DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS

Spatial Heat Transport, Polar Amplification and Climate Change Policy

William Brock

Anastasios Xepapadeas

Working Paper Series

15-15

October 2015

Spatial Heat Transport, Polar Amplification and Climate Change Policy

W. Brock[*] and A. Xepapadeas[†]

October 25, 2015

Abstract

This paper is, to our knowledge, the first paper in climate economics to consider the combination of spatial heat transport and polar amplification. We simplified the problem by stratifying the Earth into latitude belts and assuming, as in North et al. (1981), that the two hemispheres were symmetric. Our results suggest that it is possible to build climate economic models that include the very real climatic phenomena of heat transport and polar amplification and still maintain analytical tractability. We derive optimal fossil fuel paths under heat transport with and without polar amplification. We show that the optimal tax function depends not only on the distribution of welfare weights but also on the distribution of population across latitudes, the distribution of marginal damages across latitudes and cross latitude interactions of marginal damages, and climate dynamics. We also determine optimal taxes per unit of emission and show that, in contrast to the standard results suggesting spatially uniform emission taxes, poorer latitudes should be taxed less per unit emissions than richer latitudes.

JEL Classification: Q54, Q58, C61

Keywords: Climate change, Heat transport, Polar Amplification, Welfare maximization, fossil fuels, optimal taxation.

[*]Vilas Research Professor Emeritus, Economics Department, University of Wisconsin and Research Professor, University of Missouri.

[†]Athens University of Economics and Business. Xepapadeas's research has been co-financed by the European Social Fund and Greek national funds through the NSRF Research Funding Program: Thalis, AUEB, "Optimal Management of Dynamical Systems of the Economy and the Environment". The authors thank Joan Stefan for technical editing.

1 Introduction

While spatial heat transport and polar amplification are well-established phenomena in the science of climate change, they have been largely ignored in the economic modeling of climate change. This paper introduces spatial heat transport and polar amplification in a simple spatial climate economics model, in which the climate model is based upon the work of Alexeev et al. (2005) and Langen and Alexeev (2007).

In this work the strength of spatial poleward heat transport from the lower latitudes to the higher latitudes depends upon the level of global mean average temperature, The spatial transport effect causes polar amplification due to increased meridional latent heat transport, as discussed by Alexeev et al. (2005) and further developed by Alexeev and Langen (2007) and Alexeev and Jackson (2012).

In order to exhibit the economic and climatic effects of spatial heat transport in the clearest and simplest possible way, we stratify the Earth into latitude belts and model the change in damages to each latitude belt from increased CO_2 into the atmosphere from fossil fuel use in economic production activities located at each latitude. In order to focus completely on spatial climatic heat transport, we assume that total production at latitude belt x is given by $y(x,t)\,E(x,t)^{\alpha}$, $0 < \alpha < 1$ where $y(x,t)$ grows exogenously and $E(x,t)$ denotes emissions from fossil fuel inputs, or fossil fuel use by an appropriate choice of units in total production. This simplification and abstraction away from the allocative effects of other inputs to production on the economic side of the model enables us to keep a tight focus on climatic heat transport effects. In this context our approach and contributions can be summarized in the following way.

First, consider the usual welfare optimization problem in which a social planner chooses the latitude emissions to maximize the integral over latitudes and time of discounted weighted utilities of consumption per capita where the climate dynamics are modeled by an energy balance model with spatial heat transport. Progress in this kind of modeling of more realistic climate representations in Integrated Assessment Models (IAMs) has been hindered by the analytical difficulties in dealing with more realistic climatic heat and moisture transport dynamics across a continuum of locations, and the modeling of the carbon cycle under anthropogenic forcing.

Seeking more realistic climate representations, we use a linear approximation of Pierrehumbert (2014), which he calls a "radiative forcing kernel", to approximate the combined dynamics equations of CO_2 stock forcing,[1] the dynamics of the carbon cycle, and anthropogenic emissions into the atmosphere. We show that by using Pierrehumbert's radiative kernel, and by expanding the climate dynamics of the latitude temperature field, $T(x,t)$, into an infinite series of even numbered Legendre polynomials, the optimization model can be solved to any desired degree of accuracy for usual specifications of utility functions and latitude climatic damages. This analytical contribution enables economists to introduce climate effects of spatial transport and still retain some useful analytical tractability in climate economic models at this level of aggregation. We believe that this theoretical contribution is important for advancing analytically tractable IAM modeling by introducing more realistic climate dynamics than, for example, simple three box carbon cycle models and two box temperature dynamics models, for the climate component of IAMs. Analytic tractability enables us to understand how the climate and economic components of an IAM interact to produce outcomes. We feel that our approach contributes to this objective.

As an example, consider the case of zero income effects when it is optimal for the tax function to be uniform across latitudes. In this case, we show that an increase in the strength of poleward amplification of transport of heat energy $\hat{r}$ from $\hat{r} = 0$ to a small positive number causes the optimal tax function to shift upward or downward depending on the interaction of the distribution of welfare weights, population, and damages per capita across latitudes with the distribution of the temperature anomaly $T(x,t)$ across latitudes at each point in time. We illustrate this marginal distribution effect of increased polar amplification with data and plots of population distribution data across latitudes for potentially plausible per capita damage distributions across latitudes.[2]

It is worth noting that our analysis indicates that ignoring heat transport

[1] CO_2 forcing at each date t is logarithmic in the ratio of atmospheric $CO_2(t)/CO_2(0)$ where $CO_2(0)$ is the pre-industrial stock.

[2] E.g. we might expect that poorer latitudes will experience larger per capita damages, all other things equal (Burgess et al. (2014), Dell (2012). For example, Dell et al. (2012), have stressed the damaging effects of climatic changes in temperature and precipitation upon not only output levels but growth rates of poorer countries. Burgess et al. (2014) document increased death rates due to high temperature extremes among the poor who do not have access to adaptation strategies such as air conditioning.

and polar amplification in the standard economic models of climate change implies a potential bias in the calculation of emission taxes and emission path for policy purposes. The present work on distributional impacts of climate change suggests that it is worthwhile to generalize IAMs to include marginal distributional impacts of spatial heat and moisture transport across latitudes and longitudes. We believe that isolating this marginal impact of polar amplification of heat transport on the optimal tax function is new to our paper.

Desmet and Rossi-Hansberg (2015) have studied spatial effects in IAM models at the level of disaggregation into latitude belts as we do in this paper. They do not include heat transport effects across latitudes and do not include polar amplification effects. On the other hand, they include the important adaptation response of migration to negative climate change while we do not include this response. Their paper shows the importance of removing, or at least reducing, restrictions on the adaptive response of migration.

Second, in a world where compensatory transfers are not possible, the usual result that emission taxes should be uniform fails because of income effects. We show that "poorer" latitudes should be taxed less than "richer" latitudes due to income effects. Furthermore we conduct comparative dynamics of optimal emissions taxes w.r.t. to parameters, e.g. the strength of heat transfer, the strength of polar amplification $\hat{r}$, due to increased poleward latent heat transport, and more. Our comparative dynamics indicate that the optimal tax function depends not only on socioeconomic factors, but also on the interactions of these factors with climate dynamics as they are reflected in the heat transport process.

Third, our decomposition of the temperature field $T(x,t)$ into modes enables us to rank the modes by response times with the higher numbered modes responding faster than the lower numbered modes. This decomposition allows us to show that optimal paths may induce polar amplification.

The remainder of this paper is organized as follows. Section 2 develops the basic analytical framework used in the paper. Section 3 conducts welfare analysis and derives optimality conditions for the unified spatial climate and economic model. Section 4 studies the impact and exhibits the importance of heat transfer and polar amplification in the welfare analysis of climate change, and in particular on the social price of the climate change externality. This section shows how the comparative dynamics of heat transfer strength and polar amplification strength depend upon the interaction of climate component dynamics with the distribution of welfare weights, population, and productive capacities across latitudes. Section 5 discusses optimal fossil fuel taxes in a competitive environment with income effects; we show that optimal taxes have a spatial structure and are dependent on each latitude's output. Section 6 conducts the same type of analysis as was done for logarithmic utility in earlier sections, but for general power utility functions. We show that an increase in the coefficient of relative risk aversion will reduce the social price of the climate externality. Section 7 includes a short summary, conclusions, and suggestions for future research. An Appendix contains the proofs of the propositions.

2 Temperature Dynamics and Heat Transport

To study the evolution of local temperature and its impact on climate policy when heat transport across the globe is taken into account, we build and extend the standard one-dimensional energy balance model (EBM) developed by North (1975a,b), North et al. (1981), and Wu and North (2007). We also substantially extend the work of Brock et al. (2013, 2014) which, for the first time to our knowledge, introduced into an one-dimensional EBM the anthropogenic influence on local temperature resulting from the accumulation of carbon in the atmosphere and conducted economic optimization analysis in this type of model.

Let x denote the sine of the latitude. For simplicity we will just refer to x as "latitude", and let $T_{total}(x,t)$ denote surface (sea level) temperature measured in C$^\circ$ at latitude x and time t. We assume constant albedo across latitudes. The simplifying assumption of constant albedo allows us to cancel out the solar input and the constant in the outgoing radiation term of North et al. (1981) and decompose $T_{total}(x,t)$ into two parts: a baseline part and the temperature anomaly which is associated with human actions. Thus we define surface temperature as:

$$T_{total}(x,t) \equiv T_b(x,t) + T(x,t), \tag{1}$$

where $T_b(x,t)$ is what the temperature at (x,t) would have been if humans were not increasing the carbon content of the atmosphere beyond the preindustrial levels, and $T(x,t)$ is the temperature anomaly, which is the temperature increase attributed to the anthropogenic emissions of greenhouse gasses (GHGs).

Let $M(t), M_0$ denote the current stock of carbon content of the atmosphere (assuming well-mixing) at time t, and the preindustrial carbon content respectively. Then the basic energy balance equation for the temperature anomaly with human input added can be written as:

$$C\frac{\partial T(x,t)}{\partial t} = -BT(x,t) + D\mathcal{L}T(x,t) + \xi \ln\left(\frac{M(t)}{M_0}\right) \tag{2}$$

$$T_0(x,0) = 0, \text{given}$$

$$\mathcal{L}T(x,t) \equiv \frac{\partial}{\partial x}\left[\frac{\left(1-x^2\right)\partial T(x,t)}{\partial x}\right], \tag{3}$$

where $x=0$ denotes the Equator, $x=1$ denotes the North Pole and $x=-1$ denotes the South Pole and the heat capacity parameter "C" of North et al. (1981) is absorbed into the other parameters of (2). That is, we put $C=1$ by absorbing it into the other parameters in (2). In (2), D is a heat transport coefficient which is an adjustable parameter measured in W/(m^2)(°C) and has been calibrated to match observed temperatures across latitudes. It can also be expressed in dimensionless form as in North et al. (1981). The heat transport coefficient depends in principle on the global temperature anomaly defined as:

$$T(t) = \int_{x=-1}^{x=1} T(x,t)\,dx. \tag{4}$$

Alexeev et al. (2005) specify the heat transport coefficient as a function of the temperature anomaly as follows[3]:

$$D = D(T(t)) = D(T_b(t) + T(t)) = \tag{5}$$

$$D_{ref}\left[1 + \hat{r}\left(T_b(t) + T(t) - T_b(t)\right)\right] = D_{ref}\left[1 + \hat{r}T(t)\right] \tag{6}$$

$$D_{ref} = 0.445, T_b(t) \equiv \bar{T}_b = 15°C, \hat{r} = 0.03/K \tag{7}$$

where $T_b(t)$ is baseline global average temperature, i.e. the integral of overall latitude belts.

The operator $\mathcal{L}$ is a linear operator on the space of functions of x with

[3] Alexeev et al. (2005) specify $\hat{r}$ to be 3% per degree Kelvin.

the property that the nth Legendre polynomial $P_n(x)$ is an eigenfunction of $\mathcal{L}$, i.e. $\mathcal{L}P_n(x) = -n(n+1)P_n(x)$.[4] We use this property in the solution of the model. The term $D(T(t))\mathcal{L}T(x,t)$ models therefore the heat flux associated with the temperature anomaly. Finally the term $\Delta F = \xi \ln(M(t)/M_0)$ is the radiative forcing term due to anthropogenic emissions and ξ is a temperature forcing parameter (measured in ^{o}C per W per m^2).

We assume that dynamics of carbon content of the atmosphere under the well-mixing assumption are given by:

$$\dot{M}(t) = -mM(t) + E(t) \ , \quad M(0) = M_0 \ , \text{ given} \tag{8}$$

where

$$E(t) = \int_{x=-1}^{x=1} E(x,t)\,dx \tag{9}$$

denotes total carbon emissions generated at latitudes. Temperature dynamics (2) and carbon dynamics (8) will be used as dynamic constraints in the optimization of the objective function that reflects the economy part of our model. To enhance the tractability of the optimized model, since in (2) dynamics are described by partial differential equations (PDE) with a nonlinear diffusion term, we introduce two approximations, one from North et al. (1981) and the other from Pierrehumbert (2014, equations (1)-(3)).

North et al. (1981) note that $T(x,t)$ can be written in a series expansion in terms of Legendre polynomials, or

$$T(x,t) = \sum_{n=0,\text{even}} T_n(t)\,P_n(x) \tag{10}$$

where $P_n(x)$ is the nth Legendre polynomial. They approximate $T(x,t)$ by truncating the expansion at some finite N.[5]

Following Pierrehumbert (2014, equations (1)-(3)), the radiative forcing

---

[4] $P_n(x) = 2^n \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} \begin{pmatrix} n+k-1/2 \\ n \end{pmatrix}, P_0(x) = 1, P_2(x) = \frac{1}{2}\left(3x^2 - 1\right).$

[5] Note that $\sum_{n=0}^{\infty} T_n(0)\,P_n(0)$ does not imply that all $T_n(0)$'s are zero. Indeed, if all $T_n(0)$'s are zero then the solution of (2) would be independent of x and all spatial effects would vanish for the anomaly. As one might expect, if one is dealing with differential equations in an infinite dimensional space, one must specify an infinite number of initial conditions.

ΔF can be approximated by:[6]

$$\Delta F = \sum_{k=0}^{3} a_k X_k(t) \tag{11}$$

$$\dot{X}_k(t) = -b_k X_k(t) + E(t) \ , \ X_k(0) = 0 \text{ given, } k = 0, 1, 2, 3. \tag{12}$$

Using (10) and (11),(12) into (2), and assuming for the moment a fixed D, we can write the temperature anomaly dynamics as:

$$\frac{\partial T(x,t)}{\partial t} =$$

$$\sum_{n=0,\text{even}}^{\infty} \dot{T}_n(t) P_n(x) = -BT(x,t) + D\left[\sum_{n=0}^{\infty} \lambda_n T_n(t) P_n(x)\right] +$$

$$\sum_{k=0}^{3} a_k X_k(t) \tag{13}$$

$$\dot{X}_k(t) = -b_k X_k(t) + E(t) \ , \ X_k(0) = 0 \text{ given, } k = 0, 1, 2, 3 \tag{14}$$

$$T_0(x, 0) = \sum_{n=0}^{\infty} T_n(0) P_n(0) = 0 \ , \quad \text{given} \tag{15}$$

$$\lambda_n = n(n+1). \tag{16}$$

We can simplify (13) by using the property that the Legendre polynomials are orthogonal with respect to the inner L^2 product on the interval $x \in [-1, 1]$, which implies that

$$\langle P_n, P_m \rangle = \int_{x=-1}^{x=1} P_n(x) P_m(x) \, dx = \frac{2}{2n+1} \delta_{nm} \tag{17}$$

$$\delta_{nm} = \begin{cases} 1 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m. \end{cases} \tag{18}$$

Multiplying the temperature dynamics in (13) by $P_n(x)$, integrating over the interval $x \in [-1, 1]$, and noting that $P_0(x) = 1$, $\int_{-1}^{1} P_0(x) = 2$,

[6] It is very important to recognize that Pierrehumbert (2014) specifies specific values for $\{a_k, b_k\}_{k=0}^{3}$ which will depend upon a specific emissions path and the approximation (11-12) for the carbon cycle dynamics may depend on that specific emissions path. We assume in the following analysis that the approximation is good enough that we can treat the values $\{a_k, b_k\}_{k=0}^{3}$ as constants over the set of emissions paths that we optimize over.

$\int_{-1}^{1} P_n(x) = 0$, $n = 2, 4, 6, ...$, we obtain

$$\begin{aligned} \dot{T}_n(t) &= -(B + D\lambda_n) T_n(t), \ T_n(0) \text{ given } n = 2, 4, 6, ... \qquad (19) \\ \dot{T}_0(t) &= -BT_0(t) + \Delta F \ , \ T_0(0) \text{ given.} \qquad (20) \end{aligned}$$

The EBM can become more realistic by allowing the diffusion coefficient D to depend on the global average temperature anomaly, following Alexeev et al. (2005), since it is expected that with a warmer atmosphere more heat will be transported polewards. Using (6) with $D = D_{ref}$, we obtain

$$\begin{aligned} &\frac{\partial T(x,t)}{\partial t} = \qquad (21) \\ &-BT(x,t) + \Delta F + D\mathcal{L}T(x,t) + \hat{r}DT(x,t)\left[\mathcal{L}T_b(x,t) + \mathcal{L}T(x,t)\right]. \end{aligned}$$

Using North's approximation for the baseline local temperature and the local temperature anomaly which is

$$T(x,t) = \sum_{n,\text{even}} T_n(t) P_n(x), \ T_b(x,t) = \sum_{n,\text{even}} T_{bn}(t) P_n(x), \qquad (22)$$

we can write the dynamics of the temperature anomaly as:

$$\begin{aligned} &\frac{\partial T(x,t)}{\partial t} = \qquad (23) \\ &\sum_{n=0}^{\infty} \dot{T}_n(t) P_n(x) = -B\sum_{n=0}^{\infty} T_n(t) P_n(x) + D\left[\sum_{n=0}^{\infty} \lambda_n T_n(t) P_n(x)\right] + \\ &\Delta F + \hat{r}D\left(\sum_{n=0}^{\infty} T_n(t) P_n(x)\right)\left(\sum_{m=0}^{\infty} \lambda_m \left[T_{bm}(t) + T_m(t)\right] P_m(x)\right) \\ &T_0(x,0) = \sum_{n=0}^{\infty} T_n(0) P_n(0) = 0 \ , \quad \text{given.} \end{aligned}$$

Multiply both sides of (23) by $P_n(x)$ and integrate over $x \in [-1,1]$ to obtain, using (17-18) and the inner product notation $\int_{x=-1}^{x=1} F(x) G(x) = \langle F, G \rangle$,

$$\begin{aligned} &\dot{T}_n(t) = -\left[B + Dn(n+1)\right] T_n(t) + \Delta F \frac{\langle 1, P_n \rangle}{\langle P_n, P_n \rangle} \qquad (24) \\ &+ \frac{\hat{r}D}{\langle P_n, P_n \rangle}\left\langle P_n, -\sum_{n \text{ even}} \sum_{m \text{ even}} T_n(t) P_n(x) m(m+1)\left[T_{bm}(t) + T_m(t)\right] P_m(x)\right\rangle. \\ &n, m = 0, 2, 4, ... \end{aligned}$$

The two-mode approximation of (24) results in the following system of ordinary differential equations in which we omit (t) and (x) in order to ease notation:

$$\begin{aligned}
\dot{T}_0 &= -BT_0 + \sum_{k=0}^{3} a_k X_k - \frac{1}{2}\hat{r}D \left\langle 1, 6\left(T_0 + T_2 P_2\right)\left(T_{b2} + T_2\right) P_2 \right\rangle \quad (25)\\
&= -BT_0 + \sum_{k=0}^{3} a_k X_k - \frac{1}{2}\hat{r}D6 \left\langle 1, T_2\left(T_{b2} + T_2\right) P_2^2 \right\rangle \\
&= -BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r}DT_2\left(T_{b2} + T_2\right) \left\langle P_2, P_2 \right\rangle \quad (26)\\
\dot{T}_2 &= -\left(B + 6D\right) T_2 - \frac{6\hat{r}D}{\left\langle P_2, P_2\right\rangle} \left\langle P_2, \left(T_0 + T_2 P_2\right)\left(T_{b2} + T_2\right) P_2 \right\rangle \quad (27)\\
& \left\langle 1, P_2 \right\rangle = 0, \ \left\langle 1, P_2^2 \right\rangle = \left\langle P_2, P_2 \right\rangle, \ P_2\left(x\right) = \frac{1}{2}\left(3x^2 - 1\right). \quad (28)
\end{aligned}$$

The dynamical system (25)-(28), along with

$$\dot{X}_k\left(t\right) = -b_k X_k\left(t\right) + E\left(t\right) \ , \ X_k\left(0\right) = 0 \text{ given}, \ k = 0, 1, 2, 3, \quad (29)$$

represent the climate model that describes the evolution of temperature across latitudes. We will use this model to derive the optimal emission paths and the corresponding optimal spatial taxes.

3 Welfare Maximization under Heat Transfer

To study optimal emissions paths in the context of the one-dimensional climate model described above, we consider a simple welfare maximization problem with logarithmic utility, where world welfare is given by:

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left[\int_{x=-1}^{x=1} v\left(x\right) L\left(x\right) \ln \left[y E^{\alpha} e^{-\phi_T(x) T_{total}}\right]\right] dx dt \quad (30)$$

$$\int_{0}^{\infty} \mathrm{e}^{-\rho t} \left[\int_{-1}^{1} v\left(x\right) L\left(x\right) \ln \left[y\left(x,t\right) E\left(x,t\right)^{\alpha} e^{-\phi_T(x,t)\left[T_b(x,t) + T(x,t)\right]}\right]\right] dx dt,$$

where $y\left(x,t\right) E\left(x,t\right)^{\alpha}$, $0 < \alpha < 1$, $E\left(x,t\right)$, $T\left(x,t\right)$, $L\left(x\right)$ are output per capita, fossil fuel input, temperature anomaly and fully employed population at location (or latitude) x at date t, respectively. The term $e^{-\phi_T(x) T_{total}(x,t)}$ reflects damages to output per capita in location x from an increase in the temperature anomaly at this location. In the second equation of (30), we have allowed damages to depend upon time and have specified damages to depend upon total temperature at (x, t) which is defined by the sum of baseline temperature which would have occurred if there were no human emissions into the system, $T_b(x,t)$, and the temperature anomaly, $T(x,t)$, caused by human emissions into the atmosphere. We assume that $y(x,t)$, $L(x)$ are exogenously given and fixed. That is, we are abstracting away from the problem of optimally accumulating capital inputs and other inputs in order to focus sharply on optimal fossil fuel taxes. Finally, $v(x)$ represents welfare weights associated with location x.

Formulation (30) allows the incorporation of another very important aspect of spatially distributed damages from climate change, namely damages from precipitation. Defining total precipitation as the sum of baseline precipitation and the precipitation anomaly or $P_{total}(x,t) = P_b(x,t) + P(x,t)$, Castruccio et al. (2014) suggest the following approximation for the precipitation anomaly:[7]

$$P(x,t) = \psi(x) T(x,t). \tag{31}$$

Assuming exponential precipitation damages of the form $\exp(-\varphi(x,t)(P_b(x,t) + P(x,t)))$ and using Castruccio et al.'s (2014) approximation, we can write a welfare function that contains both temperature impacts and precipitation damages as:

$$\int_0^\infty e^{-\rho t} \int_{-1}^{1} [v(x) L(x) \ln(y(x,t) E(x,t)^\alpha) \times \tag{32}$$
$$\left(e^{-\phi(x,t)[T_b(x,t)+T(x,t)]} e^{-\varphi(x,t)[P_b(x,t)+\psi(x)T(x,t)]}\right)\Big] \, dxdt.$$

In the case where we are assuming logarithmic utility and exponential damages to output both from temperature and precipitation, we can add a baseline temperature $T_b(x,t)$ and a baseline precipitation $P_b(x,t)$ to the corresponding anomalies and still be able to assert that (32) can be replaced for optimization purposes by the equivalent problem,

$$\max_{E(x,t)} \left\{ \int_0^\infty e^{-\rho t} \int_{-1}^{1} v(x) L(x) [\alpha \ln E(x,t) - \phi(x,t) T(x,t)] \, dxdt \right\}, \tag{33}$$

where $\phi(x,t) = \phi_T(x,t) + \varphi(x,t)\psi(x)$. In the definition of $\phi(x,t)$ the term

[7] We ignore the conditional variance since we are working with a deterministic model.

$\phi_T(x,t)$ accounts for temperature damages, while the term $\varphi(x,t)\psi(x)$ allows for precipitation damages. It should be noted that to the best of our knowledge, this is the first time that climate science in terms of a spatial one-dimensional EBM that incorporates the important climate science phenomenon of heat transfer is combined with the spatial characteristics of damages from temperature and precipitation. This combination results in a model of climate economics capable of determining the value of climate externality and optimal fossil fuel taxes.

The problem of a social planner would be to choose fossil fuel paths $E(x,t)$ or equivalently, by an appropriate change in units, emissions paths $E(x,t)$ to maximize (33) subject to climate dynamics given by (25)-(28) and (29), and an additional constraint reflecting the potential exhaustibility of global fossil fuel reserves.

$$\int_{t=0}^{\infty} E(t)\,dt < R_0 \ , \ E(t) = \int_{x=-1}^{x=1} E(x,t)\,dx \ , \ \int_{x=-1}^{x=1} R_0(x) = R_0, \quad (34)$$

where R_0 denotes global fossil fuel reserves, and $R_0(x)$ fossil fuel reserves in location x.

Constraint (34) implies that the social planner is altruistic and treats fossil fuels reserves as a common property which can be transferred across locations. The alternative polar case is to assume that no transfers are possible and that each location is constrained by local fossil fuel reserves, or

$$\int_{t=0}^{\infty} E(t,x)\,dt < R_0(x) \ \text{ for all } x \in [-1,1] \ , \ \int_{x=-1}^{x=1} E(x,t)\,dx = E(t)\,. \quad (35)$$

We start with the welfare maximization problem of the altruistic planner, making the simplifying assumption that the damage parameter $\phi(x,t)$ is independent of t. The current value Hamiltonian for this problem is:

$$H = \int_{x=-1}^{x=1} \{v(x) L(x) [\alpha \ln E(x,t) - \phi(x) [T_0(t) + T_2(t) P_2(x)]] \tag{36}$$
$$-\lambda_R(t) E(t,x)\} dx +$$
$$\lambda_{T_0}(t) \left[-BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r} D T_2 (T_{b2} + T_2) \langle P_2, P_2 \rangle \right] +$$
$$\lambda_{T_2}(t) \left[-(B+6D) T_2 - \frac{6\hat{r}D}{\langle P_2, P_2 \rangle} \langle P_2, (T_0 + T_2 P_2)(T_{b2} + T_2) P_2 \rangle \right] +$$
$$\sum_{k=0}^{3} \lambda_{X_k}(t) [-b_k X_k(t) + E(t)].$$

The first order necessary conditions (FONC) resulting from the maximum principle, after suppressing the (x,t) arguments to ease notation when necessary, can be obtained as follows. The optimal emission (or fossil fuel) $E^*(x,t)$ path satisfies:

$$\frac{\alpha v(x) L(x)}{E(x,t)} = \lambda_R(t) - \sum_{k=0}^{3} \lambda_{X_k}(t) \Rightarrow \tag{37}$$

$$E^*(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(t) - \sum_{k=0}^{3} \lambda_{X_k}(t)}, \tag{38}$$

where $\xi_C(t) = -\sum_{k=0}^{3} \lambda_{X_k}(t)$ is the social price of the climate externality and $\xi_F(t) = \lambda_R(t) - \sum_{k=0}^{3} \lambda_{X_k}(t)$ is the social price of fossil fuels. Here we define social price of the climate externality to allow it to be negative, which it usually will be since it is typically a "bad".

The costate variables evolve according to

$$\dot{\lambda}_{T_0} = (\rho + B) \lambda_{T_0} + \langle vL, \phi \rangle + \lambda_{T_2} 6\hat{r} D (T_2 + T_{b2}) \tag{39}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D) \lambda_{T_2} + \langle vL, \phi P_2 \rangle + \lambda_{T_0} 3\hat{r} D (2T_2 + T_{b2}) \langle P_2, P_2 \rangle$$
$$+\lambda_{T_2} 6\hat{r} D \left[T_0 + \frac{\langle P_2, P_2^2 \rangle}{\langle P_2, P_2 \rangle} (2T_2 + T_{b2}) \right] \tag{40}$$

$$\dot{\lambda}_{X_k} = (\rho + b_k) \lambda_{X_k} - a_k \lambda_{T_0}, \quad k = 0, 1, 2, 3 \tag{41}$$

$$\dot{\lambda}_R(t) = \rho \lambda_R(t), \tag{42}$$

while temperature and externality dynamics are given by

$$\dot{T}_0 = -BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r}DT_2 (T_{b2} + T_2) \langle P_2, P_2 \rangle \tag{43}$$

$$\dot{T}_2 = -(B + 6D) T_2 - \frac{6\hat{r}D}{\langle P_2, P_2 \rangle} \langle P_2, (T_0 + T_2 P_2)(T_{b2} + T_2) P_2 \rangle \tag{44}$$

$$\dot{X}_k (t) = -b_k X_k (t) + E^* (t), \tag{45}$$

and the fossil fuel constraint satisfies

$$E^* (t) = \int_{x=-1}^{x=1} E^* (x,t)\, dx \ , \ \int_{t=0}^{\infty} \langle 1, E^* (x,t) \rangle\, dt = R_0. \tag{46}$$

If we assume that each location is constrained by local fossil fuel reserves $R_0 (x)$ and that no transfers are possible, condition (42) should be replaced by

$$\dot{\lambda}_R (x,t) = \rho \lambda_R (x,t). \tag{47}$$

Then

$$E^* (x,t) = \frac{\alpha v (x) L (x)}{\lambda_R (x,t) - \sum_{k=0}^{3} \lambda_{X_k} (t)}, \tag{48}$$

while the fossil fuel constraint becomes

$$\int_{x=-1}^{x=1} E^* (x,t)\, dx = R_0 (x). \tag{49}$$

3.1 Welfare Maximization when Heat Transfer is Ignored

To understand the impact of heat transfer on optimal fossil fuel paths (or emission paths) and optimal climate policy, it is helpful to consider at the beginning welfare optimization where heat transfer is ignored, or $D = 0$. The optimality conditions (39-42) with $D = 0$, or equivalently $\hat{r} = 0$, become:

$$\dot{\lambda}_{T_0} = (\rho + B) \lambda_{T_0} + \langle vL, \phi \rangle \tag{50}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D) \lambda_{T_2} + \langle vL, \phi P_2 \rangle \tag{51}$$

$$\dot{\lambda}_{X_k} = (\rho + b_k) \lambda_{X_k} - a_k \lambda_{T_0} \ , \ k = 0, 1, 2, 3 \tag{52}$$

$$\dot{\lambda}_R (t) = \rho \lambda_R (t), \tag{53}$$

while temperature and carbon dynamics in (43-45) are independent of D. Taking the forward solutions for the costate variables we obtain:

$$\lambda_{T_0} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B)(s-t)} \langle v(x) L(x), \phi(x) \rangle \, ds \tag{54}$$

$$\lambda_{T_2} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B)(s-t)} \langle v(x) L(x), \phi(x) P_2(x) \rangle \, ds \tag{55}$$

$$\lambda_{X_k} = \int_{s=0}^{\infty} \mathrm{e}^{-(\rho+b_k)(s-t)} a_k \lambda_{T_0}(s) \, ds \ , \ k = 0, 1, 2, 3. \tag{56}$$

Then the optimal fossil fuel path for the log utility case is given by

$$E^*(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(0) \mathrm{e}^{\rho t} - \sum_{k=0}^{3} \lambda_{X_k}(t)}. \tag{57}$$

Using the assumption that population and the damage parameter do not change with time, then the steady-state values for the costate variables implied from (50-53) are:

$$\lambda^*_{T_0} = -\frac{\langle v(x) L(x), \phi(x) \rangle}{(\rho + B)}, \lambda^*_{T_2} = -\frac{\langle v(x) L(x), \phi(x) P_2(x) \rangle}{(\rho + B)} \tag{58}$$

$$\lambda^*_{X_k} = -\frac{a_k}{(\rho + b_k)} \frac{\langle v(x) L(x), \phi(x) \rangle}{(\rho + B)} \tag{59}$$

$$\langle v(x) L(x), \phi(x) \rangle = \int_{x=-1}^{x=1} v(x) L(x) \phi(x) \, dx \tag{60}$$

$$\langle v(x) L(x), \phi(x) P_2(x) \rangle = \int_{x=-1}^{x=1} v(x) L(x) \phi(x) P_2(x) \, dx. \tag{61}$$

This means that the steady-state costate variables are independent of location x. The resource constraint implies

$$R_0 \geq \int_{t=0}^{\infty} dt \int_{x=-1}^{x=1} E(x,t) \, dx = \int_{t=0}^{\infty} \int_{x=-1}^{x=1} \left(\frac{\alpha v(x) L(x)}{\lambda_R(0) \mathrm{e}^{\rho t} - \sum_{k=0}^{3} \lambda_{X_k}(t)} \right) dx dt. \tag{62}$$

The initial value $\lambda_R(0)$ can be obtained by solving (62) for this initial value for any given value of total reserves R_0. Conditions (58)-(61) and (62) completely determine the optimal emission path for each location with the population kept constant at each location. Since the steady-state costate variables are independent of location x, the social price of the climate externality and the social price of fossil fuels are independent of location x. If we consider the case in which each location is constrained by local fossil fuel reserves $R_0(x)$, and that no transfers are possible, then the local resource constraint implies

$$\int_{x=-1}^{x=1} E^*(x,t)\,dx = \int_{x=-1}^{x=1} \left(\frac{\alpha v(x) L(x)}{\lambda_R(x,0)\,\mathrm{e}^{\rho t} - \sum_{k=0}^{3} \lambda_{X_k}(t)} \right) dx = R_0(x). \tag{63}$$

This constraint can be used to determine the initial value $\lambda_R(x,0)$ for any given value of total local reserves $R_0(x)$. In this case, although the social price of the climate externality does not depend on the location, the social price of fossil fuels depends on location through local reserves. This result is similar to an analogous result in Brock et al. (2014).

4 Heat Transport and Climate Change Policy

We move now to one of the main objectives of this paper, which is the characterization of the impact of heat transport towards the Poles on the social price of climate externality $\xi(t) = -\sum_{k=0}^{3} \lambda_{X_k}(t)$ and consequently on optimal fossil fuel paths and fossil fuel taxes. This impact is given by the derivative

$$\frac{\partial \xi(t; D, \hat{r})}{\partial \hat{r}} = -\sum_{k=0}^{3} \lambda'_{X_k}(t; D, \hat{r}). \tag{64}$$

To identify this impact, given the complexity of the model, we concentrate on the effect of a small change in $\hat{r}$ from the value of $\hat{r} = 0$. Thus we concentrate on small expansions around $\hat{r} = 0$. Note from (25)-(28) that when $\hat{r} = 0$, the temperature dynamics in the two-mode approximation are independent of heat transport and thus the costate variables are independent of D as shown in section 3.1. Thus the effects of a small increase in $\hat{r}$ from the value of zero will provide information about the derivative of interest, (64). In this expansion it seems sensible to assume that $T_b(x,t) = \bar{T}_b(x)$ since, as we have assumed, the anomaly temperature field, $T(x,t)$, is defined as the difference between the total temperature field minus the reference temperature field which is the pre-industrial temperature field, e.g. before 1750.

To obtain the impact of a small change in $\hat{r}$ around zero, we differentiate the optimality conditions with respect to $\hat{r}$ and evaluate the derivatives at $\hat{r} = 0$. We examine both polar cases. The case where the regulator treats fossil fuel reserves as a common property, and the opposite polar case where each location is constrained by its own reserves, and fossil fuel transfers are not possible.

Proposition 1 *The impact on the social price of the climate externality from a small increase in* $\hat{r}$ *from* $\hat{r} = 0$ *at a steady state is given by:*

$$-\sum_{k=0}^{3} \lambda_{X_k}^{'} = -S(x) \left[\frac{6D\bar{T}_{b2}}{(\rho + B)} \right] \sum_{k=0}^{3} \left[\frac{a_k}{(\rho + b_k)} \right] \quad (65)$$

$$S(x) = \left[\frac{\langle v(x) L(x), \phi(x) P_2(x) \rangle}{(\rho + B + 6D)} \right], \quad \bar{T}_{b2} < 0. \quad (66)$$

For proof see Appendix 1.

As shown in the proof to the proposition, $\bar{T}_{b2} < 0$. Therefore, the sign of this derivative depends on the distributions across latitudes of the welfare weights $v(x)$, the population $L(x)$, the damages $\phi(x)$ from an increase in global temperature, and $P_2(x)$ that reflects the dynamics of Nature on the spatial distribution of temperature. These effects are combined in the quantity

$$S(x) = \frac{\langle v(x) L(x), \phi(x) P_2(x) \rangle}{(\rho + B + 6D)} = \quad (67)$$

$$\frac{1}{(\rho + B + 6D)} \int_{x=-1}^{x=1} v(x) L(x) \phi(x) P_2(x) \, dx \quad (68)$$

$$\phi(x) = \phi_T(x) + \psi(x) \varphi(x). \quad (69)$$

Suppose that $S(x) < 0$ so that $-\sum_{k=0}^{3} \lambda_{X_k}^{'} < 0$. Recall that $-\sum_{k=0}^{3} \lambda_{X_k}$ is a positive quantity. Hence a marginal increase in $\hat{r}$ from $\hat{r} = 0$ shifts the social price of the climate externality down at every (x, t). However the sign of $S(x)$ is not straightforward since it is determined by socioeconomic factors $(v(x), L(x), \phi(x))$ and Nature dynamics $P_2(x)$.

To obtain some insight about the potential sign of $S(x)$, we consider the general function

$$f(x) = \frac{(1-x)^{\alpha_0} (1+x)^{\beta_0} \left(\gamma_0 + \delta_0 x^2\right)}{\int_{x=-1}^{x=1} (1-x)^{\alpha_0} (1+x)^{\beta_0} \left(\gamma_0 + \delta_0 x^2\right) dx}, \quad (70)$$

which can be used to approximate the distributions for distributional weights and damages across latitudes (see Brock et al. 2013). Following work by e.g.

Mendelsohn et al. (2006) or Burgess et al. (2014), we assume that climate change is expected to be most severe in poor countries surrounding the equator, with a skew towards southern latitudes. Thus we approximate the distribution of $\phi_T(x)$ by setting the parameters $\alpha_0 = 4, \beta_0 = 3, \gamma_0 = 1, \delta_0 = 0$. Regarding welfare weights we consider three possible alternatives: (i) equal weights for all x with $v(x) = 1$, (ii) Negishi-type weights set according to GDP per capita across latitudes as determined by Kummu and Varis (2011). These weights are obtained by using the parameterization $\alpha_0 = 2, \beta_0 = 2.2, \gamma_0 = 0.01, \delta_0 = 1$ in (70), and (iii) weights where the most importance is given to latitudes around the equator obtained by the normal probability density function with zero mean and standard deviation equal to 0.2. Finally, for the population, we use a density function compatible with evidence suggesting that roughly 88% of the world's population lives in the northern hemisphere, and about half the world's population lives north of 27°N.[8] These densities are presented in figure 1 below.

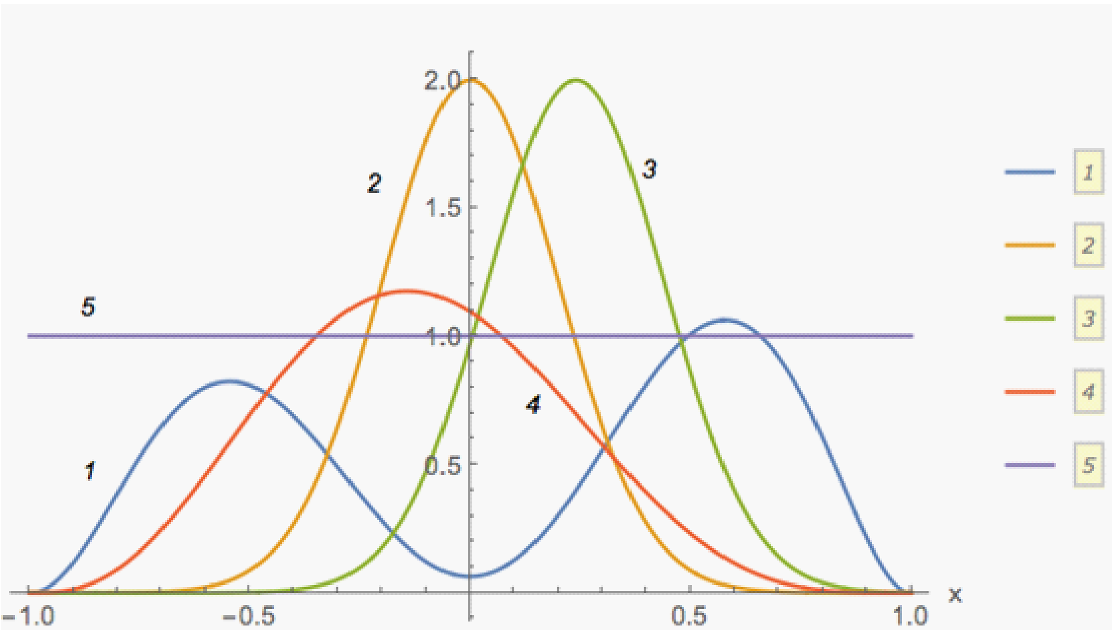

Figure 1: Negishi-type weights 1, Importance given to the Equator 2, Population 3, Damages 4, Equal weights 5.

The precipitation impact across latitudes is more difficult to assess. Fig-

[8]See http://visual.ly/worlds-population-2000-latitude-and-longitude.

ure 7 of Castruccio et al. (2014) suggests that the slope $\psi(x)$ may be negative. For the damage distribution $\varphi(x)$, we consider two alternative distributions: one the same as $\phi_T(x)$, and the other with a skew towards the northern latitudes ($\alpha_0 = 3, \beta_0 = 4, \gamma_0 = 1, \delta_0 = 0$). In most of the simulations, the term $S(x)$ turns out to be negative, while with Negishi-type weights and a steep negative slope for $\psi(x)$, the term $S(x)$ becomes positive. The sign of $S(x)$ is an empirical issue that requires further research. The important message, however, is that taking into account the spatial dynamics of nature emerging by heat transport across latitude, which is a well-documented natural process, changes the social price of the climate externality. The direction of the change depends on the natural parameters reflected in $P_2(x)$, but also on socioeconomic parameters reflected in the distribution of population, climate change damages and welfare weights.

The change in the social price of the climate externality due to heat transport effects implies that the optimal emission paths will also be affected. This can be shown easily by taking the total derivative of $E^*(x,t)$ in (37) and assuming that fossil fuel reserves are infinite so that $\lambda_R(t) = 0$ for all $t \geq 0$. Denoting $\frac{\partial E^*(x,t;D,\hat{r})}{\partial \hat{r}} = E^{*'}(x,t)$, we obtain

$$E^{*'}(x,t) = \frac{(E^*(x,t))^2}{\alpha v(x) L(x)} \left(\sum_{k=0}^{3} \lambda'_{X_k}(t; D, \hat{r}) \right). \tag{71}$$

Since $\left[(E^*(x,t))^2 / \alpha v(x) L(x) \right] > 0$, the direction of the impact of spatial heat transport on the optimal emission path is the same as the impact of spatial heat transport on social price of the climate externality.

4.1 Welfare Optimization, Polar Amplification, and Cross Latitude Effects

Having established that taking into account that heat transport across latitudes affects the social price of the climate externality and consequently optimal emission paths and taxes through socioeconomic and natural factors, our next step is to examine the same impact on the optimal temperature paths.

The identification of such a potential impact is important since our spatial model allows us to determine the characteristics of the temperature anomaly at the Poles, i.e. at $x = \pm 1$. An increase in the temperature anomaly at the Poles is related to the phenomenon of Polar amplification, which causes loss of Arctic sea ice. That in turn has consequences for melting land ice and other effects. There is growing evidence suggesting a link between rapid Arctic warming relative to the Northern hemisphere mid-latitudes. This phenomenon has been called Arctic amplification and is expected to increase the frequency of extreme weather events (Francis and Vavrus 2014). Melting land ice associated with a potential meltdown of Greenland and West Antarctica ice sheets due to polar amplifications might cause serious global sea level rise. It is estimated that the Greenland ice sheet holds an equivalent of 7 metres of global sea level rise, while the West Antarctica ice sheet holds the potential for up to 3.5 metres of global sea level rise (see Lenton et al. 2008).[9] On the other hand the loss of Arctic sea ice due to the Arctic amplification may generate economic benefits by making possible the exploitation of natural resources and fossil fuel reserves which are not accessible now because of the sea ice. Thus any polar or Arctic amplification implied by welfare maximization in the context of the spatial climate model should be taken into account.

Proposition 2 *Assuming infinite fossil fuel reserves, an increase in* $\hat{r}$ *in the neighborhood of* $\hat{r} = 0$ *will cause polar amplification for the socially optimal temperature path if the increase in* $\hat{r}$ *reduces the social price of the climate externality. If the increase in* $\hat{r}$ *increases the social price of the climate externality, there is no polar amplification. The impact from an increase in* $\hat{r}$ *on the Equator's temperature* ($x = 0$) *is ambiguous.*

For proof see Appendix 2.

The impact of heat transport on the social price of the climate externality depends on socioeconomic as well as natural factors. Therefore polar amplification may emerge from an optimization model as a result of specific choices like welfare weights or existing conditions, such as the distribution of population or production damages from climate change across latitudes.

The potential generation of extra costs and benefits to mid-latitudes due to polar amplification resulting from the optimizing model should be taken into account by fine tuning the spatial damage function. A damage function

[9] In the discussion about tipping points it has been stressed that the time scale of melting of the Greenland ice sheet is much longer than Arctic sea ice melting. However the Antarctic ice sheet could melt very fast once it gets started, but it will take an increase of 5°C of surface temperature for a serious destabilization.

which includes damages to latitude x caused by spillovers from temperature increases at other latitudes z, e.g. melting of land ice and, maybe indirect effects caused by melting of sea ice, can be written as:

$$\phi\left(x;\left\{T_0\left(t\right)+T_2\left(t\right)P_2\left(z\right)\right\}_{z=-1}^{z=+1}\right)\left[T_0\left(t\right)+T_2\left(t\right)P_2\left(x\right)\right]. \tag{72}$$

Damages from increased melting of land ice is a flow variable rather than a stock variable, so the flow of damages should depend upon the flow of melted land ice which depends, in turn, on the volume of available ice to melt. Consider the following high-latitude belt temperature index:

$$I\left(T_0\left(t\right),T_2\left(t\right);z_c\right)\equiv\int_{|z|>|z_c|}\left(T_0\left(t\right)+T_2\left(t\right)P_2\left(z\right)\right)dz= \tag{73}$$

$$2\int_{z\in[z_c,1]}\left(T_0\left(t\right)+T_2\left(t\right)P_2\left(z\right)\right)dz=2\left(1-z_c\right)\left[T_0\left(t\right)+\frac{T_2\left(t\right)}{2}\right]z_c\left(1+z_c\right).$$

Then the damage function (72) where the high-latitude temperature anomaly affects mid-latitude damages can be specified as:

$$\phi\left(I\left(T_0\left(t\right),T_2\left(t\right);z_c\right);x\right)\left[T_0\left(t\right)+T_2\left(t\right)P_2\left(x\right)\right]. \tag{74}$$

It is plausible to assume that $\phi\left(\cdot\right)$ is positive and increasing in the index $I\left(T_0\left(t\right),T_2\left(t\right);z_c\right)$ for latitudes in the set $\left\{|z|:|z|\leq|z_c|\right\}$. Note that $\phi\left(\cdot\right)$ might even be negative for some high latitudes because of the potential opening of new shipping lanes and the potential opening of access to previously inaccessible natural resources and fossil fuel reserves. Polar amplification effects could become substantial if warming continues, i.e. $T_0\left(t\right)$ continues to increase.

Using (74) the current value Hamiltonian (36) becomes

$$
\begin{aligned}
H = & \int_{x=-1}^{x=1} \{v(x) L(x) [\alpha \ln E(x,t) - \phi (I (T_0(t), T_2(t); z_c); x) [T_0(t) + T_2(t) P_2(x)]] \\
& -\lambda_R(t) E(t,x)\} dx + \\
& \lambda_{T_0}(t) \left[-BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r} D T_2 (T_{b2} + T_2) \langle P_2, P_2 \rangle \right] + \\
& \lambda_{T_2}(t) \left[-(B + 6D) T_2 - \frac{6\hat{r}D}{\langle P_2, P_2 \rangle} \langle P_2, (T_0 + T_2 P_2)(T_{b2} + T_2) P_2 \rangle \right] + \\
& \sum_{k=0}^{3} \lambda_{X_k}(t) [-b_k X_k(t) + E(t)] . \qquad (75)
\end{aligned}
$$

Polar amplification affects the costate variables for the two temperature modes T_0, T_2 which are now modified, relative to (39)-(40) and evolve according to

$$
\begin{aligned}
\dot{\lambda}_{T_0} = & (\rho + B) \lambda_{T_0} + \langle vL, \phi \rangle + \qquad (76) \\
& \left\langle vL, \frac{\partial \phi}{\partial I} 2 z_c (1 - z_c)(1 + z_c)(T_0 + T_2 P_2) \right\rangle + \lambda_{T_2} 6\hat{r} D (T_2 + T_{b2}) \\
\dot{\lambda}_{T_2} = & (\rho + B + 6D) \lambda_{T_2} + \langle vL, \phi P_2 \rangle + \left\langle vL, \frac{\partial \phi}{\partial I} z_c (1 - z_c)(1 + z_c)(T_0 + T_2 P_2) \right\rangle \\
& + \lambda_{T_0} 3\hat{r} D (2T_2 + T_{b2}) \langle P_2, P_2 \rangle + \lambda_{T_2} 6\hat{r} D \left[T_0 + \frac{\langle P_2, P_2^2 \rangle}{\langle P_2, P_2 \rangle} (2T_2 + T_{b2}) \right] . \qquad (77)
\end{aligned}
$$

The impact of polar amplification is captured by the terms

$$
\left\langle vL, \frac{\partial \phi}{\partial I} 2 z_c (1 - z_c)(1 + z_c)(T_0 + T_2 P_2) \right\rangle \qquad (78)
$$

$$
\left\langle vL, \frac{\partial \phi}{\partial I} z_c (1 - z_c)(1 + z_c)(T_0 + T_2 P_2) \right\rangle . \qquad (79)
$$

Although it is difficult to provide analytical results at this stage, it is clear that the polar amplification will affect the shadow values of the two temperature modes and through them the social price of the climate externality and the optimal temperature path. It is worth noting that polar amplification effects are determined by socioeconomic factors and nature dynamics. Calibration might provide a quantification of all these effects but the insight obtained is clear.

4.2 Heat transport and the social price of fossil fuels

Another issue that needs further analysis is the potential impact of heat transport on the social price of fossil fuels, i.e. on $\xi_F = \lambda_R(t) - \sum_{k=0}^{3} \lambda_{X_k}(t)$, when the social planner can allocate global reserves without cost, or on $\xi_F = \lambda_R(x,t) - \sum_{k=0}^{3} \lambda_{X_k}(t)$ when each location owns finite reserves. This is important since the impact of heat transport on fossil fuel paths and the optimal tax was derived under the simplifying assumption that the fossil fuel reserve was infinite so that the corresponding costates λ_R for fossil fuels were zero. Having already determined the impact on the social price of the externality, we need - in order to determine the impact on the social price of fuel - to evaluate the derivative of $\lambda_R(x,t)$ with respect to $\hat{r}$. We evaluate the derivative for the costate variable that corresponds to the case where each location owns finite reserves. We denote this derivative by $\lambda_R'(x,t)$.

Proposition 3 *Let $\bar{\xi} = -\sum_{k=0}^{3} \lambda_{X_k}$ be the steady-state social price of the climate externality which is independent of heat transfer when $\hat{r} = 0$. Then the sign of $\lambda_R'(x,0)$ is opposite to the sign of $\bar{\xi}'$.*

For the proof see Appendix.

Thus when the social price of the climate externality goes down, the social price of a finite fossil fuel reserve should go up because there is a tendency to extract more and vice versa.

5 Optimal Fossil Fuel Taxes

The solution of the welfare maximization problem allows us to obtain some insight into the structure of optimal fuel taxes, or equivalently, optimal carbon emission taxes. A representative firm produces output using emissions or, equivalently, fossil fuels according to the production function $y(x,t)E(x,t)^{\alpha}$ and faces a fossil fuel tax (or carbon tax) $\tau(x,t)$.[10] Then the profit maximizing path of fossil fuel use $E(x,t)$ is determined by

$$E^0(x,t) = \arg\max_{E(x,t)} \left\{ y(x,t)E(x,t)^{\alpha} - \tau(x,t)E(x,t) \right\}, \tag{80}$$

[10] To simplify things we assume that competitive markets exist so that output is sold at a competitive world price normalized to one, while fossil fuels are bought at a competitive world price p_F that satisfies the arbitrage condition $(\dot{p}_F(t)/p_F(t)) = r(t)$, where $r(t)$ are world interest rates. Thus τ should be interpreted as including the exogenously determined fossil fuel price.

with

$$E^0(x,t) = \left(\frac{\tau(x.t)}{\alpha y(t,x)}\right)^{\frac{1}{\alpha=1}} \quad \text{and} \tag{81}$$

$$y(x,t)\,E^0(x,t)^{\alpha} - \tau(x,t)\,E^0(x,t) = \tag{82}$$

$$(1-\alpha)\,y(t,x)^{\frac{1}{1-\alpha}}\,\tau(x,t)^{\frac{\alpha}{\alpha=1}}\,\alpha^{\frac{\alpha}{\alpha=1}}. \tag{83}$$

Consider now the problem of the social planner whose objective is to maximize

$$\int_{x=-1}^{x=1} v(x)\,L(x)\left[\ln C(x,t) - \phi\left(I\left(T_0(t), T_2(t); z_c\right); x\right)\left[T_0(t) + T_2(t)\,P_2(x)\right]\right] dx, \tag{84}$$

subject to climate and resource availability constraints, where $C(t,x)$ is per capita consumption at latitude x and time t. The planner chooses an emission tax $\tau(x,t)$ for each latitude and then the representative firm in each latitude takes this tax as parametric and determines fossil fuel use to maximize latitude payoff according to (81). Taxes collected are given by $\tau(x,t)\,E^0(x,t)$. In a competitive equilibrium the lump sum transfers from the social planner back to the consumers at latitude x at date t are equal to the taxes collected at this latitude and they are given by

$$Tr(x,t) = \tau(x,t)\,E^0(x,t) = \tau(x,t)\left(\frac{\tau(x.t)}{\alpha y(t,x)}\right)^{\frac{1}{\alpha=1}}. \tag{85}$$

Hence in equilibrium consumption at latitude x is

$$C(t,x) = \left[y(x,t)\,E^0(x,t)^{\alpha} - \tau(x,t)\,E^0(x,t)\right] + Tr(x,t) \Rightarrow \tag{86}$$

$$C(t,x) = y(t,x)\left(\frac{\tau(x.t)}{\alpha y(t,x)}\right)^{\frac{\alpha}{\alpha=1}} = y(t,x)^{\frac{1}{1-\alpha}}\,\tau(x,t)^{\frac{\alpha}{\alpha=1}}\,\alpha^{\frac{\alpha}{\alpha=1}}. \tag{87}$$

With consumption determined in terms of the fossil fuel tax by (87), the social planner acting as a Stackelberg leader chooses the spatiotemporal path for the fossil fuel tax $\tau(x,t)$ to maximize the integral of discounted values of optimized objectives (84), subject to climate and resource availability constraints. The current value Hamiltonian function for this problem is defined as:

$$H = \int_{x=-1}^{x=1} v(x) L(x) \left\{ \ln \left[y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha=1}} \alpha^{\frac{\alpha}{\alpha=1}} \right] - \right. \tag{88}$$
$$-\phi\left(I\left(T_0(t), T_2(t); z_c\right); x\right)\left[T_0(t) + T_2(t) P_2(x)\right] - \lambda_R(t) E(t,x) \Big\} dx +$$
$$\lambda_{T_0}(t) \left[-BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r} D T_2 \left(T_{b2} + T_2\right) \langle P_2, P_2 \rangle \right] +$$
$$\lambda_{T_2}(t) \left[-(B + 6D) T_2 - \frac{6\hat{r}D}{\langle P_2, P_2 \rangle} \langle P_2, \left(T_0 + T_2 P_2\right)\left(T_{b2} + T_2\right) P_2 \rangle \right] +$$
$$\sum_{k=0}^{3} \lambda_{X_k}(t) \left[-b_k X_k(t) + \left(\frac{\tau(x.t)}{\alpha y(t,x)} \right)^{\frac{1}{\alpha=1}} \right].$$

To provide a first insight into the optimal tax, we consider the simplest possible case where there are infinite reserves and damages are independent of the high-latitude index $I\left(T_0(t), T_2(t); z_c\right)$. In this case the optimal tax is determined as

$$\tau^*(x,t) = \arg\max_{\tau} \left\{ v(x) L(x) \ln \left[y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha=1}} \alpha^{\frac{\alpha}{\alpha=1}} \right] \right. \tag{89}$$
$$\left. + \sum_{k=0}^{3} \lambda_{X_k}(t) y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{1}{\alpha=1}} \alpha^{\frac{1}{\alpha=1}} \right\}, \tag{90}$$

which results in

$$\tau^*(x,t) = \alpha^{\alpha} \left(v(x) L(x)\right)^{\alpha-1} y(x,t) \left(-\sum_{k=0}^{3} \lambda_{X_k}(t) \right)^{1-\alpha}. \tag{91}$$

For the simplest case where $\hat{r} = 0$, the optimality conditions from (88) imply that at a steady state,

$$\lambda_{T_0} = -\frac{\langle vL, \phi \rangle}{(\rho + B)}, \quad \lambda_{X_k} = \frac{a_k}{(\rho + b)}, \tag{92}$$

and therefore

$$\tau^*(x,t) = \left[\alpha^{\alpha} \left(v(x) L(x)\right)^{\alpha-1} \left(-\sum_{k=0}^{3} \frac{-\langle vL, \phi \rangle a_k}{(\rho + B)} \right)^{1-\alpha} \right] y(x,t). \tag{93}$$

Thus although the steady-state social price of the climate externality, i.e., $-\sum_{k=0}^{3} \frac{a_k}{(\rho+b)}$, is independent of location, the optimal steady-state fossil fuel tax is linear in $y(x,t)$ which can be interpreted as the output-productivity component of location x. Thus there are two sources of spatial dependence for the optimal fossil fuel tax. The first is through the proportionality factor of $y(t,x)$ and it is the result of different welfare weights and population across latitudes. The second is the output-productivity component $y(x,t)$. Note that even if welfare weights and population differences across latitudes are ignored, e.g. $v(x)L(x)=1$, the spatial differentiation of the fossil fuel tax is introduced by spatial differences in the output-productivity component.

When damages depend on the high-latitude index, i.e. we have the case $\phi(x,I)$, then the steady state values for λ_{T_0}, λ_{X_k} are

$$\lambda_{T_0} = -\frac{\left\langle vL, \phi + T_0 \frac{\partial \phi}{\partial I} \right\rangle}{(\rho + B)} \ , \ \lambda_{X_k} = \frac{a_k}{(\rho + b)}, \tag{94}$$

and the optimal fossil fuel tax will be adjusted by the term $T_0 \frac{\partial \phi}{\partial I}$ which reflects polar amplification effects.[11]

It should be noted that for any given distribution of welfare weights $v(x)$ and population $L(x)$, poorer latitudes, i.e., latitudes with a relatively lower output-productivity component $y(t,x)$, are taxed less per unit emissions than richer latitudes. This result should be contrasted with the result derived under the standard assumption of compensatory transfers which indicates that a unit of emissions is taxed the same no matter which latitude belt emitted it.

5.1 The Finite Reserve Case

For the finite reserve case we choose $\tau(x,t)$ to maximize

$$\begin{aligned} &\Big\{ v(x) L(x) \ln \left[y(t,x)^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{\alpha}{\alpha=1}} \alpha^{\frac{\alpha}{\alpha=1}} \right] \\ &+ \sum_{k=0}^{3} \lambda_{X_k}(t) (\alpha y(t,x))^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{1}{\alpha=1}} - \lambda_R(t) \left[(\alpha y(t,x))^{\frac{1}{1-\alpha}} \tau(x,t)^{\frac{1}{\alpha=1}} \right] \Big\}. \end{aligned} \tag{95}$$

Following the same computations as above, we obtain the solution

[11] Note that since $T_0(t)$, the global mean yearly temperature is positive and is likely to be positive for most latitudes except possibly the higher latitudes, we expect that the added term will be positive and $\left\langle vL, \phi + T_0 \frac{\partial \phi}{\partial I} \right\rangle$ will be increased over the quantity $\langle vL, \phi \rangle$ but this does not have to be the case.

$$\tau^*(x,t) = \left[\alpha^\alpha (v(x) L(x))^{\alpha-1} \left(-\sum_{k=0}^{3} \lambda_{X_k}(t) + \lambda_R(t)\right)^{1-a}\right] y(x,t). \quad (96)$$

Computing optimal emissions, we obtain

$$E^*(x,t) = \left(\frac{\tau^*(x.t)}{\alpha y(t,x)}\right)^{\frac{1}{\alpha=1}} = \frac{\alpha v(x) L(x)}{\left(-\sum_{k=0}^{3} \lambda_{X_k}(t) + \lambda_R(t)\right)}. \quad (97)$$

In the finite reserves case we can determine $\lambda_R(t)$ from the resource constraint as:

$$\int_{x=-1}^{x=1} \int_{t=0}^{\infty} E^*(x,t)\, dxdt = \int_{x=-1}^{x=1} \int_{t=0}^{\infty} \left(\frac{\alpha v(x) L(x)}{\left(-\sum_{k=0}^{3} \lambda_{X_k}(t) + \lambda_R(t)\right)}\right) dxdt = R_0. \quad (98)$$

When we consider the full model with $\hat{r} > 0$, the approach for determining the optimal tax is the same, but calculation of the social price of the externality requires numerical approaches. It should be noted, however, that our analysis in Proposition 1 remains valid for the Hamiltonian (88) with damages $\phi(x)$ independent of the high-latitude index $I(T_0(t), T_2(t); z_c)$. This is because $\tau(x,t)$ enters as a control and does affect the evolution of the state variable associated with temperature dynamics. With proposition 1 valid, it is clear that accounting for the heat transport shifts the spatial distribution of the optimal fossil fuel taxes upwards or downwards. The direction of the shift depends on the interactions between the spatially differentiated socioeconomic factors and Nature's spatiotemporal dynamics. When reserves are finite, then the impact from an increase in $\hat{r}$ is given by combining the results of Propositions 1 and 3. In this case,

$$\frac{\partial \tau^*(x,t)}{\partial \hat{r}} = \Big[(1-\alpha) y(x,t) \alpha^\alpha (v(x) L(x))^{\alpha-1} \quad (99)$$

$$\times \left(-\sum_{k=0}^{3} \lambda_{X_k}(t) + \lambda_R(t)\right)^{-\alpha} \left(-\sum_{k=0}^{3} \lambda'_{X_k}(t) + \lambda'_R(t)\right)\Bigg], \quad (100)$$

where $\lambda'_{X_k}(t), \lambda'_R(t)$ are determined in Propositions 1 and 3.

5.2 Heat Transport and Optimal Taxation

An interesting question arising from the introduction of heat transport and polar amplification is whether heat transport from the equator to the Poles increases or reduces optimal fossil fuel taxes. From (72) and (75), total damages from the temperature anomaly can be written as:

$$
\begin{aligned}
TC(t) &= \int_{x=-1}^{x=1} v(x) L(x) \phi\left(x, \hat{T}(t)\right) T(t, x)\, dx && (101)\\
\hat{T}(t) &= \{T_0(t) + T_2(t) P_2(z)\}_{z=-1}^{z=1} && (102)\\
T(x, t) &= T_0(t) + T_2(t) P_2(x). && (103)
\end{aligned}
$$

The marginal cost of a temperature rise at location x', dropping t to ease notation, can be defined as

$$
\begin{aligned}
MC(x') &= \frac{\partial TC}{\partial T(x')} = \int_{x=-1}^{x=1} v(x) L(x) \left\{ \left[\frac{\partial \phi\left(x, \hat{T}\right)}{\partial T(z)} \right]_{z=x'} \right\} T(x)\, dx + \\
&\quad v(x') L(x') \phi\left(x', \hat{T}\right) && (104)
\end{aligned}
$$

When heat transfer increases, which in our model means an increase in $\hat{r}$, then heat moves from lower latitudes to higher latitudes. If at the higher latitudes the marginal costs $MC(x')$s are small relative to lower latitudes or even negative, the optimal emissions tax should be smaller and hence more fossil fuels should be burned. Smaller marginal costs at high latitudes are supported by arguments suggesting that in high latitude zones there are few people to be damaged by more heat there, while more heat may open trade routes, allow crops to be grown in previously frigid zones, lengthen growing seasons, and allow access to valuable natural resources. On the other hand, if $MC(x')$s are larger at the high latitudes than the lower latitudes, the effect on taxes is reversed. Higher marginal costs at lower latitudes are supported by arguments suggesting that more heat at high latitudes may destabilize ice sheets or release carbon in the permafrost, thus creating substantial damages to lower latitudes. In this case taxes should rise sharply.

6 Climate Externality Price and "Safety First" Utility

The results obtained above were based on the tractability advantages of the logarithmic utility function. In this section we seek to identify the impact on the social price of climate externality and the socially optimal use of fossil fuel under a more general utility function. In particular we investigate the class of utilities where marginal disutility increases very fast relative to the logarithmic utility as consumption goes towards zero.

A more general utility function results in the following welfare function:

$$\int_{t=0}^{\infty} \mathrm{e}^{-\rho t} \left[\int_{x=-1}^{x=1} v\left(x\right) L\left(x\right) U \left[\frac{y E^{\alpha} \mathrm{e}^{-\phi(x) T_{total}(x,t)}}{L\left(x\right)} \right] \right] dxdt = \tag{105}$$

$$\int_{0}^{\infty} \mathrm{e}^{-\rho t} \left[\int_{-1}^{1} v\left(x\right) L\left(x\right) U \left[\frac{y\left(x,t\right) E\left(x,t\right)^{\alpha} e^{-\phi(x,t)[T_b(x,t)+T(x,t)]}}{L(x)} \right] \right] dxdt,$$

which is maximized by choosing the optimal path $E\left(x,t\right)$, subject to the constraints imposed by Nature dynamics and fossil fuel exhaustibility. Using the two-mode approximations and the approximations of the radiating forcing term employed above, the current value Hamiltonian for the problem is:

$$\begin{aligned}
H = & \int_{x=-1}^{x=1} \left\{ v\left(x\right) L\left(x\right) U \left[\frac{y\left(x,t\right) E\left(x,t\right)^{\alpha} e^{-\phi(x)[T_0(t)+T_2(t)P_2(x)]}}{L\left(x\right)} \right] \right. \\
& \left. -\lambda_R\left(t\right) E\left(t,x\right) \right\} dx + \\
& \lambda_{T_0}\left(t\right) \left[-BT_0 + \sum_{k=0}^{3} a_k X_k - 3\hat{r} D T_2 \left(T_{b2} + T_2\right) \left\langle P_2, P_2 \right\rangle \right] + \\
& \lambda_{T_2}\left(t\right) \left[-\left(B + 6D\right) T_2 - \frac{6\hat{r}D}{\left\langle P_2, P_2 \right\rangle} \left\langle P_2, \left(T_0 + T_2 P_2\right) \left(T_{b2} + T_2\right) P_2 \right\rangle \right] + \\
& \sum_{k=0}^{3} \lambda_{X_k}\left(t\right) \left[-b_k X_k\left(t\right) + E\left(t\right) \right].
\end{aligned} \tag{106}$$

The FONC resulting from the maximum principle, after suppressing the (x, t) arguments to ease notation when necessary, are presented below. The optimal emission path $E^*\left(x,t\right)$ satisfies

$$\frac{\alpha v(x) L(x) \left[U'\left(\breve{C}(x,t)\right)\breve{C}(x,t)\right]}{E^*(x,t)} = \lambda_R(t) - \sum_{k=0}^{3} \lambda_{X_k(t)} \tag{107}$$

$$\breve{C}(x,t) = y(x) E^*(x,t)^\alpha \, \mathrm{e}^{-\phi(x)[T_0(t)+T_{b0}(t)+(T_2(t)+T_{b2}(t))P_2(x)]}. \tag{108}$$

We use $\breve{C}(x,t)$ to denote the output of the economy. We assume that this output is consumed, but the consumption value has been damaged by climate damages reflected in the exponential term. The costate variables evolve according to:

$$\dot{\lambda}_{T_0} = (\rho + B)\lambda_{T_0} + \left\langle vL, \phi U'\left(\breve{C}\right)\breve{C}\right\rangle + \lambda_{T_2} 6\hat{r}D(T_2 + T_{b2}) \tag{109}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D)\lambda_{T_2} + \left\langle vL, \phi P_2 U'\left(\breve{C}\right)\breve{C}\right\rangle + \lambda_{T_0} 3\hat{r}D(2T_2 + T_{b2})\langle P_2, P_2\rangle$$

$$+\lambda_{T_2} 6\hat{r}D\left[T_0 + \frac{\langle P_2, P_2^2\rangle}{\langle P_2, P_2\rangle}(2T_2 + T_{b2})\right] \tag{110}$$

$$\dot{\lambda}_{X_k} = (\rho + b_k)\lambda_{X_k} - a_k\lambda_{T_0}\ , \ \ k = 0,1,2,3 \tag{111}$$

$$\dot{\lambda}_R(t) = \rho\lambda_R(t). \tag{112}$$

The optimality conditions for temperature dynamics, externality dynamics and the fossil fuel constraints are the same as (43)-(47).

If the heat transport is ignored, i.e. $D = 0$, or equivalently $\hat{r} = 0$, the costate variables evolve according to:

$$\dot{\lambda}_{T_0} = (\rho + B)\lambda_{T_0} + \left\langle vL, \phi U'\left(\breve{C}\right)\breve{C}\right\rangle \tag{113}$$

$$\dot{\lambda}_{T_2} = (\rho + B + 6D)\lambda_{T_2} + \left\langle vL, \phi P_2 U'\left(\breve{C}\right)\breve{C}\right\rangle$$

$$\dot{\lambda}_{X_k} = (\rho + b_k)\lambda_{X_k} - a_k\lambda_{T_0}\ , \ \ k = 0,1,2,3 \tag{114}$$

$$\dot{\lambda}_R(t) = \rho\lambda_R(t), \tag{115}$$

with forward solutions

$$\lambda_{T_0} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B)(s-t)} \left\langle v(x)L(x), \phi(x)U'\left(\breve{C}\right)\breve{C}\right\rangle ds \tag{116}$$

$$\lambda_{T_2} = -\int_{s=0}^{\infty} \mathrm{e}^{-(\rho+B)(s-t)} \left\langle v(x)L(x), \phi(x)P_2(x)U'\left(\breve{C}\right)\breve{C}\right\rangle ds$$

$$\lambda_{X_k} = \int_{s=0}^{\infty} \mathrm{e}^{-(\rho+b_k)(s-t)} a_k\lambda_{T_0}(s)\,ds\ , \ \ k = 0,1,2,3. \tag{117}$$

Conditions (109)-(117) indicate that the neat property of the log utility function obtained above is lost because of the term $U'\left(\breve{C}\right)\breve{C}$ which emerges when general utility functions are used. In order to obtain some analytical results, we consider the class of utility functions

$$U\left(C\right) = \frac{C^{1-\gamma}}{1-\gamma}, \tag{118}$$

where γ is both the coefficient of relative risk aversion and (minus) the elasticity of marginal utility with respect to consumption, while the log utility function is the special case $\gamma = 1$. For $\gamma > 1$, we call the class of utilities "safety first" because in this case when the consumption value is damaged due to climate change, the disutility increases faster than the logarithmic utility for which $\gamma = 1$. In the same context, an increase of γ from the value of one implies an increase in the relative risk aversion. For this class of utility functions, we have $U'\left(\breve{C}\right)\breve{C} = \breve{C}^{1-\gamma}$. The main question is whether an increase in the coefficient of relative risk aversion from the value of one will have an impact on the social price of the climate externality and the socially optimal fossil fuel path.

Using (118), the optimality condition for the optimal choice of fossil fuel use becomes

$$\frac{\alpha v\left(x\right)L\left(x\right)\breve{C}\left(x,t;\hat{r},\gamma\right)^{1-\gamma}}{E\left(x,t;\hat{r},\gamma\right)} = \lambda_R\left(t\right) + \xi\left(t;\hat{r},\gamma\right) \tag{119}$$

$$\xi\left(t;\hat{r},\gamma\right) = -\sum_{k=0}^{3}\lambda_{X_k(t)}. \tag{120}$$

Differentiating (119) with respect to γ, using $\frac{\partial \breve{C}^{1-\gamma}}{\partial\gamma} = -\ln\breve{C}$, evaluating the derivatives at $(\hat{r},\gamma) = (0,1)$ and suppressing $(\hat{r},\gamma)$ to ease notation, we obtain

$$\alpha v\left(x\right)L\left(x\right)\left[-\frac{1}{E\left(x,t\right)^2}\frac{\partial E\left(x,t\right)}{\partial\gamma} - \frac{\ln\breve{C}}{E\left(x,t\right)}\right] = \frac{\partial\lambda_R\left(t\right)}{\partial\gamma} + \frac{\partial\xi\left(t\right)}{\partial\gamma}. \tag{121}$$

To identify the impact of increasing γ from the value $\gamma = 1$ on the social price of the climate externality, we consider expansions of any endogenous variable $\zeta\left(t;\hat{r},\gamma\right)$ of our model with respect to $(\hat{r},\gamma)$ around the point $(\hat{r}, \gamma) = (0, 1)$, or

$$\zeta(t; \hat{r}, \gamma) = \zeta(t; 0, 1) + \frac{\partial \zeta(t; 0, 1)}{\partial \hat{r}} \hat{r} + \frac{\partial \zeta(t; 0, 1)}{\partial \gamma} (\gamma - 1) + o(\hat{r}, |\gamma - 1|). \quad (122)$$

Since we are interested in the social price of the climate externality and the use of fossil fuels, we consider the following expansions

$$\begin{aligned} \xi(t; \hat{r}, \gamma) &= \xi(t; 0, 1) + \frac{\partial \xi(t; 0, 1)}{\partial \hat{r}} \hat{r} + \frac{\partial \xi(t; 0, 1)}{\partial \gamma} (\gamma - 1) + \quad (123) \\ &\quad o(\hat{r}, |\gamma - 1|) \\ E(t; \hat{r}, \gamma) &= E(t; 0, 1) + \frac{\partial E(t; 0, 1)}{\partial \hat{r}} \hat{r} + \frac{\partial E(t; 0, 1)}{\partial \gamma} (\gamma - 1) + \quad (124) \\ &\quad o(\hat{r}, |\gamma - 1|) \\ \xi(t) &= -\sum_{k=0}^{3} \lambda_{X_k}(t) > 0, \quad (125) \end{aligned}$$

which approximate the climate externality price and the fossil fuel use. Using these expansions, we can state the following result.

Proposition 4 *Assuming no serious poverty at any location at any time, so that* $\ln \breve{C}(x,t) > 0$ *for all* (x,t) *and* $\lambda_R(t) = 0$ *for all* t, *then a small increase in the coefficient of relative risk aversion* γ *from* $\gamma = 1$ *will reduce the social price of the climate externality* $\frac{\partial \xi(t;0,1)}{\partial \gamma} = -\sum_{k=0}^{3} \frac{\partial \lambda_{X_k}(t;0,1)}{\partial \gamma} < 0$ *for all* $t \geq 0$ *, where*

$$\begin{aligned} \frac{\partial \lambda_{X_k}(t; 0, 1)}{\partial \gamma} &= \int_{s=t}^{\infty} \mathrm{e}^{-\rho(s-t)} a_k \frac{\partial \lambda_{T_0}(s; 0, 1)}{\partial \gamma} ds > 0 \quad (126) \\ \frac{\partial \lambda_{T_0}(v; 0, 1)}{\partial \gamma} &= \int_{s=v}^{\infty} \mathrm{e}^{-\rho(s-v)} \left\langle vL, \phi \ln \breve{C} \right\rangle (s) ds > 0. \quad (127) \end{aligned}$$

For proof see Appendix.

Since in the safety-first class of utilities, climate damages in the utility function are realized through damages in the value of consumption, and recalling that $C = yE^{\alpha}\mathrm{e}^{-\phi T}$, and $U(C) = C^{1-\gamma}/(1-\gamma)$, it is reasonable to expect that an increase in γ from $\gamma = 1$ will reduce the price of the climate externality and the corresponding fuel tax when the stock of fossil fuels is assumed to be infinite. It should also be noted that the impact of the safety-first utility as quantified by the derivative $\frac{\partial \lambda_{X_k}(t;0,1)}{\partial \gamma}$ depends, through the derivative $\frac{\partial \lambda_{T_0}(v;0,1)}{\partial \gamma}$, on the socioeconomic factors $v(x), L(x), \phi(x)$ and the value of consumption $\breve{C}$ adjusted for climate change damages.

7 Concluding Remarks and Suggestions for Future Research

This paper is, to our knowledge, the first paper in climate economics to consider the combination of spatial heat transport and polar amplification. We simplified the problem by stratifying the Earth into latitude belts and assuming as in North et al. (1981) and Wu and North (2007) that the two hemispheres were symmetric so that solutions of the climate dynamics could be expanded into an infinite series of even numbered Legendre Polynomials.

In order to obtain analytical tractability of the climate dynamics across latitude belts and to solve the economic infinite horizon welfare economics problem, we introduced some approximations to the climate dynamics and some specializations to specific utility functions.

First we used a linear radiative kernel approximation to the carbon cycle which has been used by Hasselmann et al. (1997) and Pierrehumbert (2014).[12] Second we truncated the Legendre polynomial expansion of the climate dynamics to a small number of modes. Third, we built upon work by Alexeev et al. (2005), Langen and Alexeev (2007), and Alexeev and Jackson (2012) to motivate our specification of the heat transport function across latitudes as a function of global average temperature. This specification imparts a nonlinearity which we approximated by series expansion around the case of no polar amplification where heat transport is linear.

In this paper we use logarithmic utility and exponential specification of climate damages as a function of temperature, except in section 6 where we use a more general utility function. We analyzed spillover effects from higher latitudes onto lower latitudes because of amplification of warming on the higher latitudes. Our main contributions are the following.

First, we showed that it is possible to build climate economic models that include the very real climatic phenomena of heat transport and high latitude amplification of warming (i.e. "polar amplification") and still maintain analytical tractability.

[12]Brock thanks L.P. Hansen, A. Sanstad, V. Zhorin, and L. Han at RDCEP, University of Chicago, for stimulating conversations about radiative kernels and other ideas for solving climate economics models while working with them on climate economic models with no spatial transport.

Since analytical tractability is essential for understanding the output of more complicated and realistic models, we view this as an important - maybe the most important - contribution of this line of research. For example witness how important the work of North and others has been in showing how models with spatial transport in climate dynamics can be made analytically tractable by use of the "right" mathematics, e.g. bases of even number Legendre polynomials and spherical harmonics. This kind of work is used heavily to understand the computational output of much more complicated and realistic climate models. We view ourselves as initiating a similar line of research for the joint modeling of coupled climate dynamics and economic dynamics. We believe that our finding regarding the link between heat transfer, polar amplification, and optimal fuel taxes illustrates the importance of future research in climate change economics to study the impact of spatial energy transport across the globe.

Second, we showed that the optimal tax function, i.e. the marginal social cost of emissions, depended upon the distribution not only of welfare weights but also population across latitudes, the distribution of marginal damages across latitudes and cross latitude interactions of marginal damages, along with nature dynamics. These dynamics are reflected in the decomposition of the temperature field into modes via the expansion of the climate dynamics into a series of even numbered Legendre polynomials. The formulas we obtained are quite interpretable and comparative dynamics can be quite easily done on their components.

Third, we derived and compared optimal solutions under (i) no heat transport, (ii) heat transport but no polar amplification, (iii) both heat transport and polar amplification.

Fourth, we compared the solution for optimal taxes under the standard assumption of compensatory transfers, so that a unit of emissions is taxed the same no matter which latitude belt emitted it, with the solution for optimal taxes in which there are no compensatory transfers at all. In this latter case the poorer latitudes are taxed less per unit emissions than richer latitudes. While this is obvious for the direction of the tax, we give a formula that shows both how the interaction of the climate system with the economic system feeds into a formula for the optimal tax per unit emissions, and the way in which optimal taxes are differentiated across locations. We also discuss the possibility that an increase in the heat transfer towards the Poles may increase or reduce fossil fuels taxes. This is an important observation because it provides a direct link between climate dynamics, which are usually disregarded in IAMs, and optimal economic policy.

Future research could move in different directions. First, and most important, extensive computational work should be done to locate sufficient conditions for spatial heat transport and polar amplification to quantitatively matter significantly for welfare economics at different locations on the planet. We believe the ideal would be to conduct computational work like that of the important work of Cai et al. (2015) to assess the quantitative importance of taking into account heat transport. Second, it would be valuable to extend the results in this paper to two-dimensional space where heat transport occurs across both latitude and longitude. Brock et al. (2013) did this for the case of linear heat transport but did not include polar amplification. A third area of future research would be to extend our current paper and the Desmet and Rossi-Hansberg (2015) paper, which addresses migration responses to climate change, to include the impact of heat and moisture transport across the globe. This research could build on the work of Desmet and Rossi-Hansberg (2010), (2014), and Boucekkine et al. (2009), (2013). The results in this paper suggest that spatial heat transfer and polar amplification have a potentially important impact on climate change policy. Our conjecture is, therefore, that accounting for these natural phenomena in economic models that include policies such as adaptation to climate change, costly mitigation, back up technologies, or difference in emissions across different fossil fuels, would provide new insights into the efficient design of climate change policy.

8 Appendix

Proof of Proposition 1:

We differentiate the optimality conditions (37) - (46) with respect to $\hat{r}$. Since we are solving for the temperature anomaly it is natural to assume that initial values of the two-mode expansion are $T_0(0) = 0, T_2(0) = 0$. In the case of $\hat{r} = 0$, we obtain $T_2(t) = 0$ for all dates $t \geq 0$. Denote derivatives w.r.t. $\hat{r}$ by primes to obtain the following.

For the two-mode temperature dynamics:

$$\dot{T}_0' = -BT_0' + \sum_{k=0}^{3} a_k X_k' \ , \ T_0'(0) = 0 \tag{128}$$

$$\dot{T}_2' = -(B+6D)\,T_2' - 6T_0T_{b2} \ , \ T_2'(0) = 0. \tag{129}$$

For optimal emissions when the regulator treats fossil fuels reserves as a common property:

$$\frac{-\alpha v(x)\,L(x)\,E'(x,t)}{E(x,t)^2} = \lambda_R'(t) - \sum_{k=0}^{3} \lambda_{X_k}'(t) \tag{130}$$

$$\int_{t=0}^{\infty} \langle 1, E'(x,t) \rangle\,dt = 0. \tag{131}$$

For optimal emissions when each location is constrained by its own reserves:

$$\frac{-\alpha v(x)\,L(x)\,E'(x,t)}{E(x,t)^2} = \lambda_R'(x,t) - \sum_{k=0}^{3} \lambda_{X_k}'(t) \tag{132}$$

$$\int_{t=0}^{\infty} E'(x,t)\,dt = 0. \tag{133}$$

For the costate variables:

$$\dot{\lambda}_{T_0}' = (\rho+B)\,\lambda_{T_0}' + 6\lambda_{T_2}DT_{b2} \tag{134}$$

$$\dot{\lambda}_{T_2}' = (\rho+B+6D)\,\lambda_{T_2}' + 3\lambda_{T_0}DT_{b2}\langle P_2, P_2\rangle + 6\lambda_{T_2}D\,(T_0+T_{b2})\,\frac{\langle P_2, P_2^2\rangle}{\langle P_2, P_2\rangle} \tag{135}$$

$$\dot{\lambda}_{X_k}' = (\rho+b_k)\,\lambda_{X_k}' - a_k\lambda_{T_0}' \ , \ k = 0,1,2,3 \tag{136}$$

$$\dot{\lambda}_R'(t) = \rho\lambda_R'(t). \tag{137}$$

The forward solutions for (134) and (136) are obtained as:

$$\lambda_{X_k}' = a_k \int_{s=t}^{\infty} \mathrm{e}^{-(\rho+b_k)(s-t)}\lambda_{T_0}'\,ds \quad \text{where} \tag{138}$$

$$\lambda_{T_0}' = -6D \int_{q=s}^{\infty} \mathrm{e}^{-(\rho+B)(q-s)}\lambda_{T_2}(q)\,T_{b2}(q)\,dq. \tag{139}$$

In order to obtain the forward solution for (139) we need to determine $\lambda_{T_2}(q), T_{b2}(q)$. From (40) the forward solution of (135), and the steady state for λ_{T_2} evaluated at $\hat{r}=0$, are respectively:

$$\lambda_{T_2}(q) = -\langle v(x) L(x), \phi(x) P_2(x) \rangle \int_{q=s}^{\infty} \mathrm{e}^{-(\rho+B+6D)s} ds \quad (140)$$

$$\bar{\lambda}_{T_2} = -\frac{\langle v(x) L(x), \phi(x) P_2(x) \rangle}{(\rho+B+6D)}. \quad (141)$$

In order to evaluate $T_{b2}(q)$ we need to go back to the original two-mode expansion for the original energy balance model dynamics,

$$\dot{T}_b(x,t) = Q\hat{S}(x)\hat{a}(x) - (A + BT_b(x,t)) + D\mathcal{L}T_b(x,t) \quad (142)$$

$$\dot{T}_{b0}(t) + \dot{T}_{b2}(t) P_2(x) = Q(1 - 0.482 P_2(x)) a(x) - A - BT_{b0}(t) - BT_{b2}(t) P_2(x) - 6DT_{b2}(t) P_2(x). \quad (143)$$

Here $Q = (1/4)\, 1367.7 W/m^2$ is the mean value of the solar constant divided by 4 and the values of the other constants are taken from North (1975 a,b) as well as $\hat{S}(x)$. Since North is interested in analytical solutions with an endogenous ice line and we are not, we can approximate the co-albedo function $\hat{a}(x) \equiv 1 - \hat{\alpha}(x)$ where $\hat{\alpha}(x)$, the latitude average of albedo, by the parabola $\hat{\alpha}(x) = 0.1 + (1/2)x^2$.[13] In order to get an approximation to the likely sign of $T_{b2}(t)$, we assume constant co-albedo $\bar{a}$. From (143) we obtain:

$$\dot{T}_{b2}(t) = Q\bar{a}\frac{\langle P_2(x), (1 - 0.482 P_2(x)) \rangle}{\langle P_2(x), P_2(x) \rangle} - (B+6D) T_{b2}(t) \quad (144)$$

$$\dot{T}_{b2}(t) = -0.482 Q\bar{a} - (B+6D) T_{b2}(t). \quad (145)$$

Hence, the steady state for $T_{b2}(t)$ is obtained as:

$$T_{b2}(t) \to \bar{T}_{b2} = \frac{-0.482 Q\bar{a}}{(B+6D)} < 0 \text{ as } t \to \infty. \quad (146)$$

Then (139) and (138), evaluated at the steady state values $(\bar{\lambda}_{T_2}, \bar{T}_{b2})$ for

[13] See http://www.climatedata.info/Forcing/Forcing/albedo.htmlf Figure 1 for the graph of average albedo by latitude.

$\lambda_{T_2}(q), T_{b2}(q)$, result in:

$$\lambda'_{T_0} = \left[\frac{-6D\bar{T}_{b2}}{(\rho+B)}\right]\left[-\frac{\langle v(x)L(x), \phi(x)P_2(x)\rangle}{(\rho+B+6D)}\right] \tag{147}$$

$$\lambda'_{X_k} = \frac{a_k}{(\rho+b_k)}\lambda'_{T_0} \tag{148}$$

$$\lambda'_{X_k} = \left[\frac{a_k}{(\rho+b_k)}\right]\left[\frac{6D\bar{T}_{b2}}{(\rho+B)}\right]\left[\frac{\langle v(x)L(x), \phi(x)P_2(x)\rangle}{(\rho+B+6D)}\right]. \tag{149}$$

Therefore,

$$-\sum_{k=0}^{3}\lambda'_{X_k} = -S(x)\left[\frac{6D\bar{T}_{b2}}{(\rho+B)}\right]\sum_{k=0}^{3}\left[\frac{a_k}{(\rho+b_k)}\right] \tag{150}$$

$$S(x) = \left[\frac{\langle v(x)L(x), \phi(x)L(x)\rangle}{(\rho+B+6D)}\right] \quad , \quad \bar{T}_{b2} < 0. \tag{151}$$

□

Proof of Proposition 2:

The impact of heat transport on the optimal temperature paths requires the computation of the derivative of $T(x,t)$ with respect to $\hat{r}$ which, using the two-mode approach, is defined as:

$$T'(x,t) = T'_0(t) + T'_2(t)P_2(x). \tag{152}$$

Differentiating the optimality conditions for the state variables we obtain:

$$\dot{T}'_0(t) = -BT'_0 + \sum_{k=0}^{3}a_kX'_k \ , \ T'_0(0) = 0 \tag{153}$$

$$\dot{T}'_2(t) = -(B+6D)T'_2 - 6T_0T_{b2} \ , \ T'_2(0) = 0 \tag{154}$$

$$\dot{X}'_k(t) = -bX'_k + E'(t) \ , \ X'_k(0) = 0 \ , \ k = 0,1,2,3. \tag{155}$$

We evaluate (153)-(155) at $\bar{T}_{b2} < 0$. The solution of (155) is:

$$X'_k(t) = \mathrm{e}^{-b_kt}\left(X'_k(0) + \int_{s=0}^{t}\mathrm{e}^{b_ks}E'(s)\,ds\right) = \int_{s=0}^{t}\mathrm{e}^{b_k(s-t)}E'(s)\,ds. \tag{156}$$

Assume that the fossil fuel reserves are infinite so that $\lambda_R(t) = 0$ for all t. The derivative $E'(s)|_{s=0}^{t}$ could be either positive or negative, depending on the sign of the derivative of the social price of the externality given in Proposition 1.

Assume that socioeconomic and natural factors are such that $-\sum_{k=0}^{3}\lambda_{X_k}^{'}<0$, then $E'(s)>0$ at all locations x and at all dates s when reserves are infinite and $X_k'(t)>0$ for all k, and$t>0$. This because when the social price of the climate externality goes down, fossil fuel use goes up with infinite reserves.

Solving (153) we obtain

$$T_0'(t)=\mathrm{e}^{-Bt}\left(T_0'(0)+\int_{s=0}^{t}\mathrm{e}^{Bs}\sum_{k=0}^{3}a_kX_k'(s)\,ds\right)= \tag{157}$$

$$\int_{s=0}^{t}\mathrm{e}^{B(s-t)}\sum_{k=0}^{3}a_kX_k'(s)\,ds>0. \tag{158}$$

Recall that $T_0(t)$ is global average temperature at date t. Hence we should expect global average temperature to increase when more fossil fuels are used. Solving (154) and using $T_0(t)>0$, $\bar{T}_{b2}<0$, we obtain $T_2'(t)>0$ for all $t>0$, or

$$T_2'(t)=\mathrm{e}^{-(B+6D)t}\left(T_2'(0)-6D\int_{s=0}^{t}\mathrm{e}^{(B+6D)s}T_0(s)\,T_{b2}(s)\,ds\right) \tag{159}$$

$$=\int_{s=0}^{t-6D}\mathrm{e}^{(B+6D)(s-t)}T_0(s)\,T_{b2}(s)\,ds>0, \tag{160}$$

since $T_0(s)>0, T_{b2}(s)<0$ for all dates s. Then from the derivative (152) we obtain:

$$T'(x,t) = T_0'(t)+T_2'(t)P_2(x)=T_0'(t)+T_2'(t)\left[\frac{1}{2}\left(3x^2-1\right)\right] \tag{161}$$

$$T'(0,t) = T_0'(t)+T_2'(t)P_2(0)=T_0'(t)-T_2'(t)\left(\frac{1}{2}\right) \tag{162}$$

$$T'(\pm1,t) = T_0'(t)+T_2'(t)P_2(\pm1)=T_0'(t)+T_2'(t)>0, \tag{163}$$

i.e. temperature may fall or even rise at the Equator and rises at the poles. Hence we obtain polar amplification when $\hat{r}$ increases from $\hat{r}=0$ in the case where reserves are infinite at all locations and $-\sum_{k=0}^{3}\lambda_{X_k}^{'}<0$.

If $-\sum_{k=0}^{3}\lambda_{X_k}^{'}>0$, the signs of inequalities are reversed. That is, $E'(t)<0, X_k'(t)<0, T_0'(t)<0, T_2'(t)<0$ and

$$T'(\pm1,t)=T_0'(t)+T_2'(t)P_2(\pm1)=T_0'(t)+T_2'(t)<0. \tag{164}$$

In this case temperature may fall or even rise at the Equator and fall at the poles.□

Proof of Proposition 3:

Recall the optimality condition for the optimal emission path

$$E(x,t) = \frac{\alpha v(x) L(x)}{\lambda_R(x,t) + \xi(t)} \quad , \quad \xi(t) = -\sum_{k=0}^{3} \lambda_{X_k}(t) . \tag{165}$$

Combining this condition with the constraint of finite fossil fuel reserves in each location we obtain

$$\alpha v(x) L(x) \int_{s=0}^{\infty} \left[\frac{1}{\lambda_R(x,0) \mathrm{e}^{\rho t} + \xi(s)} \right] ds = R_0(x) . \tag{166}$$

We evaluate the last integral at $\hat{r} = 0$, where $\xi(s) = \bar{\xi}$ constant and solve for $\lambda_R(x,0)$ to obtain:

$$\lambda_R(x,0) = \bar{\xi} / \left\{ \exp\left[\frac{\rho \bar{\xi} R_0(x)}{\alpha v(x) L(x)} \right] - 1 \right\} , \tag{167}$$

since

$$\int_{s=0}^{\infty} \left[\frac{1}{\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi}} \right] ds = \frac{1}{\rho \bar{\xi}} \left[\ln\left(\frac{\lambda_R(x,0) + \bar{\xi}}{\lambda_R(x,0)} \right) \right] . \tag{168}$$

Differentiating (165) and (166) with respect to $\hat{r}$ we obtain

$$E'(x,t) = -\alpha v(x) L(x) \frac{\lambda_R'(x,0) \mathrm{e}^{\rho s} + \xi'(t)}{\left[\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi} \right]^2} \tag{169}$$

$$- \alpha v(x) L(x) \int_{s=0}^{\infty} \left(\frac{\lambda_R'(x,0) \mathrm{e}^{\rho s} + \xi'(t)}{\left[\lambda_R(x,0) \mathrm{e}^{\rho s} + \bar{\xi} \right]^2} \right) ds = 0. \tag{170}$$

Multiplying the nominator and denominator of the integral in (170) by $\mathrm{e}^{-\rho s}$

and solving for $\lambda_R'(x,0)$, we obtain:

$$\lambda_R'(x,0) = \tag{171}$$

$$\frac{-\int_{s=0}^{\infty}\xi'(s)\left\{1/\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2\right\}ds}{\int_{s=0}^{\infty}\left\{\mathrm{e}^{\rho s}/\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2\right\}ds} = \tag{172}$$

$$\frac{-\bar{\xi}'\int_{s=0}^{\infty}\left\{1/\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2\right\}ds}{\int_{s=0}^{\infty}\left\{\mathrm{e}^{\rho s}/\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2\right\}ds} = \tag{173}$$

$$-\bar{\xi}'\left\{\rho\lambda_R(x,0)\left[\lambda_R(x,0)+\bar{\xi}\right]\right\}\int_{s=0}^{\infty}\left\{\frac{1}{\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2}\right\}ds, \tag{174}$$

since

$$\int_{s=0}^{\infty}\left\{\mathrm{e}^{\rho s}/\left[\lambda_R(x,0)\mathrm{e}^{\rho s}+\bar{\xi}\right]^2\right\}ds = \frac{1}{\rho\lambda_R(x,0)\left[\lambda_R(x,0)+\bar{\xi}\right]}. \tag{175}$$

It follows from (174) that $\lambda_R'(x,0)$ and $\bar{\xi}'$ have opposite signs.□

Proof of Proposition 4

We differentiate the dynamical system (113)-(115) with respect to γ, using the utility function $U(C)=\frac{C^{1-\gamma}}{1-\gamma}$, to obtain:

$$\frac{\partial\dot{\lambda}_{T_0}}{\partial\gamma} = (\rho+B)\frac{\partial\lambda_{T_0}}{\partial\gamma}+\left\langle vL,-\phi\ln\breve{C}\right\rangle \tag{176}$$

$$\frac{\partial\dot{\lambda}_{T_2}}{\partial\gamma} = (\rho+B+6D)\frac{\partial\lambda_{T_2}}{\partial\gamma}+\left\langle vL,-\phi P_2\ln\breve{C}\right\rangle \tag{177}$$

$$\frac{\partial\dot{\lambda}_{X_k}}{\partial\gamma} = (\rho+b_k)\frac{\partial\lambda_{X_k}}{\partial\gamma}-a_k\frac{\partial\lambda_{T_0}}{\partial\gamma}\ ,\ k=0,1,2,3 \tag{178}$$

$$\frac{\partial\dot{\lambda}_R(t)}{\partial\gamma} = \rho\frac{\partial\lambda_R}{\partial\gamma}. \tag{179}$$

The quantity $\ln\breve{C}$ can be computed at $(\hat{r},\gamma)=(0,1)$ as

$$\ln\breve{C}(x,t;0,1) = \ln y(x,t)+\alpha\ln E(x,t;0,1)- \tag{180}$$

$$\phi(x)\left[T_0(t;0,1)+T_{b0}(t)+\left(T_2(t;0,1)+T_{b2}(t)\right)P_2(x)\right]. \tag{181}$$

Note that $T_2(t)=0$ for all dates. It is natural to put $T_{b0}(t)=\bar{T}_{b0}$, $T_{b2}(t)=\bar{T}_{b2}$ at steady state values for all t because the climate system without humans would plausibly be at the steady state. Making the no-serious-poverty assumption at any location at any time, so that $\ln \breve{C}(x,t) > 0$ for all (x,t), we can compute the forward solution for $\frac{\partial \lambda_{T_0}}{\partial \gamma}$ from (176) and use it to obtain $\frac{\partial \lambda_{X_k}}{\partial \gamma}$ from (178). Thus we have

$$\frac{\partial \lambda_{X_k}(t;0,1)}{\partial \gamma} = \int_{s=t}^{\infty} \mathrm{e}^{-\rho(s-t)} a_k \frac{\partial \lambda_{T_0}(s;0,1)}{\partial \gamma} ds > 0 \tag{182}$$

$$\frac{\partial \lambda_{T_0}(v;0,1)}{\partial \gamma} = \int_{s=v}^{\infty} \mathrm{e}^{-\rho(s-v)} \left\langle vL, \phi \ln \breve{C} \right\rangle (s) ds > 0. \tag{183}$$

Therefore

$$\frac{\partial \xi(t;0,1)}{\partial \gamma} = -\sum_{k=0}^{3} \frac{\partial \lambda_{X_k}(t;0,1)}{\partial \gamma} < 0. \tag{184}$$

□

References

1. Alexeev, V.A. and C. H. Jackson, 2012, Polar amplification: is atmospheric heat transport important?, Climate Dynamics, DOI 10.1007/s00382-012-1601-z.

2. Alexeev, V.A., P.L. Langen, and J.R. Bates (2005), Polar amplification of surface warming on an aquaplanet in "ghost forcing" experiments without sea ice feedbacks, Climate Dynamics, DOI 10.1007/s00382-005-0018-3.

3. Boucekkine, R., C. Camacho, and B. Zou, 2009, Bridging the gap between growth theory and the new economic geography: the spatial Ramsey model, Macroeconomic Dynamics 13(1): 20-45.

4. Boucekkine, R., C. Camacho, and G. Fabbri, 2013, Spatial dynamics and convergence: the spatial AK model, Journal of Economic Theory 148(6): 2719-2736.

5. Brock, W., G. Engström, and A. Xepapadeas, 2014, Spatial climate-economic models in the design of optimal climate policies across locations, European Economic Review 69: 78-103.

6. Brock, W.A., G. G. Engström, D. Grass, and A. Xepapadeas, 2013, Energy balance climate models and general equilibrium optimal mitigation policies, Journal of Economic Dynamics and Control 37(12): 2371-2396.

7. Burgess, R., D., Deschenes, D. Donaldson, and M. Greenstone, 2014, The unequal effects of weather and climate change: Evidence from mortality in India, http://www.lse.ac.uk/economics/people/ facultyPersonalPages/facultyFiles/RobinBurgess/UnequalEffectsOfWeather.

8. Cai, Y., K.L. Judd, and T. S. Lontzek, 2015, The Social Cost of Carbon with Economic and Climate Risks, Papers 1504.06909, arXiv.org, revised Apr 2015.

9. Castruccio, S., D. J. McInerney, M. L. Stein, F. Liu Crouch, R. L. Jacob, and E. J. Moyer, 2014, Statistical Emulation of Climate Model Projections Based on Precomputed GCM Runs, Journal of Climate 27(5): 1829-1844.

10. Dell, M., B. Jones, and B. Olken, 2012, Temperature shocks and economic growth: Evidence from the last half century, American Economic Journal: Macroeconomics, 66-95.

11. Desmet K., and E. Rossi-Hansberg, 2010, On Spatial Dynamics, Journal of Regional Science 50(1): 53-63.

12. Desmet, K. and E. Rossi-Hansberg, 2015, On the spatial economic impact of global warming, Journal of Urban Economics 88: 16-27.

13. Francis, J. and S. Vavrus, 2014, Evidence for a wavier jet stream in response to rapid Arctic warming, Environmental Research Letters 10: 1-12.

14. Hasselmann, K. S. R. Giering, V. Ocana, and H. V. Storch, 1997, Sensitivity Study of Optimal CO2 Emission Paths Using a Simplified Structural Integrated Assessment Model (SIAM), Climatic Change 7(2): 345-386.

15. Kummu, M. and O. Varis,O, 2011, The world by latitudes: a global analysis of human population, development level and environment across the North-South axis over the past half century, Applied Geography 31(2): 495-507.

16. Langen P.L. and V. A. Alexeev, 2007, Polar amplification as a preferred response in an idealized aquaplanet GCM, Climate Dynamics, DOI 10.1007/s00382-006-0221-x

17. Lenton, T., H. Held, E. Kriegler, J. Hall, W. Lucht, S. Rahmstorf, and H.J. Schellnhuber, 2008, Tipping elements in the Earth's climate system, PNAS 105(6): 1786-1793.

18. Mendelsohn, R., A. Dinar, and L. Williams, 2006.The distributional impact of climate change on rich and poor countries. Environment and Development Economics 11(2): 159.

19. North, G.R., 1975a, Analytical solution to a simple climate model with diffusive heat transport, Journal of the Atmospheric Sciences 32: 1301-1307.

20. North, G.R., 1975b, Theory of energy-balance climate models, Journal of the Atmospheric Sciences 32: 2033-2043.

21. North, G. R., Cahalan, and J. Coakely, 1981, Energy balance climate models, Reviews of Geophysics and Space Physics 19(1): 91-121.

22. Pierrehumbert, R.T., 2014, Short-Lived Climate Pollution, Annual Review of Earth and Planetary Sciences 42: 341-379.

23. Wu, W. and G.R. North, 2007, Thermal decay modes of a 2-D energy balance climate model, Tellus A 59(5): 618–26.