equation, the solution of which will provide the spatiotemporal evolution of the capital stock in U.

Thus, and in contrast to the existing spatial growth models, (e.g., [17], [18] [19], [20], [21]) where the transport operator is characterized by a constant diffusion coefficient, our assumption about capital flows leads to a diffusion operator with a diffusion coefficient D which is no longer constant, but depends on the capital stock k.[9] The behavior implied by the constant diffusion model is contradicted by the Lucas paradox. In terms of the present model, the basic assumption is that capital flows seek higher values of marginal productivity m. Thus capital will only move from a site z to a site z^* if $m(t,z^*) > m(t,z)$, independent of the relative concentration of capital between z^* and z. This property seems to overcome issues related to the Lucas paradox.

Another point of interest is the negative sign in front of the second order derivatives term. If $D(k) > 0$, this negative sign will correspond to "anti-diffusion," that is, the dynamics of the equation encourage concentration of capital in very few locations. $D(k) > 0$ can be associated with increasing returns to capital from a social point of view, due to Lucas-Romer-type positive spatial externalities in the production function, while $D(k) < 0$ can be associated with the standard case of diminishing returns to capital.

A final comment is that in the absence of net capital formation, i.e., if $\Phi(k,c) = 0$, then equation (5) is in divergence form, so that the total capital

[8] This transport operator is acceptable, since it is in divergence form and it can be easily seen that, using appropriate boundary conditions, the conservation property holds.

[9] To obtain the constant diffusion model as the continuous limit of the above discrete space capital stock transport scheme, we would need to assume that $\phi(t, i \to i \pm 1) = \phi(t, i \pm 1 \to i) = \frac{1}{2}$, i.e. constant and equal irrespective of the state of capital stock at the corresponding sites.

over the whole spatial domain is conserved. This is a desirable feature of the equation since, in the absence of net capital formation, total capital must be conserved at its initial value.

3.2 A generalized spatial model of capital flows

Using the previous analysis as a basis, we develop a general class of local continuous space models that enable us to characterize the spatiotemporal evolution of capital in an economic space. In what follows the spatial variable z will be considered as describing a point in a generalized notion of economic space, thus z takes values in $U \subset \mathbb{R}^d$ where d is the dimension of the economic space. We model capital flows, using the basic assumption that capital flows towards locations of relatively higher marginal productivity in combination with Gauss' divergence theorem, which is briefly described in the Appendix.

In this context, consider a domain $U \subset \mathbb{R}^d$ and let U_0 be any subset of U. Denote by $k(t, z)$ the stock of capital in this domain at time t and at spatial point z. Then the total amount of capital in U_0 at time t is given by the volume integral $\int_{U_0} k(t, z)\, dz$. Capital flows everywhere within U_0, so it is natural to consider a vector field $\mathfrak{v} = (v_1, \cdots, v_d)$, $\mathfrak{v} : U_0 \to \mathbb{R}^d$ which gives us the velocity of capital flow (flux) at every point of U_0. Then the following proposition can be stated:

Proposition 2 *The spatiotemporal evolution capital in the setting described above is given by:*

$$\frac{\partial}{\partial t} k(t, z) = -\nabla_z \cdot (\mathfrak{v}\, k(t, z)) + \Phi(t, z). \tag{6}$$

The vector field $\mathfrak{v}$ points in the direction towards which capital is flowing and its magnitude at point z shows how fast capital located at z is likely to migrate in the direction in which $\mathfrak{v}$ is pointing. In the context of a spatial growth model, the vector field $\mathfrak{v}$ must be specified by economic considerations.[10] Once the forms of $\mathfrak{v}$ and Φ are determined, equation (6) serves as a PDE which, if solved with the proper initial and/or boundary conditions, can fully determine the capital distribution $k(t, z)$ at all times t and at all points z in space.

We now consider the specification of the term $\mathfrak{v}$. The term $\mathfrak{v}$ is a vector field whose direction should point towards the direction where capital stock will have the tendency to flow under the influence of economic factors. Our basic economic assumption is that capital flows towards regions where its marginal productivity is relatively higher, but the movement is not very fast, as for example it is with financial capital, due to adjustment costs and

[10] Based on the intuition gained by our discrete spatial model, we may interpret $\mathfrak{v}$ in terms of the probability of capital to migrate. For example the component v_i is related to the probability of capital to migrate from z to the direction $z + e_i$.

institutional barriers. Furthermore, due to positive spatial spillover externalities, capital existing at location z may increase the value of the marginal product of capital arriving at z. The direction of the flow is locally (in z and t) determined in terms of the gradient of the scalar field that provides the marginal productivity of capital, i.e. it flows towards regions where $\nabla_z m(t,z) = \text{grad} m(t,z)$ is large.[11] The velocity of the flow (interpreted as the propensity of capital to relocate) may also depend on economic factors. One of them could be the size of the accumulated stock of capital at a given site. For example it may be easier for capital to flow to sites of high returns if the site from which the capital originates is characterized by high capital stock. This would mean that if returns to capital is high in a given site, then it will be easier for capital from mature economies with large capital stock to move to this site, compared to capital originating from smaller economies. These assumptions imply that the drift depends on the capital stock and the gradient of the marginal productivity, i.e. that it will be reasonable to specify $\mathfrak{v}$ as $\mathfrak{v} = v(t,z,k,m)\nabla_z m$, where v is a scalar function taking positive values. The vector field $\mathfrak{v}$ is therefore a vector field which points in the direction of $\nabla_z m(t,z) = \text{grad} m(t,z)$. The explicit dependence of the scalar function v on t may reflect time dependencies such as business cycles, whereas the explicit dependence of v on z may model geographical effects or effects related to local conditions, economic, cultural or legal, which may encourage or hamper capital mobility.

The gradient of m is defined in terms of the production function as

$$\nabla_z m(t,z) = \frac{\partial^2}{\partial k^2} f(k(t,z), A(z))\nabla_z k(t,z) + \frac{\partial^2}{\partial k \partial A} f(k(t,z), A(z))\nabla_z A(z).$$

Using the shorthand notation f_k for the partial derivative of f with respect to k (and similarly for the other or the higher partial derivatives), we may express $\mathfrak{v}$ in terms of the production function and the capital stock as

$$\mathfrak{v} = v(t,z,k,f_k(k,A))\left(f_{kk}(k,A)\nabla_z k + f_{kA}(k,A)\nabla_z A(z)\right),$$

where A is a known function of z.

Therefore, equation (6) becomes

$$\frac{\partial}{\partial t} k = -\nabla_z \cdot \Big(k v(t,z,k,f_k(k,A))\left(f_{kk}(k,A)\nabla_z k + f_{kA}(k,A)\nabla_z A\right) \Big) + \Phi(t,z), \tag{7}$$

which is a quasilinear parabolic PDE for the unknown function k, which provided the capital stock distribution in space and time. This equation simplifies if A is a constant (independent of space).

[11] Note the difference between the operator $\nabla_z \cdot = \text{div}$ which is the divergence operator and the operator $\nabla_z = \text{grad}$ which is the gradient operator. The first one acts on a vector field and yields a scalar field, while the second one acts on a scalar field and yields a vector field.

Note the difference between our model as expressed in equation (7) and the spatial growth models proposed so far in the literature. These models are in the general form

$$\frac{\partial}{\partial t}k = \nabla_z \cdot (D \nabla_z k) + \Phi(t, z), \tag{8}$$

where D is independent of k (but possibly depending on z). This is a semilinear parabolic equation. The fundamental difference between the general class of models expressed by equation (8) and our model expressed by equation (7) is that in our model, even if A is a constant, the diffusion coefficient depends on the actual state of the system, i.e., the capital stock concentration at point (t, z). In this case D is no longer independent of k but rather $D = D(k) = -kv(t, z, k, f_k(k, A))f_{kk}(k, A)$. This difference arises from the fact that our capital mobility assumption is associated with productivity differentials rather than with an application of Fick's law of thermal diffusion to capital flows. That is, our nonlinear diffusion model results from the economic assumption that capital migrates, striving to locate to regions where its marginal productivity is higher, and not simply from the (physically based) assumption that capital moves from regions of high concentrations to regions of low concentrations akin to what is observed for heat flow, chemical concentrations or population densities. As a result, we derive a modified Fick's law with a nonlinear diffusion coefficient (i.e., a local diffusion coefficient depending on the capital stock at each point) and this dependence is characterized by the production function and its derivatives. Furthermore, there are important qualitative differences between the solutions of the stock dependent diffusion (quasilinear capital flow equation) (7) and the constant diffusion model (semilinear capital flow equation) (8) which have implications for the economic interpretation of outcomes. Such differences are related, among other things, to (a) the possibility of existence of a compact support solution (corresponding to regions where capital is depleted) for (7) while this is impossible for (8), and (b) the possibility of formation of sharp spatial gradients of k (corresponding to the formation of well defined spatial patterns) for (7), while the general tendency is for spatial gradients to be smoothed out for solutions of equation (8). Note finally that the sign of the diffusion coefficient depends on the sign of the term f_{kk}. If $f_{kk} < 0$, then we have a positive diffusion coefficient (leading to a tendency for spatial gradients to smoothen) while if $f_{kk} > 0$, possibly due to Romer-Lucas externalities, we have a negative diffusion coefficient (so that the system has a tendency to develop sharp gradients). Therefore, as a general rule of thumb, in our model decreasing returns lead to a positive diffusion coefficient while increasing returns lead to negative diffusion.

In order to make the implications of our modelling for growth theory more transparent, we will focus on a particular special case of the general class of quasilinear models stemming from equation (7), which are related to the Cobb-Douglas type production function and the Solow growth model.

4 A Spatial Solow Model

In this section we consider the general class of quasilinear models proposed in equation (7) for production functions of the general form $f(k, A) = A\varphi(k)$ where $A = A(z)$ is a known function and $\varphi(k)$ is a standard Cobb-Douglas production function $\varphi(k) = k^\alpha$ for $\alpha \in (0, 1)$.

We further assume that the capital transport velocity can be defined as:

$$\mathfrak{v}(t, z) = B(z)(k(t, z))^{\rho_1}(m(t, z))^{\rho_2}\nabla_z m(t, z). \tag{9}$$

According to this velocity law, capital flows towards regions of increasing marginal productivity of capital (this is the effect of the $\nabla_z m(t, z)$ contribution) but the magnitude of the velocity of motion towards this direction depends on (a) the actual location through the term $B(z)$ (this captures local effects, legal structure, cultural traits, etc. that may facilitate or hamper capital mobility), (b) the concentration of capital $k(t, z)$ at time t at point z through the term $(k(t, z))^{\rho_1}$, where ρ_1 plays the role of the elasticity of the capital velocity with respect to capital concentration (a positive ρ_1 implies that an increase in the capital stock in location z will cause the velocity of capital movement towards locations with marginal productivity higher than z to increase), and (c) the actual marginal productivity of capital $m(t, z)$ at time t on the current point of location z through the term $(m(t, z))^{\rho_2}$ where the coefficient ρ_2 corresponds to the elasticity of the capital velocity with respect to capital concentration. A positive ρ_2 implies that an increase in the capital stock in location z will cause the velocity of capital movement towards locations with marginal productivity higher than z to increase. The above considerations suggest the choice $\rho_1 \geq 0$ and $\rho_2 \leq 0$ in the capital velocity model (9).

Using the neoclassical production function, the velocity law becomes

$$\mathfrak{v}(t, z) = B(z)(k(t, z))^{\rho_1}(A(z)\varphi'^{\rho_2} \left\{A(z)\varphi''(k(t, z))\nabla_z k(t, z) + \varphi'(k(t, z))\nabla_z A(z)\right\},$$

where $\varphi'(k) = \frac{d\varphi}{dk}(k)$ and $\varphi''(k) = \frac{d^2\varphi}{dk^2}(k)$. Omitting the explicit dependence of k on (t, z) for notational simplicity, this leads to the expression

$$k\mathfrak{v} = B(z)(A(z))^{\rho_2+1}k^{\rho_1+1}(\varphi'^{\rho_2}\varphi''(k)\left\{\nabla_z k + \frac{\varphi'(k)}{\varphi''(k)}\frac{\nabla_z A(z)}{A(z)}\right\}. \tag{10}$$

Exogenous saving rates s and constant depreciation of capital δ across locations implies that $\Phi(t, z) = f(k, A) - \delta k = sA(z)\varphi(k) - \delta k$, where s could be dependent on z. Substituting into the continuity equation (7) yields

$$\frac{\partial k}{\partial t} + \nabla_z \cdot \left(B(z)(A(z))^{\rho_2+1}k^{\rho_1+1}(\varphi'^{\rho_2}\varphi''(k)\left\{\nabla_z k + \frac{\varphi'(k)}{\varphi''(k)}\nabla_z \ln A(z)\right\}\right) = sA(z)\varphi(k) - \delta k. \tag{11}$$

This is a quasilinear parabolic PDE in terms of k, the solution of which yields the spatiotemporal evolution of capital stock in a spatial Solow model where capital moves towards the location with the higher return. This equation simplifies even further under the assumption that the drift due to the A term in the production function (modelled by the $\nabla_z \ln A(z)$ term in the evolution equation) is negligible with respect to the drift occurring from the spatial variability of capital $\nabla_z k$. Therefore, if $A(z)$ is such that $\nabla_z \ln A(z) << \frac{\varphi''(k)}{\varphi'(k)}\nabla_z k$,[12] the quasilinear model (11) simplifies to

$$\frac{\partial k}{\partial t} + \nabla_z \cdot \big(B(z)(A(z))^{\rho_2+1} k^{\rho_1+1} (\varphi'^{\rho_2}\varphi''(k)\nabla_z k\big) = sA(z)\varphi(k) - \delta k, \quad (12)$$

or in more compact form

$$\frac{\partial k}{\partial t} + \nabla_z \cdot (D(z,k)\nabla_z k) = sA(z)\varphi(k) - \delta k, \quad (13)$$

where

$$\begin{aligned} D(z,k) &= D_0(z)\psi(k), \\ D_0(z) &= B(z)(A(z))^{\rho_2+1}, \\ \psi(k) &= k^{\rho_1+1}(\varphi'^{\rho_2}\varphi''(k). \end{aligned}$$

If $\rho_1 = \rho_2 = 0$ then $\mathfrak{v}(t,z) = \nabla_z m(t,z)$, and the diffusion coefficient of the spatial Solow model is reduced to the diffusion coefficient of the model derived in section 3.1 which is, therefore, a special case of the more general model (12).

An equivalent way to express model (13) is in the form

$$\frac{\partial k}{\partial t} + \nabla_z \cdot (D_0(z)\nabla_z \ominus(k)) = sA(z)\varphi(k) - \delta k, \quad (14)$$

where $\Psi'(k) = \psi(k)$. In the special case where D_0 is independent of z or presents slow variations in z (in which case we may use an approximation analogous to the one we used for A), we may express (14) as

$$\frac{\partial k}{\partial t} + D_0(z)\Delta\Psi(k) = sA(z)\varphi(k) - \delta k, \quad (15)$$

where $\Delta = \nabla_z \cdot \nabla_z$ is the Laplace-Beltrami operator, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial z_i^2}$, with $\Delta = \frac{\partial^2}{\partial z^2}$ for the one-dimensional case. Note that $\Psi(k)$ may take negative values since for a neoclassical production function $f''(k) < 0$.

We proceed now to derive the explicit form of the model for the Cobb-Douglas case.

[12] Equivalently $\nabla_z \ln A_z << \nabla_z R(k)$, where $R(k) = \int^k \frac{\varphi''(u)}{\varphi'(u)} du$ then the drift effect due to spatial variability of A can be neglected. This assumption may hold if $\ln A$ is a slowly varying function of z, i.e. if $\ln A(z) = \bar{A}(\epsilon z)$ where $\epsilon > 0$ is a small parameter. Obviously, this assumption always holds for constant A.

Proposition 3 *Let the production function by of the Cobb-Douglas type* $f(k,A) = Ak^{\alpha}$, $\alpha \in (0,1)$, *with* A *constant or slowly varying with* z. *If the velocity law for the motion of capital is of the form* $\mathfrak{v} = B(z)k^{\rho_1}m^{\rho_2}\nabla_z m$, *then the spatiotemporal evolution of the capital stock is given by the quasi-linear degenerate PDE*

$$\frac{\partial}{\partial t}k = \bar{D}\nabla_z \cdot (D_0(z)k^{\beta}\nabla_z k) + sA(z)k^{\alpha} - \delta k, \tag{16}$$

or the equivalent form,

$$\frac{\partial}{\partial t}k = \frac{\bar{D}}{1+\beta}\nabla_z \cdot (D_0(z)\nabla_z k^{\beta+1}) + sA(z)k^{\alpha} - \delta k, \tag{17}$$

where $\bar{D} = \alpha^{\rho_2}\alpha(1-\alpha)$, $\beta = \rho_1 - (1+\rho_2)(1-\alpha)$ *and* $D_0(z) = B(z)(A(z))^{1+\rho_2}$.

The diffusion mechanism reduces to the linear diffusion mechanism in the special case where $\beta = 0$ *or equivalently in the case where the parameters of the model are such that* $\rho_1 = (1+\rho_2)(1-\alpha)$. *Two particularly interesting special cases are (a) the case where* $\rho_1 = 0$ *and* $\rho_2 = -1$, *and (b) the case where* $\alpha = 1$, $\rho_1 = 0$ *(the AK model).*

The proof is straightforward and is omitted; however, the following remarks are important. Except for the special case where $\beta = 0$, our model is a nonlinear diffusion model with diffusion coefficient $D(z,k)$ depending on the state of the system as $D(z,k) = \bar{D}_0(z)k^{\beta}$, where $\bar{D}_0$ is a known function of space. Therefore in our model, capital mobility across space, reflected in the diffusion coefficient $D(z,k)$, is determined endogenously**,** while in models of linear diffusion the fixed diffusion coefficient D is determined exogenously. A fixed diffusion coefficient in terms of our model emerges as a special case when $\rho_1 = \rho_2 = 0$ and the production function is Ak. We think that, although the degree of dependency of the diffusion coefficient on the stock of capital and the structure of capital velocity are empirical issues, our approach - by relating these factors to capital flows - provides a richer environment for studying the spatiotemporal evolution of capital stock.

Remark 1 *Capital* k *is interpreted as per capita capital without population growth or technical change. Assuming population growth at a rate* $\frac{\dot{L}(z,t)}{L(z,t)} = n(z)$ *and exogenous spatially uniform technical change at a rate* $\frac{\dot{A}(z,t)}{A(z,t)} = g$, $A(t,z) = A(z)\mathrm{e}^{gt}$, *the local capital accumulation equation can be defined in per effective worker terms with* $k(t,z) = \frac{K(t,z)}{A(z,t)A(t,z)}$. *In this case, by abusing notation* $\delta = \delta + n(z) + g$. *Our models can be reinterpreted in this case by assuming that capital per effective worker moves towards locations where the marginal productivity of capital per effective worker is relatively higher.*

Equation (16) is a generalization of the well studied porous medium equation in the sense that it is a porous medium equation with a reaction term.[13] It is interesting to note that this porous medium equation was not imposed as a modeling tool, but emerged from the assumption that capital flows seek locations of high productivity and move with a velocity which depends on capital stock and marginal productivities.

If the initial condition k_0 has support which is a compact subset of the domain, then the solution presents the finite speed of propagation property. This means that for any $t > 0$, there will be regions of the domain for which the solution is identically equal to 0, i.e., the support has a free boundary which separates the regions where $k > 0$ from the regions where $k = 0$. This phenomenon never holds for the linear diffusion case $\beta = 0$, which presents infinite speed of propagation, meaning that even if the initial condition k_0 is of compact support, the solution for any $t > 0$ will not have this property. Technical and abstract as it may sound at first, this qualitative behavior of the nonlinear diffusion may have implications from the point of view of economic theory: the compact support property may be interpreted in terms of the existence of regions where capital is completely and persistently depleted, which is in fact in economic terms a poverty trap.

Remark 2 (The spatial Solow model under linear diffusion) *Proposition 3 elucidates the role of parameter α in the capital concentration dynamics. To make the argument more transparent consider the case $\beta = 0$ and let D_0 be independent of z. Equation (16) assumes the semilinear form*

$$\frac{\partial}{\partial t}k = \bar{D}(1-\alpha)\Delta k + sA(z)k^{\alpha} - \delta\alpha, \tag{18}$$

similar to the models employed so far in the literature on spatial growth, but with an important difference: the diffusion coefficient is proportional to $1-\alpha$. Therefore, if $\alpha < 1$ (diminishing returns to capital), then the diffusion coefficient is positive which leads to a model similar to the one proposed by Boucekkine et al. [19], however within a totally different modeling framework. The positive diffusion coefficient corresponds to dynamics that tend to eliminate spatial gradients, thus leading to spatial convergence[14] phenomena. If on the other hand, $\alpha > 1$ (increasing returns to capital), then the diffusion coefficient is negative and this leads to a linear anti-diffusion model, which tends to amplify spatial gradients and leads to large capital concentration phenomena. Finally, if $\alpha = 1$, the model is reduced to a growth model with an AK production function which eliminates spatial heterogeneity.

In the relevant literature based on trade balance (e.g. [21]), the diffusion coefficient $D = \bar{D}(1-\alpha)$ is set at the value of one, so that the relevant PDE

[13] In the absence of a reaction term, the porous medium equation has been studied very actively as a paradigm for nonlinear diffusion and has served as a model for various physical or biological systems (see for example Vasquez [31]).

[14] The term convergence is used in its economic growth context.

is

$$\frac{\partial}{\partial t}k = \Delta k + sA(z)k^{\alpha} - \delta k. \tag{19}$$

Models (16), (18) or (19) can thus be regarded as candidate specifications for a spatial growth equation. The determination of the impact of spatial diffusion on capital accumulation, and therefore the choice of the appropriate model, are empirical issues related to the estimation of coefficients in capital accumulation equations like (16), (18) or (19). Numerical simulations presented later on suggest that the implication of these models for the long-run spatial distribution of capital, and convergence, in the context of the spatial Solow model, are not the same. In particular $D = 1$ combined with diminishing returns ($0 < \alpha < 1$), or constant returns ($\alpha = 1$) tend to provide spatially homogeneous capital distributions implying spatial convergence. On the other hand models of nonlinear diffusion result, with any type of returns to capital, in spatially nonhomogeneous capital distributions of various characteristics, implying that convergence is not attained in the long run.

5 Qualitative and Quantitative Aspects of the Spatial Solow Model

Following Proposition 3 the main object of this section is the study of the nonlinear spatial growth PDE characterizing the spatial Solow model under diminishing returns,

$$\frac{\partial}{\partial t}k = D\Delta k^{\beta+1} + sAk^{\alpha} - \delta k, \tag{20}$$

where $D > 0$ is a coefficient (independent of k), s is the savings ratio, $A(z)$ is a productivity parameter, δ is the rate of capital depreciation and $\alpha \in (0, 1)$ is the production elasticity. Without loss of generality, by a rescaling of the variable t, we can express the above equation in the form

$$\frac{\partial}{\partial t}k = \Delta k^{\beta+1} + c_1 k^{\alpha} - c_2 k \tag{21}$$

where

$$c_1 = \frac{sA}{D} \geq 0, \;\; c_2 = \frac{\delta}{D} > 0,$$

and

$$\beta = \rho_1 - (1 + \rho_2)(1 - \alpha).$$

It is conceivable that c_1 and c_2 depend on the spatial location z. Even if we assume D to be a constant, it is natural to assume that the saving ratio s depends on z, and the same may hold for A and δ. We further allow for regions where $c_1 = 0$ (i.e. regions where no saving is possible). The possibility of allowing for a set $U_0 \subset U \subset \mathbb{R}^d$ with the property $c_1(z) = 0$ if $z \in U_0$ may provide interesting economic implications.

The PDE (21) of the spatial Solow model will be complemented with an initial condition $k(0,z) = k_0(z)$, where $k_0 : U \subset \mathbb{R}^d \to \mathbb{R}_+$ is an initial capital stock distribution, and also with boundary conditions related to the prescribed behavior of the distribution of capital stock at certain parts of the domain U. We will consider two major types of boundary conditions: (a) consider $U = \mathbb{R}^d$ and assume that $k(t,z) \to 0$ as $|z| \to \infty$; and (b) Dirichlet boundary conditions, i.e., assume that $k(t,z) = 0$ for $z \in \partial U$ where ∂U denotes the boundary of U. Clearly, (a) can be taken as a limiting situation of (b) for large domains. Other types of boundary conditions are also possible, for example periodic boundary conditions or Neumann type boundary conditions (corresponding to regions in space where capital stock is repelled).

5.1 Steady state solutions

The starting point for our analysis will be the steady state solutions of model (21), that is, solutions which depend only on z and not on t. A steady-state solution $k^* = k^*(z)$ can be regarded as a steady-state distribution of the capital stock across space. For such solutions, the spatial Solow equation (21) simplifies to:

$$-\Delta k^{\beta+1} = c_1 k^{\alpha} - c_2 k. \tag{22}$$

To bring it into a more standard form we employ the Kirkhoff transformation, define the new variable

$$u = k^{1+\beta}, \;\; \beta \neq -1,$$

and express (22) in terms of u as

$$-\Delta u = c_1 u^q - c_2 u^p \tag{23}$$

where

$$q = \frac{\alpha}{1+\beta}, \;\; p = \frac{1}{1+\beta}.$$

We consider first the parameter range where $1+\beta > 1$ (slow diffusion case).[15]

We consider first the case where $z \in \mathbb{R}$, i.e. the one-dimensional case. Then equation (23) reduces to a second order ordinary differential equation (ODE) of the form

$$-\frac{d^2 u}{dz^2} = c_1 u^q - c_2 u^p. \tag{24}$$

[15] In this case we observe that $0 < q < p < 1$. In the linear diffusion case $q < p = 1$. Therefore, the elliptic equation (23) is always a sublinear equation except in the case of linear diffusion $\beta = 0$, where the nonlinearity is a linear combination of a sublinear and a linear term.

Under the simplifying assumption that c_1 and c_2 are independent of z, we can multiply this equation by $\frac{du}{dz}$ and integrate once over z to see that, along any solution of (24), the following equality holds:

$$\frac{1}{2}\left(\frac{du}{dz}\right)^2 + \frac{c_1}{1+q}u^{q+1} - \frac{c_2}{1+p}u^{p+1} = C, \tag{25}$$

where C is a constant of motion. This allows us to do a complete phase plane analysis for the steady-state solution, and even find closed form solutions for u by integration by quadrature. Since all possible solutions of (24) are characterized on the phase plane $(x_1, x_2) := (u, u')$ by the contours of the function $g(x_1, x_2) = \frac{1}{2}x_2^2 + \frac{c_1}{1+q}x_1^{q+1} - \frac{c_2}{1+p}x_1^{p+1}$, a phase plane analysis may provide information about the spatial distribution of the stock of capital at the steady state. Figure 5 below provides the phase diagram for (25) in terms of the the capital stock variable for the parameter choice

$$\alpha = 0.4, A = 1, s = 0.15, \delta = 0.03, \rho_1 = 1.3, \rho_2 = -0.1, \beta = 0.76, D = 0.1, \tag{26}$$

corresponding to (20).The shape of the contours around the horizontal axis suggest that the steady-state spatial distribution of the capital stock has an inverted U shape.

[Figure 5. Steady-state phase diagram in the spatial domain]

This is verified by the numerical solution of (24) shown in Figure 6.

[Figure 6. Steady state in the spatial domain]

The following proposition provides an existence and uniqueness result for the steady-state PDE (22). Clearly this PDE always has the solution $k(z) = 0$ for every $z \in U$, which will hereafter be called the trivial solution.

Proposition 4 *Let $\beta \geq 0$, $\alpha < 1$ and assume Hölder continuity properties*[16] *for the coefficients c_1, c_2. The steady-state equation (22) has a unique classical (non trivial) positive solution,*[17] *satisfying the a priori bounds*

$$0 \leq k \leq \left(\frac{\bar{c}_1}{\underline{c}_2}\right)^{\frac{1}{1-\alpha}},$$

where

$$\bar{c}_1 = \sup_{z \in U} c_1(z), \;\; \underline{c}_2 = \inf_{z \in U} c_2(z).$$

[16] A function $g : \mathbb{R}^d \to \mathbb{R}$ is called Hölder continuous when there exists $\eta < 1$ and $C > 1$ such that $|g(x) - g(y)| \leq C|x - y|^\eta$ for every $x, y \in \mathbb{R}^d$. Hölder continuity is a weaker form of uniform continuity than Lipschitz continuity.

[17] The regularity of the solution depends on the regularity of the coefficients c_1, c_2. If $c_1, c_2 \in L^\infty(U)$, then the solution is a weak solution $k \in L^1(U)$, whereas if the coefficients c_1, c_2 enjoy Hölder continuity properties, the solution is classical.

It should be noted that if c_1 and c_2 are independent of z, then the upper bound for the steady state is the steady state of the standard Solow model without diffusion of capital. Thus the spatial Solow model with nonlinear diffusion is characterized by a steady state which could either be spatially homogeneous, i.e. flat, or exhibit spatial heterogeneity.

The spatial model with nonlinear diffusion allows us to explore cases in which the spatial domain contains locations and regions where savings do not take place. This might be a realistic situation for extremely impoverished locations. We will call these regions poverty cores and define them as regions $V_0 \subset U$ with the property that $c_1(z) = 0$ if $z \in V_0$. The poverty core corresponds to regions where capital is identically zero at a steady state, implying that the steady-state distribution of the capital stock contains regions with no capital and regions with positive capital. This result suggests that economies where savings are not possible could eventually be trapped in the poverty core where their capital stock is depleted. The existence of poverty cores is verified by the existence of compact support solutions for the steady-state equation (22) and is established in the following proposition.

Proposition 5 *Let $\beta > 0$. If $c_1(z)$ vanishes at a point z, then any positive solution of the steady-state equation (22) will develop a poverty core. A poverty core never develops when $\beta = 0$, even if c_1 vanishes in subsets of U. To be more precise, assume that c_1, c_2 are Hölder continuous functions and allow c_1 to vanish at some point z_0 (corresponding to a spatial location where no savings takes place $s = 0$). Then, if $\beta > 0$, and for some $\rho > 0$, the function c_1 vanishes inside a ball centered at z_0 and of radius ρ, any non-trivial positive solution of the steady-state equation (22) will develop a poverty core, i.e. a region of total depletion of capital stock inside a ball centered at z_0 and of radius $\frac{\rho}{2}$, as long as the parameters of the problem satisfy the condition*

$$\bar{c}_1^{\frac{1-\alpha}{1+\beta}} \underline{c}_2^{-\frac{(1+\beta)(1+\beta-\alpha)}{\beta(1-\alpha)}} \leq c_0(\beta, d)\rho^{\frac{2(1+\beta)}{\beta}} \tag{27}$$

where $c_0(\beta, d)$ is a constant, depending only on β and the dimension d of economic space, given explicitly in the proof of this Proposition.

Poverty cores do not emerge in models of linear diffusion ($\beta = 0$) where capital moves from high to low abundance location. Thus nonlinear diffusion ($\beta > 0$) can help model the emergence of poverty cores where capital is depleted due to zero savings. This is because although capital moves to locations with low capital stock, since these locations are characterized by high marginal productivity of capital as capital is depleted, no part of the inflow is used for capital accumulation since nothing is saved. If model parameters are such that condition (27) holds, then insufficient accumulation will take place at this location and eventually the capital stock will be depleted.

As condition (27) shows, simply the vanishing of savings at a point is not enough to guarantee the existence of a poverty core. Due to the spatial interactions, a poverty core emerges when relation (27) - which relates the maximum value of c_1 over a wider region (i.e., savings and productivity in nearby regions) with the minimum value of c_2 over a wider region (i.e., rate of capital depreciation) as well as the characteristic of the velocity of capital flows (provided by β) within the region where savings vanishes - is satisfied. Furthermore, the procedure followed in the proof provides information on the local behavior of capital stock near a point z_0 with zero savings. Capital stock can be identically zero inside a ball of center z_0 and radius $\rho/2$, i.e. well inside the region where c_1 vanishes, but capital may start accumulating (still inside the region where $c_1 = 0$) on account of spatial effects and capital flow from nearby regions, since marginal productivity inside this ball is high. An upper bound for the spatial gradients of capital stock in this accumulation region can be provided by $k \leq C\left(\frac{2}{\rho}|z-z_0|-1\right)^{\frac{2}{\beta}}$, as long as z is such that $\frac{\rho}{2} < |z-z_0| < \rho$. Therefore, if the savings profile is such that $c_1(z) = 0$ for $z \in U$, with $|z-z_0| < \rho$, then any solution of the steady-state equation (22), $k^*(z)$ must be such that $k^*(z) = 0$ for $z \in U$ when$|z - z_0| < \frac{\rho}{2}$, while $k^*(z)$ may be non-zero for z such that $\frac{\rho}{2} < |z-z_0| < \rho$. Note that these z locations are well inside the region where productivity vanishes, but k is always bounded above by the radially symmetric profile, $k \leq C\left(\frac{2}{\rho}|z-z_0|-1\right)^{\frac{2}{\beta}}$, as long as z is such that $\frac{\rho}{2} < |z-z_0| < \rho$.

5.2 Time and space dependent solutions

Having studied the steady state we now turn our attention to the analysis of the full spatiotemporal Solow model. This means finding the spatial distribution of capital stock $k(t,z)$ at each point of time t that emerges if the fundamentals of the economies are determined by the basic assumptions of the Solow growth model and capital flows towards locations of relatively higher marginal productivity with velocity determined endogenously by local capital stock and the size of marginal productivity. The corresponding mathematical problem reads as follows: given a function $k_0 : U \to \mathbb{R}$, find $k : [0,T) \times U \to \mathbb{R}$ such that the following initial boundary value problem is satisfied:

$$\begin{cases} \frac{\partial k}{\partial t} = \Delta\Phi(k) + f(z,k) \ , \ 0 \leqslant t \leqslant T, \ z \in U, \\ k(t,z) = 0 \ , \ 0 \leqslant t \leqslant T, \ z \in \partial U, \\ k(0,z) = k_0(z) \ , \ z \in U, \end{cases} \tag{28}$$

where $\Phi : [0,\infty) \to [0,\infty)$, $f : U \times [0,\infty) \to [0,\infty)$ are generic functions. For our purposes we have

$$\Phi(k) := k^{1+\beta} \ , \ f(z,k) := c_1(z)k^{\alpha} - c_2(z)k. \tag{29}$$

Note that the determination of T, that is to find the maximal time interval for the solution, is also part of the problem.

5.2.1 Solvability

Problem (28) seems to be very well documented in the literature, see for example Levine and Sacks [32], Bandle et al. [33], Anderson [34], [35] and the references therein. The main assumptions made on functions Φ, f are the following:

A1 Φ is continuously differentiable with $\Phi(0) = \Phi'(0) = 0$ and $\Phi'(k) > 0$ for $k > 0$.

A2 f is continuous with $f(z, 0) = 0$.

The vanishing of Φ' at zero makes the problem degenerate and thus the concept of a classical solution for (28) is not appropriate. For this reason, and by following the usual procedure, we introduce a weak notion of solution. We let $U_T := (0, T) \times U$, $(\partial U)_T := (0, T) \times \partial U$ and define the space of test functions as follows:

$$\mathcal{J} := \left\{ \psi \in C(U_T) : \psi \geqslant 0,\ \psi|(\partial U)_T = 0 \text{ and } \frac{\partial \psi}{\partial t},\ \Delta\psi \in L^2(U_T) \right\}.$$

Definition 1 *Let $k_0 \in L^\infty(U)$. The function $k \in L^\infty(U_T)$, $k \geqslant 0$, is called a (weak) solution of* (28) *if*

$$\int_U k(t,z)\psi(t,z)\,dz = \int_U k_0(z)\psi(0,z)\,dz +$$
$$\int_0^t \int_U \left[k(s,z)\frac{\partial \psi}{\partial t}(s,z) + \Phi(k(s,z))\Delta\psi(s,z) + f(k(s,z))\psi(s,z) \right] dz\,ds,$$

for $0 \leqslant t < T$ and every $\psi \in \mathcal{J}$.

If we replace the equality above with $\geqslant$ or $\leqslant$, then we obtain the concept of supersolution or subsolution, respectively, of (28). By making two technical assumptions on Φ, f, which are satisfied by (29), we obtain the following existence result from Levine and Sacks [32, Theorems 2.1 and 3.1].

Proposition 6 *There exists a solution $k \in C(U_T)$ of* (28)*, where $T > 0$ depends on $\| k_0 \|_{L^\infty(U_T)}$ and f. If $f(z,k) \leqslant c(1 + \Phi(k)^\nu)$ for some constants $c \geqslant 0$, $\nu \in [0, 1)$, then the solution is defined for every $T > 0$. If f is locally Lipschitz with respect to k, then the solution is unique.*

We also have a comparison principle (Anderson [35, Theorem 2.1]).

Proposition 7 *Let k_0^1, $k_0^2 \in L^\infty(U)$, $k_0^1 \leqslant k_0^2$. Let also k^1 be a subsolution of* (28) *with initial datum k_0^1 and k^2 be a solution of* (28) *with initial datum k_0^2. Then $k^1 \leqslant k^2$.*

Corollary 1 *For the choice* (29) *and for every $k_0 \in L^\infty(U)$, the problem* (28) *has a solution defined for $t \in [0, \infty)$.*

However, the above fundamental results do not completely cover the specific form of equation (28) needed to study our spatiotemporal Solow model, in the case of a Cobb-Douglas production function, with diminishing returns. This is because when such a production function is assumed, the function f (see (29)) is not Lipschitz near 0. Therefore, we can no longer invoke Proposition 6 to guarantee uniqueness of solutions. In fact, a simple example of this non-uniqueness can be seen in the case where c_1, c_2 are constants and we have the zero initial condition, $k_0(z) = 0$. Then of course we have the trivial solution $\underline{k}(t, z) = 0$, but the function

$$\overline{k}(t, z) = \left[\frac{c_1}{c_2}\left(1 - e^{-(1-\alpha)c_2 t}\right)\right]^{\frac{1}{1-\alpha}}$$

is also non-trivial and positive for the $t > 0$ solution.

It is important to note that this non-uniqueness property, which stems from the fact that the Cobb-Douglas production function is Holder continuous rather than Lipschitz continuous near 0, also appears in the standard spatially independent Solow model. In the purely temporal case, however, this potential complication is bypassed by assuming that the neighborhood of 0 is avoided and all attention is focused on the neighborhood of the non-zero steady state. However, this assumption might not be enough in the case where the spatial allocation of capital in the Solow model is under investigation. For example, as we saw in the previous section, spatially dependent steady states that exhibit poverty cores are supported by the spatial Solow; this imposes the need to understand in detail the behavior of the system near the problematic region where capital stock becomes very small. Furthermore, non-uniqueness is an undesirable property for a model when one wants to solve the problem numerically. For this reason, we need an unambiguous way to define the solution of (28) that is consistent with the structure of problem (28) as a problem of economic growth. Thus we adopt a regularization which consists of considering a family of slightly modified problems which do not exhibit the pathology of the original system, but the pathology reappears in a properly defined limit.

In the present context this regularization procedure is as follows: The Solow model, regardless of whether it is the standard classical temporal Solow model or a spatiotemporal version, inherits its ill-posed nature from the non-Lipschitz behavior of the function f when $k \simeq 0$. As the model itself is dynamic, it is not proper to assume a priori that the solution is far from

0. What is a better strategy is to consider a family of functions f_b which are properly modified versions of the original production function near $k = 0$, in the sense that they are Lipschitz, but satisfy the condition that $f_b(0) = 0$, and coincide with the original Cobb-Douglas production function away from an infinitesimal neighborhood of $k = 0$. This neighborhood is chosen to be $(0, b]$, so that for small b we are arbitrarily close to the original production function, and in the limit as $b \to 0$ the production function f_b coincides with the original Cobb-Douglas production function.

A simple choice would be to choose the family $\tilde{f}_b$ as follows:

$$\tilde{f}_b(z,k) := \begin{cases} a_1 k + a_2 k^2 + a_3 k^3 + a_4 k^4 \ , \ 0 \leqslant k \leqslant b, \\ f(z,k) \ , \ k \geqslant b, \end{cases} \tag{30}$$

where constants a_1, a_2, a_3, a_4 are chosen is such a way that $\tilde{f}_b$ is two times continuously differentiable with respect to k. This requirement provides three equations, so one parameter can be freely chosen. We actually choose

$$a_1 := \left.\frac{\partial f}{\partial k}\right|_{k=b} \quad \text{since} \quad \left.\frac{\partial \tilde{f}_b}{\partial k}\right|_{k=0} = a_1,$$

so we mimic exactly the behavior of the derivative with respect to k of the original f as $k \to 0^+$. Clearly, $\lim_{b \to 0} \tilde{f}_b\,(k) = f\,(k)$ for every $k \in \mathbb{R}_+$.

Following ideas found in Pablo and Vazquez [36, Section 2],[18] we consider the family of problems

$$\begin{cases} \frac{\partial k}{\partial t} = \Delta \Phi(k) + \tilde{f}_b(z,k) \ , \ 0 \leqslant t \leqslant T, \ z \in U, \\ k(t,z) = 0 \ , \ 0 \leqslant t \leqslant T, \ z \in \partial U, \\ k(0,z) = k_0(z)\,, \ z \in U. \end{cases} \tag{31}$$

Since $\tilde{f}_b$ is k-Lipschitz for every $b > 0$, (31) has a unique continuous solution k_b. Then, as proven in detail in the aforementioned references, the pointwise limit

$$k(t,z) = \lim_{b \to 0^+} k_b(t,z) \tag{32}$$

exists and provides a continuous solution (called the minimal solution) of (28). Actually, it is exactly (32) that we understand as *the* solution of (28).

The above analysis can be generalized in the case of non-homogeneous Dirichlet boundary conditions, when we consider the problem

$$\begin{cases} \frac{\partial k}{\partial t} = \Delta \Phi(k) + f(z,k) \ , \ 0 \leqslant t \leqslant T, \ z \in U, \\ k(t,z) = K \ , \ 0 \leqslant t \leqslant T, \ z \in \partial U, \\ k(0,z) = k_0(z)\,, \ z \in U, \end{cases} \tag{33}$$

[18] See also the very recent paper by Bo and Ning [37] where a similar treatment appears.

where $K \geqslant 0$ is a constant. Actually, with the change of variable $k \to k - K$, the problem is reduced to the form (28). Observe incidentally, that k_0 has to satisfy the consistency condition $k_0(z) = K$, $z \in \partial U$. If f is independent of z and $k_0(z) = K$ for all $z \in U$, then the constant function $k(t, z) = K$ is the solution of (33) if and only if $f(K) = 0$, that is, in the case of (29) with c_1, c_2 independent of z,

$$K = 0 \text{ , or } K = \left(\frac{c_1}{c_2}\right)^{\frac{1}{1-\alpha}}. \tag{34}$$

Note, however, that the constant function K is always a subsolution of (33); therefore if $k_0(z) \geqslant K$, then Proposition 7 implies that the solution (32) satisfies $k(t, z) \geqslant K$.[19]

5.3 Stability of steady states

Section 5.2 provided results related to the existence of solutions for the spatial Solow model, while section 5.1 provided results related to the existence of the steady state. Combining these results, the next step is to study the stability properties of a steady state which is reached as the solution of the spatial Solow model evolves through time. Let $k^* = k^*(z)$ be a steady-state solution of (33), that is, k^* satisfies

$$\Delta\Phi(k^*(z)) + f(z, k^*(z)) = 0 \text{ , } z \in U.$$

We study the behavior of solutions of the form $k(z, t) = k^*(z) + u(z, t)$, where u is understood as a small perturbation (satisfying the homogeneous Dirichlet boundary condition). This behavior will indicate whether the small perturbation dies out or not with the passage of time. If the perturbation dies out as time advances, the steady state is locally stable; otherwise it is unstable.

Assume that $K > 0$, then the comparison principle implies that $k^*(z) \geq K$ and following stability result can be stated.

Proposition 8 *A positive and bounded away from zero steady state $k^*(z)$ of the spatial Solow model is stable.*

For proof, see Appendix.

This is in line with the non-spatial Solow model where a positive steady state is stable. The interesting result in this case is that this steady state need not be spatially homogeneous. Thus a non-flat distribution of per capita capital and output can be persistent in the long run.

[19]It is clear that $K = 0$ and $K = \left(\frac{c_2}{c_1}\right)^{\frac{1}{1-\alpha}}$ in this case are the unstable and stable, respectively, steady states of the non-spatial Solow model.

Let us now consider the case $K = 0$ and assume without loss of generality that $k^*(z)$ vanishes only at the boundary of the spatial domain and that $k^*(z) > 0$ for $z \in U$. Then as shown the Appendix (proof of Proposition 8) this steady state is stable. This implies that a poverty core which can be regarded as locations at the edge of the economic space where capital stock is depleted will persist in the long run.

5.4 The case of increasing returns

In this section we make some brief comments on the behavior of the growth process in the case of increasing returns. As shown in Proposition 3, in the case of increasing returns ($\alpha > 1$) our model assumes the form of the PDE

$$\frac{\partial}{\partial t}k = \frac{\bar{D}}{1+\beta}\nabla_z \cdot (D_0(z)\nabla_z k^{\beta+1}) + sA(z)k^{\alpha} - \delta k\%$$

where $\bar{D} = \alpha^{\rho_2}\alpha(1-\alpha)$, $\beta = \rho_1-(1+\rho_2)(1-\alpha)$ and $D_0(z) = B(z)(A(z))^{1+\rho_2}$, with $\bar{D} < 0$. This can be brought into the form of equation (20) as

$$\frac{\partial}{\partial t}k = D\Delta k^{\beta+1} + sAk^{\alpha} - \delta k \tag{35}$$

by a redefinition of coefficient $\bar{D}$, but with the important difference that $\bar{D} < 0$. Therefore, the important difference in the context of our model, when passing from the decreasing to the increasing returns case, is the change of sign of the diffusion coefficient from positive to negative. This is, however, an important qualitative difference, as it changes considerably the qualitative features of the model, turning diffusion effects into anti-diffusion effects, thus favouring the formation of large capital concentrations in a few locations.

In the steady state-case, that is, in the case of capital formations which do not change over time, we can proceed with our analysis in a similar fashion as in the decreasing returns case and use the Kirkchoff transformation, defining the new variable as

$$u = k^{1+\beta}, \;\; \beta \neq -1,$$

and express the steady state of (35) in terms of u as

$$-\Delta u = C_1 u^Q - C_2 u^P, \tag{36}$$

where now since $\bar{D} < 0$ we have that $C_1 = c_2$, $Q = p$ and $C_2 = c_1$, $P = q$ (with $\bar{D}$ replaced by $|\bar{D}|$ in the definitions of c_1 and c_2). Therefore, we have an elliptic problem which is formally the same as (23) but with the role of the exponents in the potential term interchanged. As a result, Proposition 5 for the existence of steady states and Proposition 6 for the existence of poverty traps are still valid in the increasing returns case, but with the roles of the coefficients c_1, c_2 and p, q interchanged. This of course reflects on the economics as these coefficients are related to various parameters in the primitives of the model, thus showing the important impact that the assumption of increasing returns has on the findings of the model.

The difference becomes even more pronounced in the time dependent case. There, the nature of the model changes, as we now have a diffusion type equation but with a negative diffusion coefficient. In simple terms, this means that this model will behave as a standard diffusion equation backwards in time; that is, since a diffusion equation tends to flatten out sharp spatial gradients, our model will have an opposite effect tending to attenuate spatial gradients. In the context of our model this implies that when increasing returns are assumed, large spatial capital concentrations will tend to become even larger. Note that this phenomenon is a novel aspect of our model, since it does not appear in the standard linear diffusion models in which the diffusion part is always positive and tends to dissipate spatial gradients, irrespectively of the nature of the returns.

The negative diffusion case introduces mathematical complications in the model which definitely merit further study. However, as a first look at the interesting new aspects it may introduce into the model, we sketch some new phenomena related to the linear stability of the flat steady state. Assume a flat steady state k^*.[20] Then, linearizing around it, (49) becomes the constant coefficients diffusion PDE

$$\frac{\partial u}{\partial t} = \Phi'(k^*)\Delta u + f'^*)u, \tag{37}$$

where

$$\Phi'(k^*) = \bar{D}(1+\beta)(k^*)^{\beta}, f'^*) = \alpha(k^*)^{\alpha-1} - \delta, \tag{38}$$

and $\bar{D} < 0$. For simplicity assume that $d = 1$ (one-dimensional space) and $U = [0, L]$. Then, the spectrum of the linearized operator is

$$\lambda_n = -\bar{D}n^2(1+\beta)(k^*)^{\beta} + \alpha(k^*)^{\alpha-1} - \delta. \tag{39}$$

If $\bar{D} > 0$ since $\alpha(k^*)^{\alpha-1} < 0$, then $\lambda_n < 0$ for every n so we get stability. However, if $\bar{D} < 0$ (increasing returns), then there are n for which $\lambda_n > 0$. That means that there are perturbations of the type $\sin(n\pi x/L)$ for specific n that will turn unstable in the case of increasing returns. The unstable modes will be those corresponding to $n > n^*$ where

$$n^* > \left(\frac{\alpha(k^*)^{\alpha-1} - \delta}{\bar{D}}\right)^{1/2}.$$

[20] In this case, increasing returns can emerge from Romer-Lucas type spatial externalities, by assuming for example that $y = k^{\alpha}\left(\bar{K}\right)^{\zeta}, \bar{K} = \xi k$ and $\alpha + \zeta > 1$, where $\bar{K}$ is the spatial externality.

5.5 Numerical simulations

Having established existence of solutions, steady states, and stability properties for the steady states, we turn now to some simulation results to determine the shape of the spatiotemporal distribution of capital emerging for the spatial Solow model under plausible parameter choice. Our simulations[21] numerically solve model (17), for the one-dimensional case, which in an extended form can be written as

$$\frac{\partial k(t,z)}{\partial t} = \frac{\bar{D}}{1+\beta}\left[\beta(\beta+1)k^{\beta-1}\left(\frac{\partial k}{\partial z}\right)^2 + (\beta+1)k^{\beta}\left(\frac{\partial^2 k}{\partial z^2}\right) + sAk^{\alpha} - \delta k\right] \tag{40}$$

$$\bar{D} = \alpha^{\rho_2}\alpha(1-\alpha),\ \beta = \rho_1 - (1+\rho_2(1-\alpha)).$$

Using the same parameter choice as in section 5.1, Figure 7 depicts the spatiotemporal evolution of the stock of capital with initial condition $k(0,z) = \mathrm{e}^{-z^2/4} + 0.01\sin[50\pi z] - 0.0183156$, $z \in [-4,4]$, which is a bell shaped distribution, chosen with the purpose of approximating through the initial conditions the distributions observed in Figures 1 and 2. The productivity parameter is also taken as $A(z) = \left(\mathrm{e}^{-z^2/4}\right)^{1+\rho_2}$ by assuming that the spatial distribution of this parameter is similar to the distribution of the capital stock, that is, more developed locations have a relatively higher productivity parameter which might reflect positive spatial spillovers associated with the higher concentration of the capital stock in these locations. Boundary conditions were $k(-4,t) = k(4,t) = 0$, assuming that at the edges of the economic space there are locations with no capital. The results do not change if we use $k(-4,t) = k(4,t) = k^0 > 0$.

[Figure. 7 Spatiotemporal distribution of capital]

Figure 8 depicts the evolution of the Sobolev norm defined as

$$Sb(t) = \int_{-4}^{4}\left(\frac{\partial \hat{k}(t,z)}{\partial z}\right)^2 dt$$

where $\hat{k}(t,z)$ is the solution of (20) as depicted in Figure 7.

[Figure 8. The time path of the Sobolev norm]

The convergence of the Sobolev norm to a fixed number means that the spatial gradients remain constant after a certain point in time, implying

[21] Wolfram Mathematica was used for the numerical simulations. The PDEs were solved for $t \in [0, 1000]$ with the exception of increasing returns to scale where the solution was exploding in small time due to the development of very sharp spatial gradients.

that the system converges to a spatially nonhomogeneous distribution of the stock of capital. Furthermore the peak of the distribution in Figure 7 converges for $t > 200$ to a fixed positive number. Combining this with the convergence of the Sobolev norm suggests that the growth model converges in a spatiotemporal sense to a nonhomogeneous capital stock distribution. This result is consistent with our theory about the stability of spatially nonhomogeneous steady states. Since per capita output is given by $\hat{y}(t,z) = \left(\hat{k}(t,z)\right)^{\alpha}$, per capita output also converges to a spatially nonhomogeneous distribution.

It should be noted that a spatially nonhomogeneous bell-shaped pattern persists with circle boundary conditions $k(-4,t) = k(4,t)$ and with time dependent boundary conditions $k(-4,t) = k(4,t) = \gamma t$ or $k(-4,t) = k(4,t) = \mathrm{e}^{\gamma t}$, which may reflect the assumption that locations with low capital stock at the beginning may grow fast. This is shown in Figures 9 and 10.

[Figure 9. Circle boundary conditions]
[Figure 10. Time dependent boundary conditions]

Figure 11 presents the solution for the AK model ($\alpha = 1$) with spatially nonhomogeneous initial conditions and productivity parameter.

[Figure 11. A spatiotemporal AK model]

The Sobolev norm of the solution and the peak of the distribution are monotonically increasing. Since the solution of this problem is proportional to per capita output, the result indicates sustained growth with the distribution of per capita income becoming less uniform with time. The shape of this distribution can be regarded as an approximation of the distribution depicted in Figures 1 and 2.

We also simulate the spatial Solow equation for the case of increasing returns by assuming $\alpha = 1.1$, while keeping the rest of the parameters fixed. Increasing returns lead to anti-diffusion, very sharp spatial gradients and concentration of the stock of capital in a very small number of locations as shown in Figure 12. This spatial pattern is verified by the sharp increase of the Sobolev norm, and the overall behavior is consistent with the theoretical results of the previous section.

[Figure 12. Spatiotemporal evolution under increasing returns]

The numerical simulations seem to support the theory developed in the context of a spatial Solow model regarding the spatiotemporal evolution of the capital stock and the existence of steady states for a plausible set of parameter values regarding savings rates depreciation and returns to scale.

Furthermore they seem to suggest that capital flows characterized by capital seeking locations of high returns and an endogenous flow velocity result in a persistent spatially nonhomogeneous distribution of capital and per capita output across locations. This result holds under various types of boundary conditions. The spatial distribution tends to become flatter the higher the coefficients of $\left(\frac{\partial k}{\partial z}\right)^2$ and $\left(\frac{\partial^2 k}{\partial z^2}\right)$ are in (40). The model produces a flat distribution with circle boundary conditions when there is no spatial variety in the productivity parameter.

Models with capital flows based on the trade balance which result in linear diffusion and capital accumulation of the form

$$\frac{\partial k\,(t,z)}{\partial t} = \frac{\partial^2 k}{\partial z^2} + sAk^{\alpha} - \delta k \tag{41}$$

tend to produce a flatter and in many cases spatially homogeneous distribution for the capital stock.

6 Concluding Remarks

In order to explore mechanisms underlying the temporal evolution of the cross sectional distribution of per capita capital and output across space, we develop a spatial growth model where saving rates are exogenous. Capital movements across locations are governed by having capital moving towards locations of relatively higher marginal productivity, with a velocity determined by the existing stock of capital and its marginal productivity. Considering that the spatial domain corresponds to economic space, we developed a local model in which the fundamental growth equation of the Solow model is augmented by a nonlinear diffusion term, which characterizes spatial movements.

We show that the augmented Solow equation has a solution and that steady states exist. Furthermore, under diminishing returns the growth process leads to a stable spatially non-homogeneous distribution for per capita capital and income in the long run. AK production functions and increasing returns lead to a strong persistent and increasing concentration of capital in a very few locations. Insufficient savings may lead to the emergence of poverty cores where capital stock is depleted in some locations and stability analysis indicates that a steady state with poverty cores is stable. This suggests that economies can persistently remain in the poverty core while economies in other locations will have a positive capital stock. In the spatial Solow model, zero capital stock in some locations is consistent with the long-run stability of the entire spatial distribution of the stock of capital. Numerical simulations confirm our theoretical results and provide spatial distributions that can be regarded as resembling observed distributions of per capita GDP. Our approach, by endogenizing the velocity of the capital flow, provides a rich environment for studying growth processes in a spatiotemporal context. Moreover our approach, by linking capital flows with differences in the marginal productivity of capital across locations and not with differences in the stock of capital across locations, seems not to suffer from the problems associated with the Lucas paradox. The emergence of spatial distributions where poverty cores coexist with locations where the stock of capital is high - that is, the solution of the growth equation results in a distribution with compact support - is a potentially interesting result suggesting that the nonlinear diffusion approach could provide a mechanism explaining observed outcomes.

The study of forces shaping the evolution of the entire cross sectional distribution of economies is an issue of importance for growth theory. Our approach, based on the assumption that capital seeks high productivity locations, leads to a novel growth equation augmented with nonlinear diffusion. Although the mathematical complexity is increased, the benefit is that we obtain results regarding the dynamics of the cross sectional distribution of capital and the structure of its steady states that can explain observed distributions.

Further research could explore the spatiotemporal distribution of capital by using the mechanism developed in this paper with optimizing consumers, both in the context of a social planner and a decentralized economy; by explicitly introducing spatial externalities in the production function; and by associating capital flows not with current marginal productivity but with the future discounted values of marginal productivity of capital at a particular location.

A Proofs

A.1 Proof of Proposition 1

Proof. Assume there exists a smooth enough function $k : [0,T] \times U \to \mathbb{R}$, such that $k_i(t) \simeq k(t, i\,\Delta z)$ for small enough $\Delta z = d$. For the present context $U \subseteq \mathbb{R}$. Using this assumption we rewrite the differences $m_{i+1} - m_i$ in terms of the derivatives of the production function. Then,

$$m_{i\pm1} - m_i = \pm f^{''}(k(z))k_z\, d + \frac{1}{2} f^{''}(k(z))\, k_{zz}\, d^2 + \frac{1}{2} f^{'''}(k(z))\, (k_z)^2\, d^2$$

where the subscript denotes the partial derivative. In the above we have used the Taylor expansion twice, once on the function $k(t,z)$ and once on the function $f(k)$. In the limit as $d \to 0$ (the continuous limit) and for smooth enough production functions, the terms which are quadratic in d and be neglected as much smaller with respect to the linear terms in d. The condition $m_{i\pm1} - m_i > 0$ which will result to capital migration from i to

$i \pm 1$ then becomes in the limit as $d \to 0^+$, $\pm f^{''}(k(z))k_z \geq 0$. Therefore,

$\phi(t, i \to i \pm 1) =$

$$\begin{cases} \lambda \left(\pm f^{''}(k(z))k_z\, d + \frac{1}{2} \left(f^{''}(k(z))\, k_{zz} + f^{'''}(k(z))\, (k_z)^2 \right) d^2 \right) & , \ \pm f^{''}(k(z))k_z \geq 0 \\ 0 & , \ \pm f^{''}(k(z))k_z < 0, \end{cases}$$

with similar expressions holding for $\phi(t, i \pm 1 \to i)$.

We now insert these approximations in the balance equation and keep terms up to d^2. We see that no matter what the sign of the term $f^{''}(k)k_z$ is the equation becomes

$$\frac{\partial k}{\partial t} = \Phi(k) - \alpha \frac{\partial}{\partial z} \left((f^{''}(k)k) \frac{\partial k}{\partial z}, \right)$$

where $\alpha > 0$ is a constant depending on the discretization (taken from now on equal to 1 without loss of generality). ■

A.2 Gauss divergence and capital flows 3.2

The Gauss divergence theorem provides a convenient mathematical foundation to model capital flows. We will briefly describe the theorem and show how in can be used to describe in an economic meaningful way capital flows across locations. Consider any domain $S \subset \mathbb{R}^d$ with boundary ∂S (of sufficient regularity) and consider a vector field $\mathfrak{v} : S \to \mathbb{R}^d$ with components $\mathfrak{v} = (v_1, \cdots, v_d)$. Then, the Gauss' divergence theorem connects the total contributions of the vector field $\mathfrak{v}$ on ∂S (a surface integral) with the total contributions of a related scalar field $w = \text{div}\mathfrak{v}$ (or using another common notation $w = \nabla_z \cdot \mathfrak{v}$) over the whole volume S (a volume integral). In particular, according to Gauss's divergence theorem

$$\int_{\partial S} \mathfrak{v}(s) \cdot n\, d\sigma(s) = \int_S \text{div}\mathfrak{v}(z)\, dz = \int_S \nabla_z \cdot \mathfrak{v}(z)\, dz \qquad (42)$$

where $d\sigma$ denotes a surface integral over the $d-1$-dimensional surface ∂S with n being the outward unit normal to the surface ∂S and dz denotes a volume integral in $\mathbb{R}^d$. The term $\text{div}\mathfrak{v}$ or $\nabla_z \cdot \mathfrak{v}$ in the volume integral is a scalar function $w : S \to \mathbb{R}$, called the divergence of the vector field $\mathfrak{v}$. As Gauss' theorem is based on an appropriate integration by parts argument, the divergence of a function involves the partial derivatives of the components of $\mathfrak{v}$ with respect to the coordinates of $z = (z_1, \cdots, z_d)$. It is important to note that the actual form of the divergence operator involves the geometry of S, as depicted in the coordinate system used. If Cartesian coordinates are used, then

$$\text{div}\mathfrak{v} = \nabla_z \cdot \mathfrak{v} = \sum_{i=1}^{d} \frac{\partial v_i}{\partial z_i}.$$

and in one dimension ($d = 1$) where $\mathfrak{v} = v$, a scalar, the divergence reduces to $\text{div}\mathfrak{v} = \frac{\partial v}{\partial z}$.

The surface integral in (42) sums up all the contributions of the vector valued function $\mathfrak{v}$ on the boundary of S, ∂S, but assigns a positive or a negative sign to them depending on the angle between $\mathfrak{v}$ and the outer normal n on ∂S. Therefore, if $\mathfrak{v}$ is a vector field related to the direction and amount of a particular entity which flows in space, then the surface integral in (42) is a measure of the net amount that has entered or left the volume S through its boundary ∂S. Gauss's theorem thus connects this net amount with the a scalar quantity, the divergence of the vector field $\mathfrak{v}$ which models the flux of the entity in question in space. >>

Consider a domain $U \subset \mathbb{R}^d$ and let U_0 be any subset of U. Denote by $k(t, z)$ the stock of capital in this domain at time t and at spatial point z. Then the total amount of capital in U_0 at time t is given by the volume integral $\int_{U_0} k(t, z)\, dz$. Capital flows everywhere within U_0, so it is natural to consider a vector field $\mathfrak{v} = (v_1, \cdots, v_d)$, $\mathfrak{v} : U_0 \to \mathbb{R}^d$ which gives us the velocity of capital flow (flux) at every point of U_0. One way to define this vector field is component-wise by defining

$$-k(t, z)v_i\, \Delta z \Delta t := \text{Capital that moved from}\, z \mapsto z + \Delta z\, e_i \;\text{ in time}\, \Delta t,$$

where $e_i = (\delta_{1i}, \cdots, \delta_{di})$ is the unit vector in the i-th direction. Note that v_i can take positive or negative values depending on whether capital flows from $z = (z_1, \cdots, z_i, \cdots, z_d)$ to $z + \Delta z\, e_i = (z_1, \cdots, z_i + \Delta z, \cdots, z_d)$ or vice versa. The velocity is a vector field and this models the fact that capital flow has a direction. The change of total capital in U is due to the outflow or inflow of capital from the boundary of U, ∂U. At different points on this boundary there is different rate of inflow and outflow of capital which is given in terms of a vector valued function $\mathfrak{v}$, as $k(t, z)\, \mathfrak{v}|_{z \in \partial U}$ (where we take the restriction of the vector field on ∂U). The total inflow and outflow of capital through ∂U is given by the surface integral $\int_{\partial U} k(t, s)\, \mathfrak{v} \cdot n\, d\sigma(s)$ where n is the outward unit normal of ∂U, and by s we denote that all functions are restricted on the surface ∂U. Using the divergence theorem.

A.3 Proof of Proposition 2

Proof. Through the use of the divergence theorem we can determine the total change of capital stock in the domain U_0 as:

$$\frac{d}{dt} \int_{U_0} k(t, z) dz = - \int_{\partial U_0} k(t, s) \mathfrak{v} \cdot n, d\sigma(s) = - \int_{U_0} \nabla_z \cdot (\mathfrak{v}\, k(t, z))\, dz$$

By the first and last terms of this equation, which hold for any $U_0 \subseteq U \subset \mathbb{R}^d$, we have that

$$\frac{\partial}{\partial t} k(t, z) = -\nabla_z \cdot (\mathfrak{v}\, k(t, z)).$$

This argument suggests that the transport operator $\mathfrak{T}$ is of the general form

$$\mathfrak{T}\check{k} = -\nabla_z \cdot (\mathfrak{v}\check{k}),$$

and can only be specified if the vector field $\mathfrak{v}$ is specified. Adding capital formation $\Phi(t,z)$ we obtain

$$\frac{\partial}{\partial t}k(t,z) = -\nabla_z \cdot (\mathfrak{v}\, k(t,z)) + \Phi(t,z) \tag{43}$$

Equation (43) is called a continuity equation and is nothing else but a sophisticated (continuous space) book-keeping argument for the capital flow, based on the use of Gauss' theorem. ■

A.4 Proof of Proposition 4

Proof. Let ϕ_1 be the eigenfunction which corresponds to the first positive eigenvalue σ_1 of the problem

$$\begin{aligned} -\Delta u &= \sigma u, \text{ in } B(z_0, r), \\ u &= 0, \text{ in } \partial B(z_0, r), \end{aligned}$$

where $z_0 \in U$ be a point such that $c_1(z) \geq c > 0$ for every $z \in B(z_0, r)$. It is easy to show that as long as a $\rho > 0$ is choosen small enough, then $\underline{u}$ is a subsolution of (23), while $\bar{u} = M_* = \left(\frac{\bar{c}_1}{\underline{c}_2}\right)^{\frac{1}{p-q}}$ is a supersolution such that $\underline{u} < \bar{u}$. Then, the existense of classical solutions for (23) can be shown using theorem 1.7 of [38] and a standard bootsrapping argument

Uniqueness is more involved, especially in the case $\beta > 0$. For that we use Theorem 2.1 in [39], according to which the positive (non trivial) solution is unique if there exists a function $g \in C^1(0,+\infty) \cap C([0,+\infty))$, with the properties (a) $g(s) > 0$ for $s > 0$, (b) $g'(s)$ non-increasing with $1/g$ integrable near 0 and (c) the map $u \mapsto \frac{\Phi(z,u)}{g(u)}$ is non-increasing in $(0,\infty)$ for a.e. $z \in U$. There are two obvious choices for g, (1) $g_1(u) = u^q$ and (2) $g_2(u) = u^p$. Both, satisfy conditions (a) and (b). For choice (1) condition (c) holds if $c_2 \geq 0$ while for choice (2) condition (c) holds if $c_1 \geq 0$. Therefore, the solution is unique in any possible case. ■

A.5 Proof of Proposition 5

Proof. We apply the Kirkhoff transformation and work with the transformed steady state equation (23). Our argument is inspired by [40] who studied a very similar system.

Let us start by showing why a poverty core **never** occurs for the linear diffusion case $\beta = 0$. Consider u to be a solution of the transformed steady state equation (23), and observe that

$$0 = -\Delta u - c_1 u^q + c_2 u^p < -\Delta u + Ku, \tag{44}$$

for some constant $K > 0$, as long as the inequality

$$-c_1 u^q + c_2 u^p < Ku, \ \ \forall\, u \in [0, M_*]. \tag{45}$$

holds. If we can find a constant K satisfying inequality (45), then, by (44) we see that the solution to the transformed steady state equation (23), u, will satisfy the inequality $-\Delta u + Ku > 0$ so by the strong maximum principle $u(z) > 0$ for every z in the interior of U, therefore, it **cannot** develop a poverty core. We therefore, just need to consider whether there exists $K > 0$ such that inequality (45) holds. To do that, consider the function $g : U \times [0, M_*] \to$, where $M_* = \left(\frac{\bar{c}_1}{\underline{c}_2}\right)^{\frac{1}{p-q}}$, defined by $g(z, x) = -c_1(z)x^q + c_2(z)x^p - Kx$. Observe that $g(z, 0) = 0$ so inequality (45) will hold if we may find $K > 0$ so that the function $x \mapsto g(z, x)$ is strictly increasing for any $x \in [0, M_*]$ and any $z \in U$. Taking the derivative of g with respect to x (and denoting that with a prime) we see that our problem is equivalent to finding $K > 0$ such that

$$g'(z, x) = -qc_1(z)x^{q-1} + qc_2(z)x^{p-1} - K < 0, \ \ \forall\, (z, x) \in U \times [0, M_*]. \tag{46}$$

If $\beta = 0$ then $p = 1$ and since $q < p = 1$ then such a K always exists, therefore, a poverty core will never occur as a result of the application of the strong maximum principle. The situation, however, differs dramatically in the case where $\beta > 0$ in which case $p < 1$. Then, $qc_2(z)x^{p-1} \to +\infty$ as $x \to 0$ and one may clearly find examples of functions c_1, c_2 such that (46) can not hold for any $K > 0$, therefore, there is no reason why a poverty core will not occur in the case $\beta > 0$. However, this on its own is not enough to guarantee the existence of poverty cores for any non trivial positive solution.

We then go to the next step and actually prove, that in the case $\beta > 0$, any non trivial positive solution will exhibit a poverty core. To this end, consider $z_0 \in V_0$. By the Hölder continuity of c_1 there exists $\epsilon > 0$ such that $B(z_0, 2\epsilon) \subset V_0$ (that means that $c_1(z)$ is zero for every $z \in U$ such that $\|z - z_0\| < 2\epsilon$ and of course ϵ can be arbitrarily small). Inside this region, there is no production, so capital simply "decays" with rate $-c_2 k$ (or $-c_2 u^p$ in the transformed equation) therefore, we expect that inside the region V_0, there may be a sub-region, for which the solution of (23) is identically zero. To show that we will use the following strategy: if u is the maximal solution of (23) in the interval $[0, M_*]$ where $M_* = \left(\frac{\bar{c}_1}{\underline{c}_2}\right)^{\frac{1}{p-q}}$, then we will show that we may construct a function W with the properties (a) $W(z) = 0$ for every

$z \in V_0' \subset V_0$ and (b) $u(z) \leq W(z)$ for every $z \in U$. Then, it is clear that the maximal solution vanishes also identically for every $z \in V_0'$, i.e., develops a poverty core within V_0'. To complete the argument, it remains to construct a function W with the above properties. As it suffices to construct one such function, let us choose $V_0' = B(z_0, \epsilon)$ and try to construct W in the form $W(z) = C\Psi(z)$ where C is a constant that will be chosen shortly while

$$\Psi(z) = \begin{cases} 0 & z \in B(z_0, \epsilon) & \text{Region I} \\ \psi(z) & z \in B(z_0, 2\epsilon) \setminus B(z_0, \epsilon), & \text{Region II} \\ 1 & z \in U \setminus B(z_0, 2\epsilon). & \text{Region III} \end{cases}$$

where ψ is a function, the exact form of which will be specified soon, with the properties $\psi(z) = 0$ for $z \in \partial B(z_0, \epsilon)$ and $\psi(z) = 1$ for $z \in \partial B(z_0, 2\epsilon)$, while $\frac{\partial \psi}{\partial n} \geq 0$ for $z \in \partial B(z_0, \epsilon)$. A convenient choice would be to take $\psi(z) = \bar{\psi}(|z|)$ and assume that $\bar{\psi}(\epsilon) = 0$, $\bar{\psi}(2\epsilon) = 1$. For a W of this form, we see that it is continuous, and vanishes identically for every z such that $|z - z_0| < \epsilon$, so that the maximal solution u develops a dead core for $z \in B(z_0, \epsilon)$.

To show the inequality $u \leq W$, it is convenient to consider the function $v = \max(u - W, 0) = (u - W)^+$ pointwise defined in z by $v(z) = \max(u(z) - W(z), 0)$. If $u(z) \leq W(z)$ for every z in a region then obviously $v(z) = 0$ for every z in the same region and the converse is also true. It is thus enough to show that $v = 0$ everywhere. This condition $u \leq W$ is very easy to satisfy for z in region III as long as C is large enough: since $u \leq M_*$ it suffices to choose

$$C \geq M_* = \left(\frac{\bar{c}_1}{\underline{c}_2} \right)^{\frac{1}{p-q}}. \tag{47}$$

We only have then to consider the cases of regions I and II. Since $v = 0$ in region III, if v is a constant we are done, and v is a constant if $\nabla_z v = 0$. So, we must try to show that $\nabla_z v = 0$ for z in regions I and II. This is tricky, as v depends on the solution u itself (which we do not know!) but it also depends on W which we may choose *ad lib*. It is our hope then by the proper choice of W we can make sure that $(u - W)^+ = 0$ even though we do not have explicit knowledge concerning u. As a final observation note that it is enough to show that $\int_U |\nabla_z v|^2 dz \leq 0$, since that inequality implies that $\nabla_z v = 0$ a.e. in U and by standard properties of the weak derivatives this implies $v(z) = c$ (a constant) a.e. in U. Notice furthermore that

$$\int_U |\nabla_z v|^2 = \int_U \nabla_z (u - W)^+ \cdot \nabla_z (u - W)^+ dz = \int_U \nabla_z (u - W) \cdot \nabla_z (u - W)^+ dz.$$

Showing the constancy of v through this argument which uses an integral formulation is quite handy, since as we will see allows us to use the properties of the Laplacian with respect to integration by parts.

Let $w \in W_0^{1,2}(B(z_0, 2\epsilon))$ be any weakly differentiable function which vanishes for $|z - z_0|^2 = 2\epsilon$, and calculate

$$\int_{B(z_0,2\epsilon)} \nabla_z (u - W) \cdot \nabla_z w dz =$$

$$= - \int_{B(z_0,2\epsilon)} c_2 u^p w dz + C \int_{B(z_0,2\epsilon) \setminus B(z_0,\epsilon)} \Delta \Psi w dz - C \int_{\partial B(z_0,\epsilon)} \frac{\partial \Psi}{\partial n} w dz$$

$$- \int_{B(z_0,2\epsilon) \setminus B(z_0,\epsilon)} c_2 u^p w dz + C \int_{B(z_0,2\epsilon) \setminus B(z_0,\epsilon)} \Delta \Psi w dz$$

$$- \int_{B(z_0,\epsilon)} c_2 u^p w dz - C \int_{\partial B(z_0,\epsilon)} \frac{\partial \Psi}{\partial n} w dz.$$

Since $c_2 \geq 0$, $u \geq 0$ and $w \geq 0$, the penultimate integral in the last line keeps a positive sign. If Ψ is chosen so that $\frac{\partial \Psi}{\partial n} \geq 0$ on $\partial B(z_0, \epsilon)$ then the last integral keeps a positive sign therefore we have the inequality,

$$\int_{B(z_0,2\epsilon)} \nabla_z (u - W) \cdot \nabla_z w dz \leq \int_{B(z_0,2\epsilon) \setminus B(z_0,\epsilon)} (-c_2 u^p + C \Delta \psi) w dz,$$
$$\forall\, w \in W_0^{1,2}(B(z_0, 2\epsilon)),\ w \geq 0.$$

If we now set $w = (u - W)^+$ in this inequality, this yields,

$$\int_{B(z_0,2\epsilon)} |\nabla_z (u - W)^+|^2 dz \leq \int_{B(z_0,2\epsilon) \setminus B(z_0,\epsilon)} (-c_2 u^p + C \Delta \psi)(u - W)^+ dz$$

If we manage to show that the integral on the right hand side is less or equal to 0, for the right choice of W the proof will be complete. If $u \leq W$ then this is true since $(u - W)^+ = 0$ in this case. We only have to consider thus the case $u > W$. Since $(u - W)^+ > 0$ in this case, we must choose W so that $-c_2 u^p + C \Delta \Psi \leq 0$ for $u > W$ and $z \in B(z_0, 2\epsilon) \setminus B(z_0, \epsilon)$. Since for $0 \leq W \leq u$ we have (by the positivity of c_2) that $-c_2 u^p \leq -\underline{c}_2 W^p = -\underline{c_2} C^p \psi^p$ (recall the definition of W), we obtain the estimate

$$-c_2 u^p + C \Delta \psi \leq -\underline{c}_2 C^p \psi^p + C \Delta \psi = C^p \left(-\underline{c}_2 \psi^p + C^{1-p} \Delta \psi \right).$$

It is therefore sufficient to choose ψ and C so that

$$-\underline{c}_2 \psi^p + C^{1-p} \Delta \psi \leq 0$$

This is still a PDE inequality for the choice of C and ψ which may be further simplified if we look for special solutions of the form $\psi(z) = \bar{\psi}(|z|)$. The function $\bar{\psi}$ is defined for $\epsilon < |z - z_0| < 2\epsilon$ and must be of order 1. To this end define $\bar{z} = \frac{z - z_0}{\epsilon}$ (or else simply set $z_0 = 0$ without loss of generality) and look for $\psi(z) = \bar{\psi}(|\bar{z}|)$. The Laplacian can be re-expressed in terms of the variable $\bar{z}$ by noting that $\Delta\psi(z) = \epsilon^{-2}\Delta_{\bar{z}}\bar{\psi}(|\bar{z}|)$, so that the required inequality becomes

$$-\underline{c}_2\bar{\psi}^p(\bar{z}) + \epsilon^{-2}C^{1-p}\Delta_{\bar{z}}\bar{\psi}(|\bar{z}) \leq 0.$$

A straightforward calculation yields that $\Delta_{\bar{z}}\bar{\psi}(|\bar{z}|) = \bar{\psi}''(|\bar{z}|) + \frac{d-1}{|\bar{z}|}\bar{\psi}'(|\bar{z}|)$ so that the required inequality can be expressed in terms of an ODE inequality as

$$-\underline{c}_2\bar{\psi}^p(x) + \epsilon^{-2}C^{1-p}\frac{d-1}{x}\bar{\psi}'{}^{-2}C^{1-p}\bar{\psi}''(x) \leq 0,$$

for $1 < x < 2$ and d is the spatial dimension. This inequality must be solved with boundary conditions $\bar{\psi}(1) = 0$, $\bar{\psi}(1^+) \geq 0$ and $\bar{\psi}(2) = 1$. A natural choice for the solution of this inequality will be to look for a solution of the type $\bar{\psi}(x) = (x-1)^\nu$ for some $\nu > 0$. This ansatz satisfies all the required boundary conditions. Substituting this into the inequality yields

$$(x-1)^{p\nu}\left(-\underline{c}_2 + \nu(\nu-1)\epsilon^{-2}C^{1-p}(x-1)^{\nu-2-p\nu} + \frac{d-1}{x}\nu\epsilon^{-2}C^{1-p}(x-1)^{\nu-1-p\nu}\right) \leq 0,$$

and this is greatly simplified by choosing ν so that $\nu - 1 - p\nu = 0$, i.e., $\nu = \frac{2}{1-p}$ to

$$-\underline{c}_2 + \epsilon^{-2}C^{1-p}\nu\left((\nu-1) + (d-1)\frac{x-1}{x}\right) \leq 0,$$

as long as $x \in (1,2)$. Since for this range of x, $\frac{x-1}{x} \in (0, \frac{1}{2})$ the above inequality will be valid as long as

$$-\underline{c}_2 + \epsilon^{-2}C^{1-p}\nu\left((\nu-1) + (d-1)\frac{1}{2}\right) \leq 0,$$

which provides an estimate for C of the form,

$$C \leq \left(\frac{2\epsilon^2}{\nu}\frac{1}{(d-1) + 2(1-\nu)}\right)^{\frac{1}{1-p}} \underline{c}_2^{\frac{1}{1-p}}.$$

Therefore, recalling (47), C must be chosen so that

$$\left(\frac{\bar{c}_1}{\underline{c}_2}\right)^{\frac{1}{p-q}} \leq C \leq \left(\frac{2\epsilon^2}{\nu}\frac{1}{(d-1) + 2(1-\nu)}\right)^{\frac{1}{1-p}} \underline{c}_2^{\frac{1}{1-p}}.$$

This is feasible, as long as the far left hand side of the above inequality is indeed smaller than the far right hand side, and this of course depends on the parameters of the system. A quick calculation shows that for this to be true the parameters c_1, c_2, p, q must satisfy the inequality,

$$\bar{c}_1^{\frac{1}{p-q}}\underline{c}_2^{-\frac{1-q}{(1-p)(p-q)}} \leq \left(\frac{2}{\nu(d-1) + 2\nu(1-\nu)}\right)^{\frac{1}{1-p}} \epsilon^{\frac{2}{1-p}}$$

Upon substitution of the expressions for q, p and ν we obtain condition (27) and the value of the constant $c_0(\beta, d)$ as

$$c_0(\beta, d) = \left(\frac{\beta^2}{4(1+\beta)(\beta(d-1) + 2(\beta+2))} \right)^{\frac{1+\beta}{\beta}}.$$

This completes the proof. ■

A.6 Proof of Proposition 8

Proof. By expanding into a Taylor series around k^*, we obtain

$$\Phi(k) = \Phi(k^* + u) = \Phi(k^*) + \Phi'(k^*)u + \Phi''(k^*)\frac{u^2}{2} + ...,$$
$$f(k, z) = f(k^* + u, z) = f(k^*, z) + \frac{\partial f}{\partial k}(k^*, z)u + \frac{\partial^2 f}{\partial k^2}(k^*, z)\frac{u^2}{2} + ...$$

We substitute these expansions into

$$\frac{\partial k}{\partial t} = \Delta\Phi(k) + f(z, k) \tag{48}$$

and we disregard the terms which contain terms u^2 or higher. In this way, we obtain the linearization of (48) around k^*, as

$$\frac{\partial u}{\partial t} = \Delta(\kappa(z)u) + \mu(z)u, \tag{49}$$

where $\kappa(z) = \Phi'(k^*(z))$, $\mu(z) = \frac{\partial f}{\partial k}(z, k^*(z))$. The right hand side of (49) involves a formal linear second order operator

$$L_0 w := \Delta(\kappa w) + \mu w,$$

which we formally expand as

$$L_0 w = \kappa(z)\Delta w + 2\nabla\kappa(z) \cdot \nabla w + (\Delta\kappa(z) + \mu(z))w.$$

However the last term on the right hand side vanishes, since

$$\Delta\kappa(z) + \mu(z) = \Delta\left(\frac{d}{dk}\Phi(k^*(z))\right) + \frac{\partial f}{\partial k}(z, k^*(z)) = \tag{50}$$
$$\frac{d}{dk}\Delta\Phi(k^*(z)) + \frac{\partial f}{\partial k}(z, k^*(z)) = \frac{\partial}{\partial k}\left[\Delta\Phi(k^*(z)) + f(z, k^*(z))\right] = 0. \tag{51}$$

We are now looking for special solutions of (49) of the form $u(z, t) = w(z)e^{\lambda t}$ and we are thus led to the eigenvalue problem

$$\begin{cases} L_0 w = \lambda w \text{ in } U, \\ w = 0 \text{ on } \partial U. \end{cases} \tag{52}$$

The formal definition for stability is given below.

Definition: *The steady state* $k^* = k^*(z)$ *is asymptotically stable if all eigenvalues of* L_0 *have negative real parts. If there is an eigenvalue with a positive real part, then* k^* *is unstable.*

Assume now that $K > 0$. The comparison principle implies that $k^*(z) \geqslant K$, which in turn implies $\kappa(z) = \Phi'(\kappa(z)) \geqslant \Phi'(K) > 0$ and thus L_0 can be realized as a strongly uniformly elliptic operator in $L^2(U)$, having $H_0^1(U)$ as its domain of definition; for the relevant theory we refer to Evans [41, Chapters 5, 6]. We now consider the operator $L = \kappa(z)L_0$, that is,

$$Lw = (\kappa(z))^2\Delta w + 2\kappa(z)\nabla\kappa(z)\cdot\nabla w = \nabla\cdot(p(z)\nabla w),$$

where $p(z) = (\kappa(z))^2$. After that, (52) is equivalent to the weighted eigenvalue problem

$$\begin{cases} Lw = \lambda\kappa w \text{ in } U, \\ w = 0 \text{ on } \partial U, \end{cases} \tag{53}$$

and since κ is positive, (53) reads as the usual eigenvalue problem

$$\begin{cases} \frac{1}{\kappa}Lw = \lambda w \text{ in } U, \\ w = 0 \text{ on } \partial U. \end{cases} \tag{54}$$

Then the following result can be stated

Result 1: *The eigenvalues of* (54) *are real and bounded above by* $-(\Phi')^2$. *Therefore, they are all negative.*

This can be shown as follows:

In $L^2(U)$ we consider the usual inner product

$$< w_1, w_2 >:= \int_U w_1 w_2\, dz,$$

and the equiavalent weighted inner product

$$< w_1, w_2 >_\kappa:=< w_1, \kappa w_2 >= \int_U w_1 w_2 \kappa\, dz.$$

It is trivial to check that $\frac{1}{\kappa}L$ is self-adjoint with respect to $< \cdot, \cdot >$, so it has real eigenvalues. Moreover, we have

$$< \tfrac{1}{\kappa}Lw, w >_\kappa=< Lw, w >= \int_U (\nabla\cdot(\nabla w))w\, dz = \tag{55}$$

$$-\int_U p\nabla w\cdot\nabla w\, dz \leqslant -\Phi'^2\int_U |\nabla w|^2\, dz \leqslant -\Phi'(K)\|w\|_\kappa \tag{56}$$

If w is a normalized eigenvector with respect to $\|\cdot\|_\kappa$ *i.e.*, $\|w\|_\kappa = 1$, corresponding to the eigenvalue λ of $\frac{1}{\kappa}L$, inequality (55) becomes $\lambda \leqslant -\Phi'^2$.

Thus a positive and bounded away from zero steady state is stable

When the case $K = 0$ is considered observe that p and κ in problem (53) vanish smultaneously and therefore this problem has a meaning only in the interior of the support of p, which coincides with that of κ and k^*. So, without loss of generality, we assume that k^* vanishes only at the boundary, that is $k^*(z) > 0$ for $z \in U$ and the same applies for p. We can generalize the above analysis under a technical integrability condition for k^*, but now the mathematics become rather involved and we refer to [42] for further details and references. Then the following result can be stated

Result 2: *Assume that $\frac{1}{\kappa} \in L^t(U)$ for some $t > d/2$ (d is the dimension of the space). Then L^{-1} exists as a compact self-adjoint operator.*

Now problem (53) reads

$$\begin{cases} \kappa L^{-1} w = \frac{1}{\lambda} w \text{ in } U, \\ w = 0 \text{ on } \partial U. \end{cases} \tag{57}$$

Again, κL^{-1} is a compact operator and thus it has a sequence of eigenvalues having 0 as the only limit point. That is to say, eigenvalues of (53) are calculated as inverses of eigenvalues of κL^{-1}. Since $< Lw, w > \leqslant 0$ and 0 is not an eigenvalue, we find that all eigenvalues of (53) are negative. Thus under the assumptions leading to Result 2 the steady state k^* is stable. This implies that a poverty core which can be regarded as locations at the edge of the economic space where capital stock is depleted will persist in the long run. ■

References

[1] D. Acemoglu, Introduction to modern economic growth, MIT press, 2009.

[2] R. J. Barro, X. Sala-i Martin, Economic Growth, MIT Press, Cambridge, Massachusetts, 2004.

[3] D. T. Quah, Empirics for economic growth and convergence, European Economic Review 40 (6) (1996) 1353–1375.

[4] D. T. Quah, Empirics for growth and distribution: stratification, polarization, and convergence clubs, Journal of Economic Growth 2 (1) (1997) 27–59.

[5] D. T. Quah, Regional convergence clusters across europe, European Economic Review 40 (3) (1996) 951–958.

[6] P. Martin, G. I. Ottaviano, Growth and agglomeration, International Economic Review 42 (4) (2001) 947–968.

[7] R. E. Baldwin, P. Martin, G. I. Ottaviano, Global income divergence, trade, and industrialization: the geography of growth take-offs, Journal of Economic Growth 6 (1) (2001) 5–37.

[8] R. Baldwin, R. Forslid, P. Martin, G. Ottaviano, F. Robert-Nicoud, Public policies and economic geography, PUP, Princeton.

[9] R. E. Baldwin, P. Martin, Agglomeration and regional growth, Handbook of regional and urban economics 4 (2004) 2671–2711.

[10] M. Fujita, T. Mori, Frontiers of the new economic geography, Papers in Regional Science 84 (3) (2005) 377–405.

[11] K. Desmet, E. Rossi-Hansberg, Spatial growth and industry age, Journal of Economic Theory 144 (6) (2009) 2477–2502.

[12] K. Desmet, E. Rossi-Hansberg, On spatial dynamics*, Journal of Regional Science 50 (1) (2010) 43–63.

[13] W. Isard, P. Liossatos, Transport investment and optimal space-time development, Papers in Regional Science 31 (1) (1973) 31–50.

[14] W. Isard, P. Liossatos, On optimal development over space and time, Journal of Regional Analysis and Policy 3 (1).

[15] W. Isard, P. Liossatos, Spatial Dynamics and Optimal Space-Time Development, North-Holland, Amsterdam, 1978.

[16] D. A. Carlson, A. Haurie, A. Leizarowitz, Infinite horizon optimal control: deterministic and stochastic systems, Springer-verlag, 1991.

[17] P. Brito, The dynamics of growth and distribution in a spatially heterogeneous world, ISEG–Departamento de Economia.

[18] C. Camacho, B. Zou, The spatial solow model, Econ. Bull 18 (2004) 1–11.

[19] R. Boucekkine, C. Camacho, B. Zou, Bridging the gap between growth theory and the new economic geography: The spatial ramsey model, Macroeconomic Dynamics 13 (1) (2009) 20.

[20] R. Boucekkine, C. Camacho, G. Fabbri, On the optimal control of some parabolic partial differential equations arising in economics, AMSE, WP 2013-Nr 34.

[21] R. Boucekkine, C. Camacho, G. Fabbri, Spatial dynamics and convergence: The spatial ak model, Journal of Economic Theory 148 (6) (2013) 2719–2736.

[22] W. Brock, A. Xepapadeas, Diffusion-induced instability and pattern formation in infinite horizon recursive optimal control, Journal of Economic Dynamics and Control 32 (9) (2008) 2745–2787.

[23] W. Brock, A. Xepapadeas, Pattern formation, spatial externalities and regulation in coupled economic–ecological systems, Journal of Environmental Economics and Management 59 (2) (2010) 149–164.

[24] W. A. Brock, A. Xepapadeas, A. N. Yannacopoulos, Optimal agglomerations in dynamic economics, Journal of Mathematical Economics 53 (2014) 1–15.

[25] W. A. Brock, A. Xepapadeas, A. N. Yannacopoulos, Spatial externalities and agglomeration in a competitive industry, Journal of Economic Dynamics and Control 42 (2014) 143–174.

[26] R. E. Lucas, Why doesn't capital flow from rich to poor countries?, The American Economic Review 80 (2) (1990) 92–96.

[27] R. E. Lucas, Macroeconomic priorities, American Economic Review 93 (1) (2003) 1–14.

[28] A. Razin, E. Sadka, Welfare migration: Is the net fiscal burden a good measure of its economic impact on the welfare of the native-born population?, CESifo Economic Studies 50 (4) (2004) 709–716.

[29] L. Alfaro, S. Kalemli-Ozcan, V. Volosovych, Capital flows in a globalized world: The role of policies and institutions, in: Capital Controls and Capital Flows in Emerging Economies: Policies, Practices and Consequences, University of Chicago Press, 2007, pp. 19–72.

[30] T. G. Conley, E. Ligon, Economic distance and cross-country spillovers, Journal of Economic Growth 7 (2) (2002) 157–187.

[31] J. L. Vázquez, The Porous Medium Equation: Mathematical Theory: Mathematical Theory, Oxford University Press, 2006.

[32] H. E. Levine, P. E. Sacks, Some existence and nonexistence theorems for solutions of degenerate parabolic equations, Journal of Differential Equations 52 (2) (1984) 135–161.

[33] C. Bandle, T. Nanbu, I. Stakgold, Porous medium equation with absorption, SIAM Journal on Mathematical Analysis 29 (5) (1998) 1268–1278.

[34] J. R. Anderson, Local existence and uniqueness of solutions of degenerate parabolic equations, Communications in Partial Differential Equations 16 (1).

[35] J. R. Anderson, Necessary and sufficient conditions for the unique solvability of a nonlinear reaction-diffusion model, Journal of Mathematical Analysis and Applications 228 (1998) 483–494.

[36] A. D. Pablo, J. L. Vazquez, Travelling waves and finite propagation in a reaction-diffusion equation, Journal of Differential Equations 93 (1991) 19–61.

[37] J. Bo, S. Ning, Existence and uniqueness of solutions of reaction-convection equations with non-lipschitz nonlinearity, Journal of Mathematical Analysis and Applications 410 (2014) 267–274.

[38] M. Ghergu, V. Radulescu, Singular elliptic problems, Oxford Univ. Press, 2008.

[39] M. Delgado, A. Suárez, On the uniqueness of positive solution of an elliptic equation, Applied mathematics letters 18 (10) (2005) 1089–1093.

[40] M. Delgado, A. Suárez, On the existence of dead cores for degenerate lotka-volterra models, Proceedings of the Royal Society of Edinburgh: Section A Mathematics 130 (04) (2000) 743–766.

[41] L. C. Evans, Partial Differential Equations, American Mathematical Society, Providence, 1998.

[42] A. Henrot, Extremum Problems for Eigenvalues of Elliptic Operators, Birkhauser, 2006.

Figures

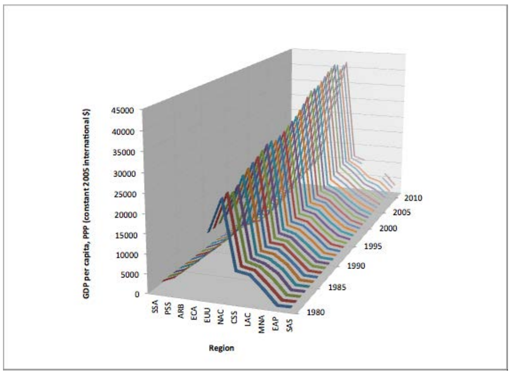

Figure 1. Spatial distribution of GDP per capita, by world regions

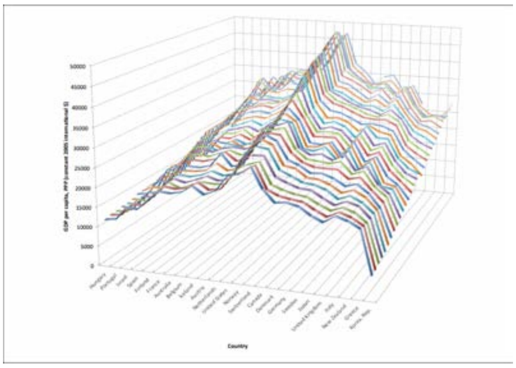

Figure 2. Spatial distribution of GDP per capita, developed countries

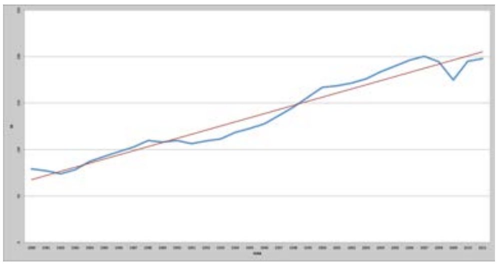

Figure 3. Regional inhomogeneity measure

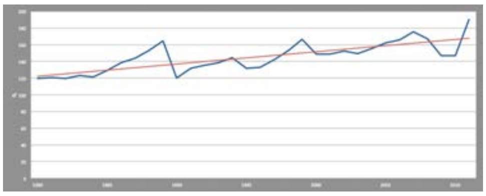

Figure 4. Inhomogeneity measure, developed countries

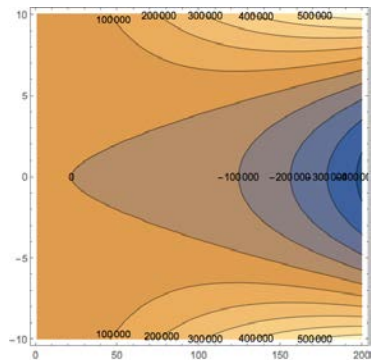

Figure 5. Steady-state phase diagram in the spatial domain

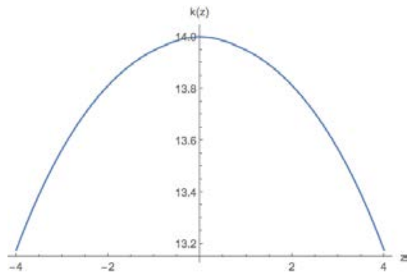

Figure 6. Steady state in the spatial domain

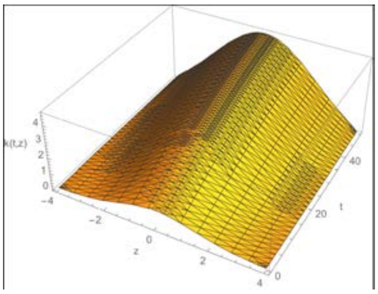

Figure 7. Spatiotemporal distribution of capital

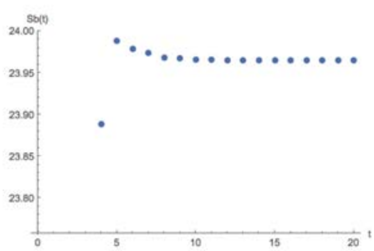

Figure 8. The time path of the Sobolev norm

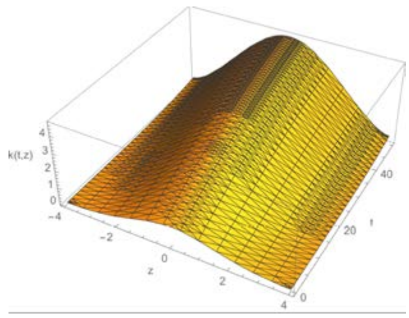

Figure 9. Circle boundary conditions

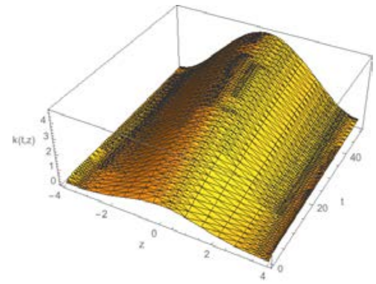

Figure 10. Time dependent boundary conditions

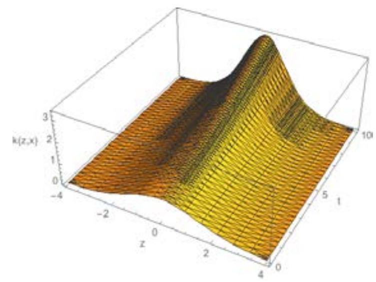

Figure 11. A spatiotemporal AK model

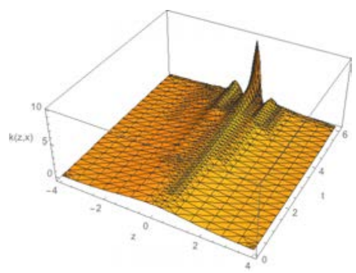

Figure 12. Spatiotemporal evolution under increasing returns