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SPATIAL RESOURCE MANAGEMENT UNDER POLLUTION EXTERNALITIES

ANASTASIOS XEPAPADEAS

ATHANASIOS YANNACOPOULOS

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Spatial Resource Management under Pollution Externalities*

A. Xepapadeas† and A. N. Yannacopoulos‡

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Abstract

Variables describing the state of an environmental system such as resources (renewable or exhaustible), pollutants, greenhouse gases have a profound spatial dimension. This is because resources or pollutants are harvested, extracted, emitted, or abated in a specific location or locations, the impacts of environmental variables, whether beneficial or detrimental, have a strong spatial dimension, and there is transport of environmental state variables across geographical space due to natural processes. In this paper we study dynamic optimization for the joint management of resources and pollution when pollution affects resource growth and when spatial transport phenomena both for the resources and the pollution are present. We present approaches that deal with dynamic optimization in infinite dimensional spaces which can be used as tools in environmental and resource economic. We also present methods which can be used to study the emergence of spatial patterns in dynamic optimizations models. Our methods draw on the celebrated Turing diffusion induced instability but are different from Turing's mechanism since they apply to forward-optimization models. We believe that this approach provides the tools to analyze a wide range of problems with explicit spatial structure which are very often encountered in environmental and resource economics.

Keywords: Spatial transport, renewable resource, pollution, optimization, infinite dimensional spaces, Tiring instability, patter formation, policy design.

JEL Classification: C61, Q20, Q52

1 Introduction

Resource management - renewable or exhaustible resources - is usually analyzed in terms of dynamic models where the resource stock is a state variable that evolves in time, and harvesting or extraction per unit time is a control variable. The evolution of the state variables under the influence of resource growth functions and harvesting or extraction is modeled in general by dynamical systems consisting of nonlinear ordinary differential equations (ODEs).

Pollution management problems are dynamic when pollution has stock and not flow characteristics (e.g. accumulation of phosphorus in a lake that may cause eutrophication, accumulation of airborne particles and pollutants from combustion creating "brown clouds"). The system's evolution is described in this case by dynamical systems of ODEs.

However, variables describing the state of an environmental system such as resources (renewable or exhaustible), pollutants, greenhouse gases (GHGs), heat, and precipitation have a profound spatial dimension in addition to their temporal dimension. This is because:

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†Athens University of Economics and Business, Department of International and European Economic Studies, xepapad@aueb.gr

‡Athens University of Economics and Business, Department of Statistics, ayannaco@aueb.gr

(i) Resources or pollutants are harvested, extracted, emitted, or abated in a specific location or locations.

(ii) The impacts of environmental variables, whether beneficial or detrimental, have a strong spatial dimension. Polar amplification suggests that the temperature increases faster in the Poles than the Equator because of heat transfer polarwards. Polar amplification is an established natural phenomenon since the recorded temperature anomaly is consistently higher at the Poles that the Equator and results in the loss of sea ice and land ice. Loss of sea ice could be beneficial because of the potential opening of new shipping lanes and the potential opening of access to previously inaccessible natural resources and fossil fuel reserves Loss of land ice may cause damages to lower latitudes because of sea level rise. Thus there is strong spatial dimension in climate change policies. The Atmospheric Brown Clouds (ABC) can be regarded as reflecting the spatial structure of air pollution. As stated in recent a UNEP study (Ramanathan et al. (2008)), ABC consist of particles (or primary aerosols) and pollutant gases, such as nitrogen oxides (NOx), carbon monoxide (CO), sulphur dioxide (SO2), ammonia (NH3), and hundreds of organic gases and acids. ABC plumes which result from the combustion of biofuels from indoors; biomass burning outdoors and fossil fuels, are found in all densely inhabited regions and oceanic regions downwind of populated continents.[1] Fisheries crashes have a profound spatial dimension e.g. the Peruvian coastal anchovies fisheries crash on the 1970s, or the collapse of the Atlantic northwest cod fishery off the cost of Newfoundland in early 1990s.

(iii) There is transport of environmental state variables across geographical space due to natural processes.

   - Energy balance climate models (EBCMs) explicitly account for the transport of heat across the globe from the Equator to the Poles (e.g. North et al. (1981))
   - There is horizontal heat and moisture transport across the globe in more general EBCMs.
   - Air-borne contaminants are transported in the atmosphere from the source of emissions due to turbulent eddy motion and wind.
   - Renewable resources move in a given spatial domain.

When forward-looking optimizing economics agents that take decisions regarding resource management or emissions ignore transport effects, they essentially ignore the impact of their own actions on the utility or profits of agents located at different sites. This is a spatial externality, which is not internalized. Therefore efficient policy seeking to maximize social welfare should involve mechanisms to internalize spatial spillovers, along with potential temporal spillovers.

It should be noted that although the spatial dimension is important in resource management not much research in the spatial aspect of environmental and resource economics has been undertaken, although there are notable exceptions in several cases such as:

- Spatially dependent taxes (e.g. Xabadia et al. (2004), Goetz and Zilberman (2007))
- Spatial resource models and spatial fishery models (e.g. Wilen (2007), Smith et al. (2009), Desmet and Rossi-Hansberg (2010), Brock et al. (2014b), Brock et al. (2014a), Behringer and Upmann (2014), Camacho and Pérez-Barahona (2015))
- Spatial models of climate and the economy (e.g. Brock et al. (2013), Brock et al. (2014c), Hassler and Krusell (2012), Desmet and Rossi-Hansberg (2015))

[1] Five regional ABC hotspots around the world have been identified: i) East Asia, ii) Indo-Gangetic Plain in South Asia, iii) Southeast Asia, iv) Southern Africa; and v) the Amazon Basin.

The lack of substantial literature incorporating spatial issues in environmental and resource economics can be attributed to the technical difficulties involved when the mathematics of optimal control theory is extended to infinite dimensional state spaces that naturally emerge when optimization takes place in a spatiotemporal domains. Exceptions try to overcome the mathematical complication by imposing a certain structure to the problem that allows simplifications and sometimes closed form solutions. However, the importance of transport phenomena in environmental and resource economics, and the need to design regulation for internalizing spatial externalities emerging from these transport phenomena, make it necessary to extend dynamic optimization methods into spatial settings.

In this context we study dynamic optimization for the joint management of resources and pollution when pollution affects resource growth and when spatial transport phenomena both for the resources and the pollution are present. We present approaches that deal with dynamic optimization in infinite dimensional spaces which can be used as tools in environmental and resource economics, along with examples of their application. We also present methods which can be used to study the emergence of spatial patterns in dynamic optimizations models.

Our methods draw on the celebrated Turing diffusion induced instability but are different from Turing's mechanism since they apply to forward-optimization models. We believe that this approach provides the tools to analyze a wide range of problems with explicit spatial structure which are very often encountered in environmental and resource economics.

2 The model

Consider a two sector economy, consisting of an industrial sector generating emissions (pollution) and a harvesting sector. The harvesting sector specializes in a single species. This is modelled by a density function v whose spatiotemporal evolution is given by the partial differential equation (PDE)

$$\frac{\partial}{\partial t} v = f(v, K) + d_1 \Delta v - H,$$

where f is a nonlinear function modelling the dynamics of the species, K is the carrying capacity of the environment and H is a harvesting function. A typical example for the function f is the logistic function $f(v, K) = a\, v\, \left(1 - \frac{v}{K}\right)$. The term $d_1 \Delta n$ models the spatial transport of the species in terms of a diffusion mechanism, whereas h is a harvesting term. Importantly, the carrying capacity of the environment is not considered to be a constant but rather a varying quantity modelled by a function $K : D \times \mathbb{R}_+ \to \mathbb{R}_+$. Depending on the level of complexity of the model we may discard the spatial dependence on K and leave only the temporal dependence.

The industrial sector of the economy interacts with the harvesting sector via the carrying capacity K. The underlying idea is that the industry sector generates pollution externalities through emissions which have a negative effect on the harvesting sector by decreasing the carrying capacity K thus reducing the population and having a negative overall effect on the harvesting.

Emissions are generated by agents at different locations $x \in D$, and by $s(t, x)$ we denote the emissions at time t and location x. Emissions generate benefits locally through a benefit function B, in particular, $B(x, s(t, x))$ is the benefit generated from emissions at point (t, x). The benefit function is assumed to be increasing and strictly concave.

The emissions aggregate and create an emissions stock S. The emissions stock displays spatiotemporal dynamics of the form

$$\frac{\partial}{\partial t} S = d_2 \Delta S - \delta S + s,$$

where $d_2 \Delta s$ models the spatial transport of the emissions and the term $-\delta s$ models the natural tendency of the environment to absorb emissions and restore to its initial state.

The carrying capacity is negatively affected by the emissions stock S. This effect depends on some weighted spatial average of the total stock i.e. is assumed to depend on

$$(\mathcal{T}S)(t) = \int_D k(x')S(t,x')dx',$$

where k is a kernel function modelling the effect of total emissions stock on the carrying capacity. The effect of the total emissions stock on the carrying capacity is then modelled by

$$K(t) = K((\mathcal{T}S)(t)) = K(\int_D k(x')S(t,x')dx'),$$

where $K : \mathbb{R} \to \mathbb{R}$ is a decreasing function which is positive and bounded away from zero.

Furthermore, harvesting must necessarily be a fraction of the total population, so it is more natural to express $H = hv$ where $h : \mathbb{R}_+ \times D \to [0,1]$ and $h(t,x)$ is the fraction of the population which is harvested at location x at time t.

Then we get a coupled system of the form

$$\begin{aligned} \frac{\partial}{\partial t}v &= f(v, K(\mathcal{T}(S)) + d_1\Delta v - hv, \\ \frac{\partial}{\partial t}S &= d_2\Delta S - \delta S + s, \end{aligned} \tag{1}$$

where e.g. $f(v,S) = \alpha v\left(1 - \frac{v}{K(\mathcal{T})}\right)$. Our state equations is a coupled nonlocal reaction diffusion system. The variables (v,S) are the state variables and the variables (h,s) are the control variables.

The decision maker (social planner) solves the following problem. She wishes to maximize the total profit from both sectors, the harvesting sector and the industrial sector. The profit from the harvesting sector depends on h and is a concave function of hv. The profit from the industrial sector is a concave function of emissions s and there is also a damage function Z from the total emissions stock. We may consider therefore that the decision maker, who can be regarded as a regulator managing the coupled system, chooses (h,s) so as to maximize

$$J(h,s) = \int_0^\infty \int_D e^{-rt}\left(U(h\,v) + B(s) - Z(S)\right)dxdt,$$

under the dynamic constraints

$$\begin{aligned} \frac{\partial}{\partial t}v &= f(v,\mathcal{T}) + d_1\Delta v - hv, \\ \frac{\partial}{\partial t}S &= d_2\Delta S - \delta S + s, \end{aligned} \tag{2}$$

subject to homogeneous Neumann boundary conditions.

The control variable h is constrained to take values in $[0,1]$ and the control variable s in constrained to take values in $[0,\bar{s}]$ where $\bar{s}$ is the maximum possible pollution rate allowed by the production capacity of the economy. We assume that the control variables (h,s) lie in the convex subset of $H := L^2(0,\infty;D) \times L^2(0,\infty;D)$,

$$C = \{(h,s) \in H,\;\; h(t,x) \in [0,1],\;\; s(t,x) \in [0,\bar{s}],\;\; (t,x)\, a.e.\, (0,\infty) \times D\}.$$

Proposition 2.1. *The state equation* (2) *is well posed for any choice of control procedure* $(h,s) \in K$*.*

3 A necessary condition in terms of the Pontryagin maximum principle

In this section, assuming the existence of a solution to this optimal control system, we derive a necessary condition that allows for the identification of the optimal path and the optimal control procedure.

In order to express the maximum principle we need to define the adjoint variables (p, q), which are solutions of the backward system

$$
\begin{aligned}
\frac{\partial p}{\partial t} &= -d_1 \Delta p + (h - f_v)p + rp + U'(hv)h, \\
\frac{\partial q}{\partial t} &= -d_2 \Delta q - \mathcal{T}^*(f_K K'(\mathcal{T} S)p) + \delta q + rq - Z'(S),
\end{aligned}
\tag{3}
$$

where $\mathcal{T}^*$ is the adjoint operator of $\mathcal{T}$.

This system is solved for homogeneous Neumann boundary conditions and with a transversality condition of the form

$$\lim_{t \to \infty} e^{-rt} \int_D (v(t,x)p(t,x) + S(t,x)q(t,x))dx = 0, \quad a.e.\, t \in (0, \infty).$$

An equivalent way of expressing the adjoint system is in terms of the adjoint variables $(\mathfrak{p}, \mathfrak{q}) = (-p, -q)$. This leads us to the equivalent backward system,

$$
\begin{aligned}
\frac{\partial \mathfrak{p}}{\partial t} &= -d_1 \Delta \mathfrak{p} + (h - f_v)\mathfrak{p} + r\mathfrak{p} - U'(hv)h, \\
\frac{\partial \mathfrak{q}}{\partial t} &= -d_2 \Delta \mathfrak{q} - \mathcal{T}^*(f_K K'(\mathcal{T} S)\mathfrak{p}) + \delta \mathfrak{q} + r\mathfrak{q} + Z'(S),
\end{aligned}
\tag{4}
$$

where we used the linearity of the adjoint system in (p, q). The transversality condition remains the same, but now is expressed in terms of $(\mathfrak{p}, \mathfrak{q})$. The same applies for the boundary conditions. The state equation is unchanged.

The following proposition provides the maximum principle.

Proposition 3.1. *Let (v, S) be the optimal path and (h, s) be the optimal control protocol. Then, there exists a pair of processes (p, q) which along with the quadruple (v, S, h, s) satisfy the set of forward-backward PDEs* (2)-(3) *along with the maximality condition*

$$\int_D [(U'(hv) + p)v\eta + (B'(s) - q)\sigma]dx \leq 0, \quad \forall\ (\eta, \sigma) \in C_0, \tag{5}$$

where C_0 is the relative interior of C.

An equivalent form of the maximum principle is that there exists a pair of processes $(\mathfrak{p}, \mathfrak{q})$ which along with the quadruple (v, S, h, s) satisfy the set of forward-backward PDEs (2)-(4) *along with the maximality condition*

$$\int_D [(U'(hv) - \mathfrak{p})v\eta + (B'(s) + \mathfrak{q})\sigma]dx \leq 0, \quad \forall\ (\eta, \sigma) \in C_0, \tag{6}$$

Proof. See Appendix A. □

Note that condition (5) can be interpreted as a condition for static maximization of the Hamiltonian

$$H(h, s) = \int_D \{(U(hv) + B(s) - D(S)) + p(d_1 \Delta v + f(v, K(\mathcal{T} S)) - hv) + q(d_2 \Delta S - \delta S + s)\}dx,$$

over $(h,s) \in C$, where v, S, p, q are considered as given functions (of space and time) and treated in the above optimization problem as fixed functional parameters. In this sense condition (5) can be written in equivalent form as

$$DH(h,s) \in N_C(h,s),$$

where by $N_C(h,s)$ we denote the normal cone of the convex set C at point (h,s). By the structure of the set C it can be seen that the normal cone $N_C(h,s)$ is characterized as follows

$$\begin{aligned} D_h U(hv) + vp > 0, &\quad \text{if } h = 0, \\ D_h U(hv) + vp = 0, &\quad \text{if } h \in (0,1), \\ D_h U(hv) + vp < 0, &\quad \text{if } h = 1, \end{aligned}$$

and

$$\begin{aligned} D_s B(s) + q > 0, &\quad \text{if } s = 0, \\ D_s B(s) + q = 0, &\quad \text{if } s \in (0,\bar{s}), \\ D_s B(s) + q < 0, &\quad \text{if } s = \bar{s}, \end{aligned}$$

where all the above are considered to hold a.e. in $(0,\infty)\times D$, i..e., $D_h U(h(t,x)v(t,x)) + v(t,x)p(t,x) > 0$ if $h(t,x) = 0$ and similarly for the other conditions.

To simplify the exposition a little, assume that the functions U and B are such that the maximum is attained for $h \in (0,1)$ and $s \in (0,\bar{s})$ respectively. A condition that will certainly guarantee that $h > 0$ and $s > 0$ is to assume that $U'(z)$ and $B'(z)$ tend to $+\infty$ as $z \to 0^+$. This condition holds for e.g. logarithmic function. A similar condition can be imposed on the other end of the interval.

Under this assumption, and defining by I_1 the inverse function of U' and I_2 the inverse function of B', we can express the maximality condition as

$$\begin{aligned} U'(hv) - p &= 0, \\ B'(s) + q &= 0, \end{aligned}$$

which yields,

$$\begin{aligned} hv &= I_1(p), \\ s &= I_2(-q). \end{aligned}$$

This allows to express the necessary condition in terms of (v,S,p,q) only, and characterize the optimal path and the optimal control procedure in terms of the solution of the forward-backward PDE system

$$\begin{aligned} \frac{\partial v}{\partial t} &= d_1 \Delta v + f(v, K(\mathcal{T}(S)) - I_1(-p), \\ \frac{\partial S}{\partial t} &= d_2 \Delta S - \delta S + I_2(q), \\ \frac{\partial p}{\partial t} &= -d_1 \Delta p + (r - f_v(v, K(\mathcal{T}(S)))p, \\ \frac{\partial q}{\partial t} &= -d_2 \Delta q - \mathcal{T}^*(f_K(v, K(\mathcal{T}(S))K'(\mathcal{T}(S))p) + (r+\delta)q - Z'(S), \end{aligned} \tag{7}$$

with homogeneous Neumann boundary conditions for all the variables, initial conditions for the state variables (v,S) and the transversality condition for the adjoint variables (p,q).

In terms of the adjoint variables $(\mathfrak{p}, \mathfrak{q})$, the Hamiltonian is transformed to

$$\mathfrak{H}(h,s) = \int_D \{(U(hv) + B(s) - D(S)) + \mathfrak{p}(d_1 \Delta v + f(v, K(\mathcal{T}S)) - hv) + \mathfrak{q}(d_2 \Delta S - \delta S + s)\} dx,$$

and in view of the discussion above, the connection between the adjoint variables $(\mathfrak{p}, \mathfrak{q})$ and the optimal control protocol becomes,

$$\begin{aligned} U'(hv) - \mathfrak{p} &= 0, \\ B'(s) + \mathfrak{q} &= 0, \end{aligned}$$

which yields,

$$\begin{aligned} hv &= I_1(\mathfrak{p}), \\ s &= I_2(-\mathfrak{q}). \end{aligned}$$

This allows us to understand $\mathfrak{p}$ as a price for the resource and $\mathfrak{q}$ as a cost for pollution. The necessary condition when expressed in terms of $(v, S, \mathfrak{p}, \mathfrak{q})$ becomes

$$\begin{aligned} \frac{\partial v}{\partial t} &= d_1 \Delta v + f(v, K(\mathcal{T}(S)) - I_1(\mathfrak{p}), \\ \frac{\partial S}{\partial t} &= d_2 \Delta S - \delta S + I_2(-\mathfrak{q}), \\ \frac{\partial \mathfrak{p}}{\partial t} &= -d_1 \Delta \mathfrak{p} + (r - f_v(v, K(\mathcal{T}(S)))\mathfrak{p}, \\ \frac{\partial \mathfrak{q}}{\partial t} &= -d_2 \Delta \mathfrak{q} - \mathcal{T}^*(f_K(v, K(\mathcal{T}(S))K'(\mathcal{T}(S))\mathfrak{p}) + (r+\delta)\mathfrak{q} + Z'(S), \end{aligned} \tag{8}$$

with homogeneous Neumann boundary conditions for all the variables, initial conditions for the state variables (v, S) and the transversality condition for the adjoint variables $(\mathfrak{p}, \mathfrak{q})$.

The equation (7) (or its equivalent version (8)) is a system of nonlinear differential integral equations (non-local PDEs). We will write them in explicit form in order to make the effect of the kernel k, more transparent. In order to do so, we recall the nature of the actions of the averaging operator $\mathcal{T}$ and its adjoint $\mathcal{T}^*$ on any function $u : [0, \infty) \times D \to \mathbb{R}$ (see Appendix A):

$$\begin{aligned} (\mathcal{T}u)(t,x) &= \int_D k(x')u(t,x')dx', \\ (\mathcal{T}^*u)(t,x) &= k(x) \int_D u(t,x')dx'. \end{aligned}$$

Note that while the action of $\mathcal{T}$ on any spatiotemporally varying function u will deliver a function $U := \mathcal{T}u$ which depends **only** on t, the action of $\mathcal{T}^*$ will in general deliver a function $U^* := \mathcal{T}^*u$ that depends on **both** time and space! In the special case where the averaging kernel $k = k_0$ is a constant, the action of $\mathcal{T}^*$ will deliver a function which depend **only** on time. This physically corresponds to the case where the effect of pollution on the carrying capacity of the species is obtained by an unweighted average of the pollution stock over the whole spatial domain D (i.e. all regions contribute equally on the effect of pollution on the carrying capacity).

In view of the above observation we may write (7) in a more explicit form. We will use the shorthand notation

$$S_a(t) := \mathcal{T}(S) = \int_D k(x')S(t,x')dx'$$

and

$$K_a(t) := K(\mathcal{T}(S)) = K(S_a(t)),$$
$$K_a'(t) := K'(\mathcal{T}(S)) = K'(S_a(t)).$$

Using this notation and the definition of the operator $\mathcal{T}^*$ we can express the non-local term in the last equation of (7) as

$$\begin{aligned}\left(\mathcal{T}^*(f_K(v, K(\mathcal{T}(S))K'(\mathcal{T}(S))p)\right)(t,x) &= k(x)\int_D \left(f_K(v(t,x'), K_a(t))K_a'(t)p(t,x')\right)dx' \\ &= k(x)K_a'(t)\int_D \left(f_K(v(t,x'), K_a(t))p(t,x')\right)dx'\end{aligned}$$

We can therefore express (7) in the more explicit form

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) &= d_1\Delta v(t,x) + f(v(t,x), K_a(t)) - I_1(-p(t,x)),\\ \frac{\partial S}{\partial t}(t,x) &= d_2\Delta S(t,x) - \delta S(t,x) + I_2(q(t,x)),\\ \frac{\partial p}{\partial t}(t,x) &= -d_1\Delta p(t,x) + (r - f_v(v(t,x), K_a(t)))p(t,x),\\ \frac{\partial q}{\partial t}(t,x) &= -d_2\Delta q(t,x)' + (r+\delta)q(t,x) - Z'(S(t,x))\\ &\quad - k(x)K_a'(t)\int_D \left(f_K(v(t,x'), K_a(t))p(t,x')\right)dx,\end{aligned} \tag{9}$$

where the terms $K_a(t)$ and $K_a'(t)$ have a non-local dependence on S via

$$\begin{aligned}K_a(t) &= K\left(\int_D k(x')S(t,x')dx'\right),\\ K_a'(t) &= K'\left(\int_D k(x')S(t,x')dx'\right).\end{aligned} \tag{10}$$

The equation (7) is therefore expressed in explicit nonlinear nonlocal PDE form in terms of the system (9)-(10). For most of the remaining part of the paper, we will favour the more compact formulation (7) which uses the operator notation.

Similarly, we can express (8) in the more explicit form

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) &= d_1\Delta v(t,x) + f(v(t,x), K_a(t)) - I_1(\mathfrak{p}(t,x)),\\ \frac{\partial S}{\partial t}(t,x) &= d_2\Delta S(t,x) - \delta S(t,x) + I_2(-\mathfrak{q}(t,x)),\\ \frac{\partial \mathfrak{p}}{\partial t}(t,x) &= -d_1\Delta \mathfrak{p}(t,x) + (r - f_v(v(t,x), K_a(t)))\mathfrak{p}(t,x),\\ \frac{\partial \mathfrak{q}}{\partial t}(t,x) &= -d_2\Delta \mathfrak{q}(t,x)' + (r+\delta)\mathfrak{q}(t,x) + Z'(S(t,x))\\ &\quad - k(x)K_a'(t)\int_D \left(f_K(v(t,x'), K_a(t))\mathfrak{p}(t,x')\right)dx,\end{aligned} \tag{11}$$

where the terms $K_a(t)$ and $K_a'(t)$ have a non-local dependence on S via

$$\begin{aligned}K_a(t) &= K\left(\int_D k(x')S(t,x')dx'\right),\\ K_a'(t) &= K'\left(\int_D k(x')S(t,x')dx'\right).\end{aligned} \tag{12}$$

The equation (8) is therefore expressed in explicit nonlinear nonlocal PDE form in terms of the system (11)-(12). For most of the remaining part of the paper, we will favour the more compact formulation 8 which uses the operator notation.

It is highly unlikely that the system (7) (or the equivalent form (9)-(10)) – respectively (8) (or the equivalent form (11)-(12)) – can be solved analytically, in closed form and we need to resort to numerical techniques. However, there is still a lot that can be said from the analysis point of view, such as for instance the existence of solutions, uniqueness issues as well as important information on the qualitative nature of the solutions. Such issues will be the main concern of the remaining part of the paper.

4 The steady state solution and its properties

We are now interested in special solutions of the optimally controlled system, for which there is no time dependence, which will be here after called the steady state solution. These can be understood as fixed points of the forward-backward infinite dimensional dynamical system which is presented in (7), and is the infinite dimensional analogue of a saddle point. Such a saddle point however, in principle will have still a spatial structure, i.e., it will be a function of space. A steady state will be the solution of the system of non-local partial differential equations (integro-differential equations)

$$
\begin{aligned}
0 &= d_1\Delta v + f(v, K(\mathcal{T}(S)) - I_1(-p),\\
0 &= d_2\Delta S - \delta S + I_2(q),\\
0 &= -d_1\Delta p + (r - f_v(v, K(\mathcal{T}(S)))p,\\
0 &= -d_2\Delta q - \mathcal{T}^*(f_K(v, K(\mathcal{T}(S))K'(\mathcal{T}(S))p) + (r+\delta)q - Z'(S),
\end{aligned}
\tag{13}
$$

with homogeneous Neumann boundary conditions, where the actions of the operators $\mathcal{T}$ and $\mathcal{T}^*$ on any function u (see Appendix A) are defined as

$$
\begin{aligned}
(\mathcal{T}u)(x) &= \int_D k(x')u(x')dx',\\
(\mathcal{T}^*u)(x) &= k(x)\int_D u(x')dx'.
\end{aligned}
$$

Note that in general, when $\mathcal{T}$ acts on any function it delivers a constant, whereas when $\mathcal{T}^*$ acts on any function it delivers a constant multiple of the spatially varying function k which is the averaging kernel. In the special case where $k = k_0$, a constant, both $\mathcal{T}$ and $\mathcal{T}^*$ deliver a constant when acting on any function. This special case, corresponds to the case where the effect of pollution on the carrying capacity of the species is obtained by an unweighted average of the pollution stock over the whole spatial domain D (i.e. all regions contribute equally on the effect of pollution on the carrying capacity).

In view of the above observation we may write (13) in a more explicit form. If the solution of (7) is not time dependent, then the functions K_a and K'_a will no longer depend on time and will be constants denoted by K_c and K'_c respectively, depending of course on the spatial distribution of S.

We can therefore express (13) in the more explicit form

$$
\begin{aligned}
0 &= d_1\Delta v(x) + f(v(x), K_c) - I_1(-p(x)),\\
0 &= d_2\Delta S(x) - \delta S(x) + I_2(q(x)),\\
0 &= -d_1\Delta p(x) + (r - f_v(v(x), K_c))p(x),\\
0 &= -d_2\Delta q(x) + (r+\delta)q(x) - Z'(S(x))\\
&\qquad - k(x)K'_c\int_D \left(f_K(v(x'), K_c)p(x')\right)dx',
\end{aligned}
\tag{14}
$$

where the terms K_c and K_c' have a non-local dependence on S via

$$K_c = K\left(\int_D k(x')S(x')dx'\right),$$
$$K_c' = K'\left(\int_D k(x')S(x')dx'\right). \tag{15}$$

A steady state solution is then a quadruple of functions $z := (v, S, p, q) : D \to \mathbb{R}^4$ which is a solution of the system of nonlinear and nonlocal elliptic PDEs (13) (or equivalently (14)-(15)). Considering z as a point in the function space $\mathbb{X}$ in which $(v, S, p, q) : D \to \mathbb{R}^4$ is properly defined (typically $\mathbb{X} = L^2(D) \times L^2(D) \times L^2(D) \times L^2(D)$). Note that unlike the case where diffusion is not present, now the steady optimal solution (the analogue of the saddle point in the temporal system) carries a spatial dependence. It is a point, when treated as an element of the infinite dimensional space $\mathbb{X}$, which is a function space.

The solvability of the steady state system is by no means a trivial task but it can be shown that under certain conditions, the system (13) (or equivalently (14)-(15)) admits a solution. We turn our attention to qualitative properties of this system.

A particularly interesting steady state is the so called flat steady state, which corresponds to a solution of (7) which is spatially independent.

We first address the question of when such a solution may exist. We claim that such a solution exists when the averaging kernel k is constant $k = k_0$ for all $x \in D$. If $k = k(x)$, i.e. if the operator $\mathcal{T}$ corresponds to a weighted averaging then we claim that a flat steady state of (7) cannot exist.

Proposition 4.1 (Spatial variability of the steady state optimal solution)**.**

(i) If the averaging kernel k is not a constant function (weighted averaging) then a flat steady state solution of (7) *does not exist. This implies that the candidate for the optimal solution necessarily presents spatial variability.*

(ii) If the averaging kernel is a constant function $k = k_0$ (unweighted averaging corresponding to $\mathcal{T}^*S = k_0 \int_D S(x')dx'$), a flat optimal steady state exists, and the corresponding optimal policy which supports it is prescribed by the solution of the system of equations

$$\bar{h} = \frac{f(\bar{v}, \bar{K})}{\bar{v}},$$
$$\bar{s} = \delta\bar{S} \tag{16}$$

while it must hold that

$$f_v(\bar{v}, \bar{K}) = r,$$
$$0 = -M f_K(\bar{v}, \bar{K})\bar{K}'\bar{p} + (\delta + r)\bar{q} - Z'(\bar{S}). \tag{17}$$

where $\bar{K} = K(M\,\bar{S})$, $\bar{K}' = K'(M\bar{S})$, and $M = \int_D k(x')dx' = \mathcal{L}(D)k_0$ and $\mathcal{L}(D)$ is the Lebesgue measure of D.[2]

Proof. The proof is given in Appendix C □

The following remark is useful: The second system (17) specifies $\bar{v}$ and $\bar{K}$, while the first system (16) gives the optimal policy (considered as an asymptotic optimal policy rule). The above rule is a sensible suggestion, since it requires that the pollution rate should equal the natural rate by which the pollution stock decays, where as the rate of regeneration of the natural resource $f_v(\bar{v}, \bar{K})$ should equal the discount rate r. This naturally holds for specific values of $\bar{v}$ and $\bar{K}$ (hence $\bar{S}$).

[2] $\mathcal{L}(D)$ is the length of D if $D \subset \mathbb{R}$, the area of D if $D \subset \mathbb{R}^2$ and the volume of D if $D \subset \mathbb{R}^3$.

Example 4.2 (The logistic growth model)**.** In order to provide a concrete example of this flat steady state solution, consider the case where the function f corresponds to logistic growth,

$$f(v,K) = \alpha v \left(1 - \frac{v}{K}\right),$$

and the carrying capacity K depends on the global pollution stock in terms of an exponential decay law as

$$K = K_m \exp\left(-\beta\, \mathcal{T}(S)\right), \quad \beta > 0.$$

In the above, K_m is the maximal carrying capacity, which decreases as the pollution stock increases ($\mathcal{T}$ is a positive operator).

In this case we see that

$$\bar{v} = \frac{\alpha - r}{2\alpha}\bar{K},$$
$$\bar{h} = \frac{\alpha + r}{2},$$

where

$$\bar{K} = K_m \exp(-\beta M \bar{S}),$$

is still to be determined.

We furthermore assume that the utility, the benefit and the damage function are all logarithmic and of the form

$$U(y) = \lambda_1 \ln(y), \;\; B(y) = \lambda_2 \ln(y), \;\; Z(y) = \lambda_3 \ln(y), \;\; \lambda_i > 0.$$

With this choice we see that

$$\bar{\mathfrak{p}} = -\bar{p} = \frac{4\alpha\lambda_1}{\bar{K}(\alpha^2 - r^2)},$$
$$\bar{\mathfrak{q}} = -\bar{q} = -\frac{\lambda_2}{\delta \bar{S}}$$

Finally, the last equation of (17) yields,

$$\bar{S} = \frac{(\lambda_2 - \lambda_3)\delta + r\lambda_2}{\lambda_1 \beta M}\frac{\alpha + r}{\alpha - r}.$$

This fully determines the flat optimal steady state.

5 Qualitative behavior near a steady state

We now consider a solution $\bar{z} := (\bar{v}, \bar{S}, \bar{p}, \bar{q})$ of the time independent problem (13). This a time independent optimal solution **not necessarily** spatially independent (an infinite dimensional analogue of a saddle point). The stable and the unstable manifolds of this saddle point can be obtained through linearization of the full time dependent problem (7).

Consider a solution of the full time dependent system (7) of the form $Z = \bar{z} + \epsilon z$ for some small ϵ, where $z = (v, S, p, q)$ denote the deviation from the time independent optimal solution $\bar{z} = (\bar{v}, \bar{S}, \bar{p}, \bar{q})$. The spatiotemporal evolution of $z = (v, S, p, q)$ is approximated for small enough ϵ by the linearized version of (7) which after some tedious algebra can be shown to admit the following form

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= d_1\Delta v + f_v(\bar z)v + f_K(\bar z)K'(\bar z)\mathcal{T}S - I_1'(-\bar p)p,\\
\frac{\partial S}{\partial t} &= d_2\Delta S - \delta S + I_2'(\bar q)q,\\
\frac{\partial p}{\partial t} &= -d_1\Delta p - \bar p f_{vv}(\bar z)v + (r - f_v(\bar z))p - \bar p f_{vK}(\bar z)K'(\bar z)\mathcal{T}S,\\
\frac{\partial q}{\partial t} &= -d_2\Delta q - Z''(\bar S)s + (\delta + r)q - \mathcal{T}^*(\bar p K'(\bar z)f_{Kv}(\bar z)v)\\
&\quad - \mathcal{T}^*(f_K(\bar z)K'(\bar z)p) - \mathcal{T}^*([\bar p f_{KK}(\bar z)(K'(\bar z))^2 + \bar p f_K(\bar z)K''(\bar z)]\mathcal{T}(S)).
\end{aligned}
\tag{18}
$$

We remark that $K(\bar z)$, $K'(\bar z)$ and $K''(\bar z)$ are shorthands for the constants

$$
\begin{aligned}
K(\bar z) &:= K\left(\int_D k(x')\bar S(x')dx'\right) = K_c,\\
K'(\bar z) &:= K'\left(\int_D k(x')\bar S(x')dx'\right) = K_c',\\
K''(\bar z) &:= K''\left(\int_D k(x')\bar S(x')dx'\right) =: K_c'',
\end{aligned}
$$

while $f_v(\bar z)$, $f_K(\bar z)$, $f_{vv}(\bar z)$, $f_{vK}(\bar z)$, $f_{KK}(\bar z)$ are shorthands for

$$
f_v(\bar z) = f_v\left(\bar v(x), K\left(\int_D k(x')\bar S(x')dx'\right)\right) = f_v(\bar v(x), K_c),
$$

and similarly for all the other partial derivatives of f calculated at $\bar z$.

Note that the linearized system (18) consists of a local and nonlocal part and can be expressed in compact form as

$$
z' = Az + Lz + Nz, \tag{19}
$$

where $z = (v, S, p, q)^{tr}$ and

$$
A = \begin{pmatrix} d_1\Delta & 0 & 0 & 0\\ 0 & d_2\Delta & 0 & 0\\ 0 & 0 & -d_1\Delta & 0\\ 0 & 0 & 0 & -d_2\Delta \end{pmatrix}
$$

$$
L = \begin{pmatrix} f_v(\bar z) & 0 & -I_1'(-\bar p) & 0\\ 0 & -\delta & 0 & -I_2'(\bar q)\\ -\bar p f_{vv}(\bar z) & 0 & r - f_v(\bar z) & 0\\ 0 & -Z''(\bar S) & 0 & \delta + r \end{pmatrix}
$$

$$
N = \begin{pmatrix} 0 & f_K(\bar z)K'(\bar z)\mathcal{T} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & -\bar p f_{vK}(\bar z)K'(\bar z)\mathcal{T} & 0 & 0\\ -\mathcal{T}^*\left(\bar p K'(\bar z)f_{Kv}(\bar z)I\right) & -\mathcal{T}^*\left(\bar p\left[f_{KK}(\bar z)(K'(\bar z))^2 + f_K(\bar z)K''(\bar z)\right]\mathcal{T}\right) & -\mathcal{T}^*\left(f_K(\bar z)K'(\bar z)I\right) & 0 \end{pmatrix}
$$

This is a linear system consisting of a local part $A + L$ and a nonlocal part L.

If $\bar z$ is a **spatially dependent** solution of the time independent problem (7) then the operator $A+L$ is a local operator with spatially dependent coefficients, while N is a nonlocal (integral) operator again with spatially dependent coefficients.

If $\bar{z}$ is **spatially independent**, i.e. corresponds to a flat steady state then all the operators, local and nonlocal are constant coefficient operators, and the analysis simplifies considerably. Note that by Proposition 4.1, this case can **only** happen if $k = k_0$, a constant function. In this case $f_v(\bar{z})$, $f_K(\bar{z})$, $f_{vv}(\bar{z})$, $f_{vK}(\bar{z})$, $f_{KK}(\bar{z})$ are constants (i.e. independent of x) and the operator N achieves the reduced form

$$N = \begin{pmatrix} 0 & f_K(\bar{z})K'(\bar{z})\mathcal{T} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\bar{p} f_{vK}(\bar{z})K'(\bar{z})\mathcal{T} & 0 & 0 \\ -\bar{p}K'(\bar{z})f_{Kv}(\bar{z})\mathcal{T}^* & -\bar{p}\left[f_{KK}(\bar{z})(K'(\bar{z}))^2 + f_K(\bar{z})K''(\bar{z})\right]\mathcal{T}^*\mathcal{T} & -f_K(\bar{z})K'(\bar{z})\mathcal{T}^* & 0 \end{pmatrix}$$

An important question that arises is the stability of the steady state $\bar{z}$. This is an optimal procedure leading to the optimal state $(\bar{v}, \bar{S})$ which is supported by an optimal control policy $\bar{h} = \frac{1}{\bar{v}} I_1(-\bar{p})$ and $\bar{s} = I_2(\bar{q})$. Is that steady state stable, i.e., what happens if we deviate from $\bar{z} = (\bar{v}, \bar{S}, \bar{p}, \bar{q})$ (hence equivalently from $(\bar{v}, \bar{S}, h^{(0)}, s^{(0)})$) by a small deviation that will lead to new path $\bar{z} + \epsilon z = (\bar{v} + \epsilon v, \bar{S} + \epsilon S, \bar{p} + \epsilon p, \bar{q} + \epsilon q)$. Under smoothness assumptions, the deviation z is given by the solution of the linearized, nonlocal system

$$z' = (A + L)z + Nz.$$

In order to study the local dynamics we need to consider the linear eigenvalue problem

$$\lambda u = (A + L)u + Nu,$$

with homogeneous Neumann boundary conditions on ∂D.

The eigenmodes that correspond to eigenvalues with positive real part are unstable eigenmodes, whereas the eigenmodes that correspond to eigenvalues with negative real part are stable eigenmodes. This provides a generalization of the familiar picture of a saddle point, usually encountered in finite dimensional problems, however, care must be taken with subtle technical issues arising from the infinite dimensional nature of the dynamics.

The existence of stable and unstable eigenmodes can be obtained using perturbation theory for linear operators.

6 Turing type instability formation for the controlled system

6.1 Turing instability of a flat steady state optimal solution

Assume now that the steady state solution $\bar{z} := (\bar{v}, \bar{S}, \bar{p}, \bar{q})$ is **spatially independent**, hence a constant. We will call that a flat steady state optimal solution. By Proposition 4.1 this requires the special case $k = k_0$, a constant.

In this case, the linear operator L is a constant matrix multiplication operator of the type

(20)
$$L = \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & a_4 \\ a_5 & 0 & a_6 & 0 \\ 0 & a_7 & 0 & a_8 \end{pmatrix}$$

and the nonlocal operator simplifies to

(21)
$$N = \begin{pmatrix} 0 & A_1\mathcal{T} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_2\mathcal{T} & 0 & 0 \\ A_3\mathcal{T}^* & A_4\mathcal{T}^*\mathcal{T} & A_5\mathcal{T}^* & 0 \end{pmatrix},$$

where a_i, A_i are constants defined by

$$
\begin{aligned}
&a_1 = f_v(\bar{z}),\ \ a_2 = -I_1'(-\bar{p}),\ \ a_3 = -\delta,\ \ a_4 = -I_2'(\bar{q}),\\
&a_5 = -\bar{p} f_{vv}(\bar{z}),\ \ a_6 = r - f_v(\bar{z}),\ \ a_7 = -Z''(\bar{S}),\ \ a_8 = \delta + r,\\
&A_1 = f_K(\bar{z})K'(\bar{z}),\ \ A_2 = -\bar{p} f_{vK}(\bar{z})K'(\bar{z})\ \ A_3 = -\bar{p}K'(\bar{z})f_{Kv}(\bar{z}),\\
&A_4 = -\bar{p}(f_{KK}(\bar{z})(K'(\bar{z}))^2 + f_K(\bar{z})K''(\bar{z})),\ \ A_5 = -f_K(\bar{z})K'(\bar{z}),
\end{aligned}
\tag{22}
$$

Note that $A_1 = -A_5$ and that under smoothness assumptions on f, $A_2 = A_3$.

Example 6.1. For the logistic growth case of Example 4.2 the above constants are

$$
a_1 = r,\ \ a_2 = -\frac{K^2(a^2 - r^2)}{16a^2\lambda_1},\ \ a_3 = -\delta,\ \ a_4 = \frac{\delta^2}{\lambda_2}\frac{\gamma^2}{\beta^2 M^2}\frac{(a+r)^2}{(a-r)^2},
$$

$$
a_5 = -\frac{8a^2}{K^2}\frac{\lambda_1}{a^2 - r^2},\ \ a_6 = \lambda_3\frac{\beta^2 M^2}{\gamma^2}\frac{(a-r)^2}{(a+r)^2},\ \ a_7 = \delta + r
$$

In order to study the spatiotemporal evolution of a flat optimal steady state, we need to study the constant coefficient nonlocal system

$$
z' = Az + Lz + Nz,
\tag{23}
$$

subject to the initial condition $z(0) = Z_0$ (this is the known initial deviation from the flat steady state $\bar{z}$; it is a known function of space) and Neumann boundary conditions.

To study the dynamics of (23) we will use a Galerkin approach to reduce the nonlocal problem (23) to a countable system of ODEs, whose structure may reveal the qualitative features of the dynamics and importantly, the possible spatial structures that may arise for the deviation z from the desired flat optimal steady state $\bar{z}$.

Let $\{\phi_n, \lambda_n\}$, $n \in \mathbb{N}_0 := \{0, 1, 2, \cdots\}$, be the solutions to the eigenvalue problem

$$
\begin{aligned}
-\Delta\phi_n &= \lambda_n \phi_n,\ \ \text{in}\ \ D,\\
\nabla\phi_n \cdot \eta &= 0,\ \ \text{on}\ \ \partial D
\end{aligned}
$$

The eigenvalue $\lambda_0 = 0$ corresponds to the constant eigenfunction $\phi_0 = C$, where C is chosen so that ϕ_0 is normalized. In particular, we choose C so that $\int_D \phi_0^2 dx = 1$ which means $C = (\mathcal{L}(D))^{-1/2}$, where $\mathcal{L}(D)$ is the Lebesgue measure of D (the length of D in dimension 1, the area of D is dimension 2 and the volume of D in dimension 3).

The set $\{\phi_n\}$ forms a complete orthonormal system for $L^2(D)$ so that expansions for (v, S, p, q) of the form

$$
\begin{aligned}
&v = \sum_{n\in\mathbb{N}_0} v_n\phi_n,\ \ S = \sum_{n\in\mathbb{N}_0} S_n\phi_n,\ \ p = \sum_{n\in\mathbb{N}_0} p_n\phi_n,\ \ q = \sum_{n\in\mathbb{N}_0} q_n\phi_n,\\
&v_n = \int_D v\phi_n dx,\ \ S_n = \int_D S\phi_n dx\ \ p_n = \int_D p\phi_n dx,\ \ q_n = \int_D q\phi_n dx,
\end{aligned}
\tag{24}
$$

are guaranteed. The orthonormality means that $\int_D \phi_n\phi_m dx = \delta_{n,m}$ for every $n, m \in \mathbb{N}_0$.

Using this basis, the nonlocal equation (23) reduces to a countable set of ODEs for the expansion modes (v_n, S_n, p_n, q_n), $n \in \mathbb{N}_0$ of the form (see Appendix D for details)

$$
\begin{aligned}
v_0' &= a_1 v_0 + a_2 p_0 + A_1\mu_0 S_0,\\
S_0' &= a_3 S_0 + a_4 q_0,\\
p_0' &= a_5 v_0 + a_6 p_0 + A_2\mu_0 S_0,\\
q_0' &= a_7 S_0 + a_8 q_0 + \left\{A_3\mu_0 v_0 + A_4\mu_0^2 S_0 + A_5\mu_0 p_0\right\}
\end{aligned}
\tag{25}
$$

and

$$(26)\qquad \begin{aligned} v_n' &= -d_1\lambda_n v_n + a_1 v_n + a_2 p_n,\\ S_n' &= -d_2\lambda_n S_n + a_3 S_n + a_4 q_n,\\ p_n' &= d_1\lambda_n p_n + a_5 v_n + a_6 p_n,\\ q_n' &= d_2\lambda_n q_n + a_7 S_n + a_8 q_n \end{aligned}$$

where $\mu_0 = k_0\mathcal{L}(D)$.

Note that this is a system of uncoupled ODEs that may be solved for each n separately to provide the full solution for (v_n, S_n, p_n, q_n), $n \in \mathbb{N}_0$ and then from that we may obtain by summing the representations (24) the full spatiotemporal dependence.

These equations ($n \in \mathbb{N}_0$) can be expressed in compact form in terms of the family of matrices $G(n)$ defined in terms of the family of matrices $A(n)$ which is the related to the diagonalization of the operator A in the basis of eigenfunctions of the Laplacian,

$$A(n) = \begin{pmatrix} -d_1\lambda_n & 0 & 0 & 0\\ 0 & -d_2\lambda_n & 0 & 0\\ 0 & 0 & d_1\lambda_n & 0\\ 0 & 0 & 0 & d_2\lambda_n \end{pmatrix}$$

so that

$$(27)\qquad G(n) = A(n) + L = \begin{pmatrix} a_1 - d_1\lambda_n & 0 & a_2 & 0\\ 0 & a_3 - d_2\lambda_n & 0 & a_4\\ a_5 & 0 & a_6 + d_1\lambda_n & 0\\ 0 & a_7 & 0 & a_8 + d_2\lambda_n \end{pmatrix}, \quad n \in \mathbb{N}.$$

Note that when $n = 0$, the matrix $G(0)$ admits a different form

$$(28)\qquad G(0) = \begin{pmatrix} a_1 & A_1\mu_0 & a_2 & 0\\ 0 & a_3 & 0 & a_4\\ a_5 & A_2\mu_0 & a_6 & 0\\ A_3\mu_0 & a_7 + A_4\mu_0^2 & A_5\mu_0 & a_8 \end{pmatrix}$$

In terms of the above definitions and using the notation $z_n = (v_n, S_n, p_n, q_n)^{tr} \in \mathbb{R}^4$ for any n, we may rewrite the set of equations (25)-(26) as

$$z_n' = G(n) z_n, \quad n \in \mathbb{N}_0.$$

The general solution of these is expressed in terms of the exponential of the matrices $G(n)$, as

$$z_n(t) = \exp(tG(n)) z_n(0), \quad n \in \mathbb{N}_0.$$

We note that $z_n(0)$ is the expansion in the complete basis of eigenfunctions of the Laplacian of the initial conditions $z(0)$, and only the part $v_n(0), S_n(0)$ is this vector is known to us. The second part which corresponds to the adjoint variables $p_n(0), q_n(0)$ is in principle not known but this is not crucial for our arguments that follow.

The general spatiotemporal evolution of any initial deviation $z(0) = \sum_n z_n(0)\phi_n$ from the optimal policy and path which corresponds to the flat optimal steady state is expressed

$$z(t) = \sum_n z_n(t)\phi_n = \sum_{n\in\mathbb{N}_0} \exp(tG(n)) z_n(0)\phi_n$$

where in the above sum, for each summand, the first term $z_n(t) = \exp(tG(n))z_n(0)$ is only time dependent whereas the second term contains spatiotemporal variation (the spatial dependence is contained in the ϕ_n terms).

This expression allows us to consider the problem of pattern formation for the controlled system. As all spatial variation comes from the second term we need first to consider carefully the function

$$X(t,x) = \sum_{n \neq 0} \exp(tG(n))z_n(0)\phi_n(x),$$

which will specify the spatial patterns that may develop in the controlled system. This is a sum of matrix exponentials, the asymptotic behavior of which is determined from the spectrum of the matrices $G(n)$. Let us denote by $\sigma_i(n)$, $i = 1, 2, 3, 4$ the eigenvalues of the matrix $G(n)$, $n \neq 0$.

Consider the sets

- $\mathcal{S} = \{n \in \mathbb{N} \setminus \{0\} \ : \ \max_{i \in \{1,2,3,4\}} Re(\sigma_i(n)) < 0\}$
- $\mathcal{P} = \{n \in \mathbb{N} \setminus \{0\} \ : \ Re(\sigma_i(n) \in [0, \frac{r}{2}], \ i \in \{1,2,3,4\}\}$

We then have that

$$\lim_{t \to \infty} \sum_{n \in \mathcal{S}} \exp(tG(n))z_n(0)\phi_n(x) \to 0,$$

whereas

$$\lim_{t \to \infty} \sum_{n \in \mathcal{P}} \exp(tG(n))z_n(0)\phi_n(x) \to \infty,$$

however at an exponential rate which is allowed by the transversality condition.

We therefore conclude that possible spatial patterns allowed by the transversality condition hence viable even under the effect of control will be of the form

$$P(t,x) = \sum_{n \in \mathcal{P}} \exp(tG(n))z_n(0)\phi_n(x)$$

This resembles a Turing pattern, a phenomenon which is often encountered in uncontrolled systems. What is interesting here is that the pattern forming instability may be triggered by the actions of the controller. The spatial content of the pattern is determined by the shape of the relevant eigenfunctions ϕ_n for $n \in \mathcal{P}$.

We thus arrive at the following result

Proposition 6.2. *Spatiotemporal patterns may occur for the controlled system for small enough perturbations $z(0,x)$ from the flat optimal steady state, as long as there exists $n \in \mathcal{P}$ such that $\Pi_n z := \int_D z(0,x)\phi_n(x)dx \neq 0$. A condition for the occurrence of patterns is the existence of $n \in \mathbb{N} \setminus \{0\}$, such that $\min_{i \in \{1,2,3,4\}} \sigma_i(n) \in [0, \frac{r}{2}]$.*

6.2 Turing patterns arising from a flat non optimal steady state

In the case where $k = k(x)$ is spatially dependent, we have seen that an optimal flat steady state cannot exist. In this case we may consider variations of the Turing mechanism of pattern formation, by considering perturbations about flat steady states which are not optimal for the controlled system. One such case, may be to consider a flat steady state which is optimal for the spatially independent problem, of maximizing

$$J(h,s) = \int_0^\infty e^{-rt} \left(U(h\,v) + B(s) - Z(S)\right) dt,$$

under the dynamic constraints

$$
\begin{aligned}
v' &= f(v, K(S)) - hv, \\
S' &= -\delta S + s,
\end{aligned} \tag{29}
$$

Then we will take $(\bar{v}, \bar{S}, \bar{p}, \bar{q})$ to be saddle point of the above spatially independent problem and consider linearizations of the optimality conditions around this flat steady state. Now we no longer need k to be a constant, so this will lead to a linearized problem which is slightly more complicated than the one treated in the previous section. However, the qualitative nature of the results will be similar.

If we assume that $d_1 = d_2 = 0$ and that the averaging kernel normalized such that $k(x) = k_0 = (\mathcal{L}(D))^{-1}$ then the optimality condition forward-backward PDE (9) simplifies to

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x) &= f(v(t,x), K_a(t)) - I_1(-p(t,x)), \\
\frac{\partial S}{\partial t}(t,x) &= -\delta S(t,x) + I_2(q(t,x)), \\
\frac{\partial p}{\partial t}(t,x) &= (r - f_v(v(t,x), K_a(t)))p(t,x), \\
\frac{\partial q}{\partial t}(t,x) &= (r+\delta)q(t,x) - Z'(S(t,x)) \\
&\quad - k_0 K_a'(t) \int_D \left(f_K(v(t,x'), K_a(t)) p(t,x') \right) dx,
\end{aligned} \tag{30}
$$

where the terms $K_a(t)$ and $K_a'(t)$ have a non-local dependence on S via

$$
\begin{aligned}
K_a(t) &= K\left(k_0 \int_D S(t,x') dx' \right), \\
K_a'(t) &= K'\left(k_0 \int_D S(t,x') dx' \right).
\end{aligned} \tag{31}
$$

Assume that we look for only temporally dependent solutions. Then

$$
\begin{aligned}
K_a(t) &= K\left(k_0 \int_D S(t) dx' \right) = K\left(k_0 \mathcal{L}(D) S(t) \right) = K(S(t)), \\
K_a'(t) &= K'\left(k_0 \int_D S(t) dx' \right) = K'\left(k_0 \mathcal{L}(D) S(t) \right) = K'(S(t)),
\end{aligned}
$$

by the choice of the constant k_0. Furthermore,

$$
\begin{aligned}
k_0 K_a'(t) \int_D \left(f_K(v(t), K_a(t)) p(t) \right) dx &= k_0 \mathcal{L}(D) K_a'(t) f_K(v(t), K_a(t)) p(t) \\
&= K_a'(t) f_K(v(t), K_a(t)) p(t),
\end{aligned}
$$

by the choice of the constant k_0. Suppressing the explicit time dependence to ease notation for such a temporal solution (32) becomes

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= f(v, K(S)) - I_1(-p), \\
\frac{\partial S}{\partial t} &= -\delta S + I_2(q), \\
\frac{\partial p}{\partial t} &= (r - f_v(v, K(S)))p, \\
\frac{\partial q}{\partial t} &= (r+\delta)q - Z'(S) - K'(S) f_K(v, K(S)),
\end{aligned} \tag{32}
$$

which is readily recognized as the optimality condition for the temporal optimization problem. The saddle point for this problem will correspond to the steady state solution which will be the solution to the set of algebraic equations

$$
\begin{aligned}
&0 = f(\bar{v}, K(\bar{S})) - I_1(-\bar{p}),\\
&0 = -\delta \bar{S} + I_2(\bar{q}),\\
&0 = (r - f_v(\bar{v}, K(\bar{S})))\bar{p},\\
&0 = (r+\delta)\bar{q} - Z'(\bar{S}) - K'(\bar{S}) f_K(\bar{v}, K(\bar{S})),
\end{aligned} \tag{33}
$$

We now consider the case where $d_1 = \epsilon D_1$, $d_2 = \epsilon D_2$ and $k(x) = k_0 + \epsilon \hat{k}(x)$ with $k_0 = \mathcal{L}(D)^{-1}$, and consider the optimality condition forward-backward PDE (9) looking for spatiotemporal solutions of the form $Z := \bar{z} + \epsilon z$ where $z = (v, S, p, q)$. We express the nonlocal term in operator form as $\mathcal{T} = \mathcal{T}_0 + \epsilon \mathcal{T}_1$, where, using u as a proxy for v, S, p, q.

$$
\begin{aligned}
(\mathcal{T}_0 u)(x) &= \int_D k_0 u(x')dx' = k_0 \int_D u(x')dx',\\
(\mathcal{T}_1 u)(x) &= \int_D \hat{k}(x) u(x')dx'.
\end{aligned}
$$

According to Appendix B the adjoint admits the form $\mathcal{T}^* = \mathcal{T}_0^* + \epsilon \mathcal{T}_1^*$ where

$$
\begin{aligned}
(\mathcal{T}_0^* u)(x) &= k_0 \int_D u(x')dx' = (\mathcal{T}_0 u)(x),\\
(\mathcal{T}_1 u)(x) &= \hat{k}(x) \int_D u(x')dx'.
\end{aligned}
$$

The action of $\mathcal{T}$ and $\mathcal{T}^*$ to a function of the form $U(t, x) = \bar{u} + \epsilon u(t, x)$ is

$$
\begin{aligned}
\mathcal{T} U &= (\mathcal{T}_0 + \epsilon \mathcal{T}_1)(\bar{u} + \epsilon u) = \mathcal{T}_0 \bar{u} + \epsilon\,(\mathcal{T}_0 u + \mathcal{T}_1 \bar{u}) + O(\epsilon^2),\\
\mathcal{T}^* U &= (\mathcal{T}_0^* + \epsilon \mathcal{T}_1^*)(\bar{u} + \epsilon u) = \mathcal{T}_0^* \bar{u} + \epsilon\,(\mathcal{T}_0^* u + \mathcal{T}_1^* \bar{u}) + O(\epsilon^2).
\end{aligned}
$$

We now insert an ansatz of the form $Z = \bar{z} + \epsilon z$ into the optimality condition (7) where $\bar{z} = (\bar{v}, \bar{S}, \bar{p}, \bar{q})$ is the flat steady state solution of the temporal model (32), and expand in powers of ϵ keeping only terms up to the first order. We observe that the zeroth order term is (32) calculated at $\bar{z}$ which vanishes, since $\bar{z}$ is chosen to be a solution of (32) .

After some rather tedious algebra we obtain the linearized equation

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= d_1 \Delta v + f_v(\bar{z}) v + f_K(\bar{z}) K'(\bar{z}) \mathcal{T}_0 S - I_1'(-\bar{p}) p + F^{(1)},\\
\frac{\partial S}{\partial t} &= d_2 \Delta S - \delta S + I_2'(\bar{q}) q + F^{(2)},\\
\frac{\partial p}{\partial t} &= -d_1 \Delta p - \bar{p} f_{vv}(\bar{z}) v + (r - f_v(\bar{z})) p - \bar{p} f_{vK}(\bar{z}) K'(\bar{z}) \mathcal{T}_0 S + F^{(3)},\\
\frac{\partial q}{\partial t} &= -d_2 \Delta q - Z''(\bar{S}) s + (\delta + r) q - \mathcal{T}_0^*(\bar{p} K'(\bar{z}) f_{Kv}(\bar{z}) v) - \mathcal{T}_0^*(f_K(\bar{z}) K'(\bar{z}) p)\\
&\quad - \mathcal{T}_0^*([\bar{p} f_{KK}(\bar{z})(K'(\bar{z}))^2 + \bar{p} f_K(\bar{z}) K''(\bar{z})] \mathcal{T}_0(S)) + F^{(4)}
\end{aligned} \tag{34}
$$

where the source terms $F := (F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)})$ are known functions of space depending on the steady state and of the form

$$
\begin{aligned}
F^{(1)} &:= +f_K(\bar z)K'(\bar z)\mathcal{T}_1\bar S,\\
F^{(2)} &:= 0\\
F^{(3)} &:= -\bar p f_{vK}(\bar z)K'(\bar z)\mathcal{T}_1\bar S,\\
F^{(4)} &:= -\mathcal{T}_1^*(\bar p K'(\bar z)f_{Kv}(\bar z)\bar v) - \mathcal{T}_1^*(f_K(\bar z)K'(\bar z)\bar p)\\
&\quad - \mathcal{T}_0^*([\bar p f_{KK}(\bar z)(K'(\bar z))^2 + \bar p f_K(\bar z)K''(\bar z)]\mathcal{T}_1(\bar S)) - \mathcal{T}_1^*\left(\bar p K'(\bar z)f_{Kv}(\bar z)\right).
\end{aligned}
$$

The linearized system is can be brought into the general compact form

$$
z' = Az + Lz + Nz + F \tag{35}
$$

where the operators L and N are of the same form as in (20), (21) with the same constants as in (22), the only difference being that $\mathcal{T}_0$ replaces the operator $\mathcal{T}$. Therefore, (23) is of the same form as (19) but now with the addition of a known spatially depending source term, which it terms of this notation simplifies to

$$
\begin{aligned}
F^{(1)}(x) &= A_1\mathcal{T}_1(\bar S) = A_1\int_D \hat k(x')\bar S dx' = A_1\bar S\int_D \hat k(x')dx',\\
F^{(2)}(x) &= 0\\
F^{(3)}(x) &= A_3\mathcal{T}_1(\bar S) = A_3\bar S\int_D \hat k(x')dx',\\
F^{(4)}(x) &= A_3\mathcal{T}_1^*(\bar v) + A_5\mathcal{T}_1^*(\bar p) + A_4\mathcal{T}_0^*(\mathcal{T}_1\bar S) + A_3\mathcal{T}_1^*(1)\\
&= A_3\mathcal{L}(D)\bar v\hat k(x) + A_5\mathcal{L}(D)\bar p\hat k(x) + A_4\mathcal{L}(D)k_0\bar S\int_D \hat k(x')dx' + A_3\mathcal{L}(D)\hat k(x)
\end{aligned}
$$

We now consider the complete orthonormal set $\{\phi_n\}$, $n \in \mathbb{N}_0$ consisting of the eigenfunctions of the Neumann Laplacian (as before) and assume that the constant eigenfunction is chosen normalized to $\phi_0 = C$, $C = (\mathcal{L}(D))^{-1/2}$. Then the averaging kernel $\hat k$ admits an expansion as

$$
\hat k(x) = \sum_{n\in\mathbb{N}_0} \hat k_n\phi_n(x),
$$

$$
\hat k_n = \int_D \hat k(x)\phi_n(x)dx, \;\; n \in \mathbb{N}_0.
$$

In terms of these we can expand the source term in this basis as

$$
\begin{aligned}
&F^{(1)}(x) = A_1\mathcal{L}(D)\bar S\hat k_0\phi_0,\\
&F^{(2)}(x) = 0\\
&F^{(3)}(x) = A_3\mathcal{L}(D)\bar S\hat k_0\phi_0,\\
&F^{(4)}(x) = A_3\mathcal{L}(D)\bar v\hat k(x) + A_5\mathcal{L}(D)\bar p\hat k(x) + A_4(\mathcal{L}(D))^2k_0\hat k_0\bar S\phi_0 + A_3\mathcal{L}(D)\hat k(x)\\
&= \mathcal{L}(D)\left(A_3\bar v + A_5\bar p + A_4(\mathcal{L}(D))k_0\bar S + A_3\right)\hat k_0\phi_0 + \sum_{n=1}^{\infty}\mathcal{L}(D)(A_3\bar v + A_5\bar p + A_3)\hat k_n\phi_n(x).
\end{aligned}
$$

which means that

$$
F^{(i)}(x) = \sum_{n\in\mathbb{N}_0} F_n^{(i)}\phi_n(x), \;\; i = 1,2,3,4,
$$

with

$$\begin{aligned}
F_n^{(1)} &= A_1\mathcal{L}(D)\bar{S}\hat{k}_0\delta_{n,0},\\
F_n^{(2)} &= 0,\\
F_n^{(3)} &= A_3\mathcal{L}(D)\bar{S}\hat{k}_0\delta_{n,0},\\
F_0^{(4)} &= \mathcal{L}(D)\left(A_3\bar{v} + A_5\bar{p} + A_4(\mathcal{L}(D))k_0\bar{S} + A_3\right)\hat{k}_0,\\
F_n^{(4)} &= \mathcal{L}(D)(A_3\bar{v} + A_5\bar{p} + A_3)\hat{k}_n
\end{aligned}$$

Working in a similar fashion as in the previous case, we may reduce 35 to a countable system of ODEs for the expansion modes (v_n, S_n, p_n, q_n), $n \in \mathbb{N}_0$ of the form (see Appendix D for details)

$$\begin{aligned}
v_0' &= a_1v_0 + a_2p_0 + A_1\mu_0S_0 + F_0^{(1)},\\
S_0' &= a_3S_0 + a_4q_0,\\
p_0' &= a_5v_0 + a_6p_0 + A_2\mu_0S_0 + F_0^{(3)},\\
q_0' &= a_7S_0 + a_8q_0 + \left\{A_3\mu_0v_0 + A_4\mu_0^2S_0 + A_5\mu_0p_0\right\} + F_0^{(4)},
\end{aligned} \tag{36}$$

and

$$\begin{aligned}
v_n' &= -d_1\lambda_nv_n + a_1v_n + a_2p_n,\\
S_n' &= -d_2\lambda_nS_n + a_3S_n + a_4q_n,\\
p_n' &= d_1\lambda_np_n + a_5v_n + a_6p_n,\\
q_n' &= d_2\lambda_nq_n + a_7S_n + a_8q_n + F_n^{(4)}
\end{aligned} \tag{37}$$

where $\mu_0 = k_0\mathcal{L}(D)$.

Using for any $n \in \mathbb{N}_0$ the notation $z_n = (v_n, S_n, p_n, q_n)$ and $F_n = (F_n^{(1)}, F_n^{(2)}, F_n^{(3)}, F_n^{(4)})$, these are brought into the compact form

$$z_n' = G(n)z_n + F_n, \;\; n \in \mathbb{N}_0 \tag{38}$$

where the matrices $G(n)$ are the same as in (27) and (28) .

The analysis of the Turing pattern instability remains essentially the same from there. The main difference is that now there is a constant in time part of the solution which may contain spatial dependence. Consider the steady states solutions of the above which are $z_n = -G(n)^{-1}F_n$. The pattern

$$P_0(x) := \sum_n (-G(n)^{-1}F_n)\phi_n(x),$$

is a steady state spatial pattern.

For the exponentially growing patterns Proposition 6.2 covers us completely.

6.3 Turing Instability and Optimal Turing Instability

The results of the two previous subsections relate to the Turing diffusion-induced instability which underlies spatial pattern formation in reaction-diffusion systems which are either:

- not controlled at all, or
- controlled by some arbitrary rule but not controlled optimally.

In this paper, where the reaction-diffusion system is optimally controlled, the following possibilities arise when we compare optimally controlled with uncontrolled systems.

1. The non-optimally controlled system generates spatial patterns through Turing instability, but the optimally controlled system has an optimal flat steady state which is stable to spatial perturbations. Optimal control acts as a spatial stabilizing or spatial homogenizing force.

2. The non-optimally controlled system is spatially stable in the sense of Turing instability, but a flat steady state of the optimally controlled system is unstable to spatial perturbations. Optimal control acts as a generator of optimal spatial patterns.

3. The non-optimally controlled system generates spatial patterns through Turing instability and a flat steady state of the optimally controlled system is unstable to spatial perturbations. Optimal control acts as a generator of optimal spatial patterns which are different from Turing's pattern for the non-optimally controlled system.

The spatial structure of the optimal pattern in the last two cases (cases 2 and 3) is determined by the shape of the relevant eigenfunctions ϕ_n for $n \in \mathcal{P}$. We call these two cases *optimal Turing instability.*

Optimal Turing instability generates spatial patterns in the state-costate space, or equivalently in the space of stocks (the states) and their corresponding shadow values (the costates). Furthermore, the optimal Turing instability generates spatial patterns in the state-control space. The spatiotemporal patterns either in the state-costate space or the state control space can be used by a regulator seeking to internalize the spatial externality to design optimal spatial policies in terms of prices (costates) or quantities (controls).

7 Concluding Remarks

Transport phenomena, local or non-local, are very closely associated with environmental and resource systems. Spatial transport in renewable resources, air or water pollution, and heat transfer towards the Poles, are some of the issues that should be taken into account in environmental and resource management. Most of the times we tend to ignore these issues and design policies assuming that spatially homogeneity is a good approximation and spatial heterogeneities are not that important. However, this might not be the case.

In this paper we model spatial interactions in a coupled system of a renewable resource and industrial pollution and study the optimal control of this system, in the sense of attaining an optimal solution for a regulator Our results suggests that optimal policies may have a spatial structure and that the emergence of Turing type optimal instability indicates the potential existence of optimal agglomerations in the environmental systems. Optimal agglomerations imply that spatially homogenous policies which are derived when spatial interactions and transport phenomena are ignored may not be the optimal policies, but instead spatial structured policies are required.

The study of spatial interactions in urban and regional economics, mainly through non-local interactions, has recently led to very interesting results on the structure of cities and agglomeration dynamics. In this paper we provide analytical tools to study the qualitatively similar issues of spatial patterns and agglomeration dynamics associated with environmental and resource systems by introducing spatial transport phenomena which are an empirically relevant. The type of research presented here could therefore provide new insights into the spatial dimension of environmental policy. Spatially differentiated instruments - price or quantities - zoning systems, reserves and no-take areas could be cases where spatially heterogenous policies are appropriate.

A Proof of Proposition 3.1

Consider the problem of maximizing the functional J under the dynamic constraints defined by the nonlocal nonlinear PDEs (2). Let $\mathcal{U}$ be the set of admissible controls (h, s), and assume that $(h_*, s_*) \in \mathcal{U}$ is a maximizer of the functional $J : \mathcal{U} \to \mathbb{R}$ under the stated constraints. Consider any perturbation of the optimal protocol $(u_* + \epsilon u, s_* + \epsilon s)$ chosen so that $(u_* + \epsilon u, s_* + \epsilon s) \in \mathcal{U}$. The new control protocol is not optimal so it leads to a path for the state equation which is not optimal. Let us denote the optimal path by (v_*, S_*), which is nothing but the solution of the state equation (2) when $h = h_*$ and $s = s_*$. The adoption of the perturbed protocol $(u_* + \epsilon u, s_* + \epsilon s)$ will lead to a new path (non-optimal in general) which we will denote by $(v_* + \epsilon v, S_* + \epsilon S)$ and will be the solution of system (2) when we substitute the control procedure $(u_* + \epsilon u, s_* + \epsilon s)$. We are interested in small deviations, from the optimal path, so we will assume that $\epsilon \to 0$. Under technical regularity conditions for the state equation (2) (typically smoothness of the nonlinearities) for small ϵ, the evolution of the deviation from the optimal path is given by the solution of the linearized system

$$
\begin{aligned}
&\frac{\partial}{\partial t} v = d_1 \Delta v + (f_v(v_*, K(\mathcal{T} S_*)) - h_*) v + f_K(v_*) f_v(v_*, K(\mathcal{T} S_*)) K'(\mathcal{T} S_*) \mathcal{T} S - v_* h \\
&\frac{\partial}{\partial t} S = d_2 \Delta S - \delta\, S + s,
\end{aligned}
\tag{39}
$$

with Neumann boundary conditions and initial conditions $v(0, x) = 0$ and $S(0, x) = 0$. This is an approximation of the deviation from the optimal path, which under smoothness assumptions can be shown to be a good approximation up to $O(\epsilon^2)$. To simplify the notation we will define

$$
\begin{aligned}
F_1 &:= f_v(v_*, K(\mathcal{T} S_*)) - h_*, \\
F_2 &:= f_K(v_*) f_v(v_*, K(\mathcal{T} S_*)) K'(\mathcal{T} S_*).
\end{aligned}
$$

We furthermore, assume smoothness assumptions for U, B and Z and we calculate $J(h_* + \epsilon h, s_* + \epsilon s)$ which using the Taylor approximation around (h_*, s_*) we obtain that up to $O(\epsilon)$,

$$
\begin{aligned}
\frac{1}{\epsilon}(J(h_* + \epsilon h, s_* + \epsilon s) - J(h_*, s_*)) = &\int_0^\infty e^{-rt} \int_D \Big\{ U'(h_* v_*) v_* h + B'(s_*) s \Big\} dx dt \\
&+ \int_0^\infty e^{-rt} \int_D \Big\{ U'(h_* v_*) h_* v - Z'(s S_*) S \Big\} dx dt.
\end{aligned}
\tag{40}
$$

The first integral in the above expression is in a desirable form since it contains only the optimal quadruple (h_*, s_*, v_*, S_*) and the arbitrary perturbation (h, s). The second integral however is not so nice, as it contains the optimal path (h_*, s_*, v_*, S_*) again but now the deviation (v, S) from the optimal path, rather than the deviation of the policy (h, s). Clearly, (v, S) is connected to (h, s) through the solution of the linearized system (39) and this is dependence is furnished in general via the action of the solution operator whose action is defined as $(h, s) \mapsto (v, S)$. Our aim is to make this connection explicit and this is accomplished by the construction of the adjoint system.

To motivate the construction of the adjoint system, we introduce two auxiliary variables (p, q) (one for each state variable) and introduce the auxiliary functional

$$
\bar{I} = \int_D (v\, p + S\, q) dx.
$$

Since this functional involves only integration over space, its value depends on time and taking the time derivative and substituting $(\frac{\partial}{\partial t} v, \frac{\partial}{\partial t} S)$, by their evolution given by the linearized system (39), we obtain that

$$
\frac{d\bar{I}}{dt} = \int_D \Big\{ \Big(d_1 \Delta v + F_1 v + F_2\, \mathcal{T} S - v_* h \Big) p + \Big(d_2 \Delta S - \delta\, S + s \Big) q + v \frac{\partial p}{\partial t} + S \frac{\partial q}{\partial t} \Big\} dx
\tag{41}
$$

We use the Green's formulae

$$\int_D \Delta v\, p dx = \int_D \Delta p\, v dx,$$
$$\int_D \Delta S\, q dx = \int_D \Delta q\, S dx,$$

as well as the definition of the adjoint operator of $\mathcal{T}$,

$$\int_D F_2 p(\mathcal{T} S) dx = \int_D \mathcal{T}^*(F_2 p) S dx,$$

to bring the expression (41) into the equivalent form,

$$\frac{d\bar{I}}{dt} = \int_D \Big(d_1 \Delta p + F_1 p + \frac{\partial p}{\partial t} \Big) v dx + \int_D \Big(d_2 \Delta q - \delta\, S + \mathcal{T}^*(F_2 p) + \frac{\partial q}{\partial t} \Big) S dx + \int_D \Big(-v_* h p + s q \Big) dx \tag{42}$$

We have so far left (p,q) completely unspecified. We now assume that (p,q) are solutions of the evolution equations

$$d_1 \Delta p + F_1 p + \frac{\partial p}{\partial t} = G_1,$$
$$d_2 \Delta q - \delta\, S + \mathcal{T}^*(F_2 p) + \frac{\partial q}{\partial t} = G_2,$$

with Neumann boundary conditions and (G_1, G_2) to be specified shortly. This yields,

$$\frac{d\bar{I}}{dt} = \int_D (G_1 v + G_2 S) dx + \int_D (-v_* h p + s q) dx.$$

Since (G_1, G_2) are left unspecified, we will choose them in such a way that $\int_D (G_1 v + G_2 S) dx$ resembles or reproduces the second problematic integral in (40). To this end we choose

$$G_1 = U'(h_* v_*) h_* + r\, p,$$
$$G_2 = -Z'(S_*) + r\, q.$$

For this choice, (42) becomes

$$\frac{d\bar{I}}{dt} = r\bar{I} + \int_D \Big(U'(h_* v_*) h_* v - Z'(S_*) S \Big) dx + \int_D (-v_* h p + s q) dx,$$

where we used the definition of $\bar{I}$, and integrating over time between $t = 0$ and $t = T$ (for T arbitrary) we get after taking the limit as $T \to \infty$ that,

$$\lim_{T\to\infty} e^{-rT} \bar{I}(T) - \bar{I}(0) = \int_0^\infty e^{-rt} \int_D \Big(U'(h_* v_*) h_* v - Z'(S_*) S \Big) dx\, dt + \int_0^\infty e^{-rt} \int_D (-v_* h p + s q) dx\, dt. \tag{43}$$

By the definition of the functional $\bar{I}$ and the chosen initial conditions for (39), $I(0) = 0$. If we choose

$$\lim_{T\to\infty} e^{-rT} \bar{I}(T) := \lim_{T\to\infty} e^{-rT} \int_D (v(T,x) p(T,x) + S(T,x) q(T,x)) dx = 0,$$

then (43) yields,

$$\int_0^\infty e^{-rt} \int_D \Big(U'(h_* v_*) h_* v - Z'(S_*) S \Big) dx\, dt = - \int_0^\infty e^{-rt} \int_D (-v_* h p + s q) dx\, dt,$$

so that we have managed to express the problematic second integral in (40) in terms of the variation of the optimal protocol (h, s) at the cost of introducing the adjoint variables (p, q).

Combining these results we conclude that

$$\frac{1}{\epsilon}(J(h_*+\epsilon h, s_*+\epsilon s) - J(h_*, s_*)) = \int_0^\infty e^{-rt}\int_D \left\{\Big(U'(h_* v_*)v_* - v_* p\Big)h + \Big(B'(s_*) - q\Big)s\right\}dxdt,$$

and since $J(h_*, s_*)$ is assumed to be a maximum, for any admissible (h, v) , and any $\epsilon > 0$ we have that $\frac{1}{\epsilon}(J(h_*+\epsilon h, s_*+\epsilon s) - J(h_*, s_*)) \leq 0$, hence we conclude that

$$\int_0^\infty e^{-rt}\int_D \left\{\Big(U'(h_* v_*)v_* - v_* p\Big)h + \Big(B'(s_*) - q\Big)s\right\}dxdt \leq 0, \ \ \forall\,(h, s) \tag{44}$$

This holding for any (h, s) can be seen to hold a.e. on D and this can be recognized as an optimality condition. To make this point more clear assume, for the time being, that any variation $(h, s) \in L^2(D)\times L^2(D)$ is possible (meaning that we may take as variations $(h, 0)$ and $(-h, 0)$ for any $h \in L^2(D)$, as well as $(0, s)$ and $(0, -s)$ for any $s \in L^2(D)$). Then condition (44) yields,

$$v_*(U'(h_* v_*) + p) = 0, \tag{45}$$

$$B'(s_*) - q = 0, \tag{46}$$

with the above holding a.e. on D, which can be recognized as the first order conditions for the maximization of

$$\bar{H}_1(h) := U(hv) + vhp,$$
$$\bar{H}_2(s) := B(s) - qs,$$

with respect to h and s. These are considered are static optimization problems, since the optimality conditions hold for any (fixed) (t, x) a.e. on $[0, T]\times D$. In the general case where constraints are to be taken into account on the admissible set for the allowed (h, s), we recognize (45) as the condition for maximization of $\bar{H}_1$ and $\bar{H}_2$ with respect to h and s subject to the constraints that $(h_*+\epsilon h, s_*+\epsilon s) \in \mathcal{U}$. Note that the optimality condition (45) (or the simplest special case version (45) connects the adjoint variables (p, q) with the optimal policy (h_*, s_*).

We therefore conclude that if (v_*, S_*, h_*, s_*) is an optimal quadruple, then it is connected with the solution (p, q) for the adjoint system

$$d_1\Delta p + F_1 p + \frac{\partial p}{\partial t} = U'(h_* v_*)h_* + r\,p,$$
$$d_2\Delta q - \delta\,S + \mathcal{T}^*(F_2 p) + \frac{\partial q}{\partial t} = -Z'(S_*) + r\,q,$$

subject to Neumann boundary conditions and the transversality condition, (v_*, S_*) is the solution of the state equation (2) for $h = h_*$, $s = s_*$, and that $(v_*, S_*, p, q, h_*, s_*)$ must be connected through the optimality condition (45).

B Adjoint operator

Assume that $\mathcal{T}$ is the integral operator defined by

$$(\mathcal{T}S)(x) := \int_D k(x, x')S(x')dx'$$

The adjoint $\mathcal{T}^*$ is defined by

$$\int_D (\mathcal{T}S)(x)u(x)dx = \int_D S(x)(\mathcal{T}^*u)(x)dx, \ \forall\, u \in L^2(D). \tag{47}$$

Start from the left hand side,

$$\begin{aligned}\int_D (\mathcal{T}S)(x)u(x)dx &= \int_D \left(\int_D k(x,x')S(x')dx'\right) u(x)dx \\ = \int_D \left(\int_D k(x',x)S(x)dx\right) u(x')dx' &= \int_D \left(\int_D k(x',x)u(x')dx'\right) S(x)dx,\end{aligned} \tag{48}$$

where we first "relabelled" x and x' by interchanging them and then we used Fubini-Tonelli to interchange the order of integration. Comparing (47) with (48) we see that the action of the operator $\mathcal{T}^*$ is as follows:

$$(\mathcal{T}^*u)(x) := \int_D k(x',x)u(x')dx',$$

i.e. it is an integral operator again by now with kernel $k(x',x)$ in lieu of $k(x,x')$. If k is symmetric, e.g. if $k(x,x') = \phi(|x-x'|)$ as for instance in most of the cases we use in the past then $\mathcal{T} = \mathcal{T}^*$, i.e. $\mathcal{T}$ is self adjoint.

Suppose now that $\mathcal{T}$ is just a global spatial averaging operator,

$$(\mathcal{T}S)(x) := \int_D k(x')S(x')dx'$$

i.e. we get the same answer for all x. We repeat the above

$$\begin{aligned}\int_D (\mathcal{T}S)(x)u(x)dx &= \int_D \left(\int_D k(x')S(x')dx'\right) u(x)dx \\ = \int_D \left(\int_D k(x)S(x)dx\right) u(x')dx' &= \int_D k(x)\left(\int_D u(x')dx'\right) S(x)dx,\end{aligned} \tag{49}$$

so that comparing (47) with (49) we see that the action of the operator $\mathcal{T}^*$ is as follows:

$$(\mathcal{T}^*u)(x) := k(x)\int_D u(x')dx',$$

When acting on spatiotemporal functions, then the operator only affects the spatial part, i.e.

$$(\mathcal{T}^*u)(t,x) := k(x)\int_D u(t,x')dx',$$

So for example,

$$(\mathcal{T}^* f_K K'(\mathcal{T}S)p)(t,x) := k(x)\int_D f_K(t,x')K'(t)p(t,x')dx' = k(x)K'(t)\int_D f_K(t,x')p(t,x')dx'$$

where

$$K'(t) = K'\left(\int_D q(x')S(t,x')dx'\right),$$

$$f_K(t,x') = f_K\left(v(t,x'), \int_D q(z)S(t,z)dz\right)$$

C Proof of Proposition 4.1

Proof. (i) Assume on the contrary that it does, i.e. there exists a constant vector $(\bar{v}, \bar{S}, \bar{p}, \bar{q})$ that solves (7). The action of $\mathcal{T}$ on a constant function yiels a constant, i.e., $\mathcal{T}\bar{S} = M\bar{S}$ where $M = \int_D k(x')dx'$. The action of $\mathcal{T}^*$ on a constant function on the other hand, does not yield a constant function unless k is constant. In fact, $\mathcal{T}^* C = k(x) \int_D C dx = C\mathcal{L}(D)k(x)$, where $\mathcal{L}(D)$ is the Lebesgue measure of D.[3] That implies,

$$K_c = K(M\bar{S}), \;\; K'_c = K'(M\bar{S}),$$

while

$$f_v(v(x'), K_c) = f_v(\bar{v}, K(M\bar{S})), \;\; f_K(v(x'), K_c) = f_K(\bar{v}, K(M\bar{S})), \;\; \forall\, x' \in D,$$

so that taking into account the action of the $\mathcal{T}^*$ operator (and under the assumption $p(x') = \bar{p}$ for all $x' \in D$, we see that system (13) for the assumed solution reduces to the following system

$$\begin{aligned} 0 &= f(\bar{v}, \bar{K}) - I_1(-\bar{p}), \\ 0 &= -\delta\bar{S} + I_2(\bar{q}), \\ 0 &= (-f_v(\bar{v}, \bar{K}) + r)\bar{p}, \\ 0 &= -\left(\mathcal{L}(D) f_K(\bar{v}, \bar{K})\bar{K}'\bar{p}\right)\, k(x) + (\delta + r)\bar{q} - Z'(\bar{S}), \end{aligned} \tag{50}$$

where $\bar{K} = K(M\,\bar{S})$ and $\bar{K}' = K'(M\bar{S})$. System (51) must be true for every $x \in D$, which is clearly impossible, hence the claim is proved. (ii) We now consider the case where $k = k_0$ is the constant function. Then, condition for existence of a flat optimal steady state reduces to the solvability of the system of algebraic equations

$$\begin{aligned} 0 &= f(\bar{v}, \bar{K}) - I_1(-\bar{p}), \\ 0 &= -\delta\bar{S} + I_2(\bar{q}), \\ 0 &= (-f_v(\bar{v}, \bar{K}) + r)\bar{p}, \\ 0 &= -M f_K(\bar{v}, \bar{K})\bar{K}'\bar{p} + (\delta + r)\bar{q} - Z'(\bar{S}), \end{aligned} \tag{51}$$

where $\bar{K} = K(M\,\bar{S})$, $\bar{K}' = K'(M\bar{S})$, and $M = \int_D k(x')dx' = \mathcal{L}(D)k_0$. That implies, that the optimal state is given by the following control strategy

$$\begin{aligned} \bar{h} &= \frac{f(\bar{v}, \bar{K})}{\bar{v}}, \\ \bar{s} &= \delta\bar{S} \end{aligned} \tag{52}$$

while it must hold that

$$\begin{aligned} f_v(\bar{v}, \bar{K}) &= r, \\ 0 &= -M f_K(\bar{v}, \bar{K})\bar{K}'\bar{p} + (\delta + r)\bar{q} - Z'(\bar{S}). \end{aligned} \tag{53}$$

This completes the proof. □

[3] $\mathcal{L}(D)$ is the length of D if $D \subset \mathbb{R}$, the area of D if $D \subset \mathbb{R}^2$ and the volume of D if $D \subset \mathbb{R}^3$.

D Expansion of the linearized system into eigenmodes

The action of the operator $\mathcal{T}$ on ϕ_n is as follows

$$\mathcal{T}\phi_n = \mu_n \phi_0, \;\; n \in \mathbb{N}_0,$$

where

$$\mu_n = \frac{1}{C}\int_D k(x)\phi_n(x)dx. \;\; n \in \mathbb{N}_0.$$

Note that even though ϕ_0 is just a constant ($\phi_0 = C = (\mathcal{L}(D))^{-1/2}$) we express the action of the operator $\mathcal{T}$ in the above form so that it is clear that this operator contributes along the direction of ϕ_0 and **not** along any other eigenspace or linear subspace spanned by ϕ_n with $n \neq 0$. This observation is important in the Galerkin type expansion that follows. Since in this section we consider the special case where $k(x) = k_0$, a constant (this is the only case where the flat optimal steady state exists!) we may calculate μ_n explicitly to obtain

$$\mu_n = \frac{1}{C}\int_D k_0\phi_n(x)dx = \frac{k_0}{C^2}\int_D \phi_n(x)\phi_0(x)dx = \frac{k_0}{C^2}\delta_{0,n} = k_0\mathcal{L}(D)\delta_{0,n},$$

which means that for this special case only the μ_0 term survives and all the other $\mu_n = 0$, $n = 1, 2, \cdots$. Our final result then is that when $k(x) = k_0$ we have that

$$\mathcal{T}\phi_0 = \mu_0\phi_0, \;\; \mathcal{T}\phi_n = 0, \;\; n \neq 0, \tag{54}$$

where

$$\mu_0 = k_0\mathcal{L}(D).$$

The action of the operator $\mathcal{T}^*$ on ϕ_n is as follows:

$$(\mathcal{T}^*\phi_n)(x) = k(x)\int_D \phi_n(x)dx = k(x)\frac{1}{C}\int_D \phi_n(x)\phi_0 dx = \frac{1}{C}k(x)\delta_{n,0} = \frac{1}{C^2}k(x)\phi_0\delta_{n,0},$$

and in the special case where $k(x) = k_0$ a constant, we obtain that

$$(\mathcal{T}^*\phi_n)(x) = \frac{k_0}{C^2}\phi_0\delta_{n,0} = k_0\mathcal{L}(D)\phi_0\delta_{n,0},$$

i.e.

$$\mathcal{T}^*\phi_0 = k_0\mathcal{L}(D)\phi_0 = \mu_0\phi_0, \;\; \mathcal{T}^*\phi_n = 0, \;\; n \neq 0. \tag{55}$$

Consider now the action of $\mathcal{T}^*$ on $S = \sum_{n\in\mathbb{N}_0} S_n\phi_n$. For the special case $k(x) = k_0$, using linearity and (54) we find that

$$(\mathcal{T}S)(t,x) = \mathcal{T}\left(\sum_{n\in\mathbb{N}_0} S_n(t)\phi_n(x)\right) = \sum_{n\in\mathbb{N}_0} S_n(t)(\mathcal{T}\phi_n)(x) = k_0\mathcal{L}(D)S_0(t)\phi_0 = \mu_0 S_0(t)\phi_0, \tag{56}$$

i.e. the action of the operator $\mathcal{T}$ on any spatiotemporal field $S(t,x) = \sum_{n\in\mathbb{N}_0} S_n(t)\phi_n(x)$ annihilates all the ϕ_n components except ϕ_0 leading to a term which only depends on t while, using again linearity and (55) we obtain that

$$(\mathcal{T}^*S)(t,x) = \mathcal{T}^*\left(\sum_{n\in\mathbb{N}_0} S_n(t)\phi_n(x)\right) = \sum_{n\in\mathbb{N}_0} S_n(t)(\mathcal{T}^*\phi_n)(x) = k_0\mathcal{L}(D)S_0(t)\phi_0 = \mu_0 S_0(t)\phi_0,$$

i.e., the action of $\mathcal{T}^*$ on any $S(t,x) = \sum_{n\ in\mathbb{N}_0} S_n(t)\phi_n(x)$ is the same as that of $\mathcal{T}$,

$$(\mathcal{T}^*S)(t,x) = (\mathcal{T}S)(t,x) = k_0\mathcal{L}(D)S_0(t)\phi_0 = \mu_0 S_0(t)\phi_0,$$

where $S_0(t) = \int_D S(t,x)\phi_0 dx = C\int_D S(t,x)dx$. Finally, when $k(x) = k_0$, we have

$$\mathcal{T}^*\mathcal{T}S = \mathcal{T}^*\left(k_0\mathcal{L}(D)S_0\phi_0\right) = k_0\mathcal{L}(D)S_0\mathcal{T}^*\phi_0 = k_0^2\mathcal{L}(D)^2 S_0\phi_0 = \mu_0^2 S_0\phi_0,$$

meaning of course that

$$(\mathcal{T}^*\mathcal{T}S)(t,x) = k_0^2\mathcal{L}(D)^2 S_0(t)\phi_0 = \mu_0^2 S_0(t)\phi_0,$$

Our aim is to substitute the expansions in (24) into the linearized system (19) and then project along the eigenspaces spanned by ϕ_n for each $n \in \mathbb{N}_0$, to obtain an infinite set of ODEs that will govern the temporal evolution of the expansion coefficients (v_n, S_n, p_n, q_n), $n \in \mathbb{N}_0$. The full spatiotemporal dependence can be regained by summing up the expansions in (24).

Consider now the system (19). We start from the first equation which is expressed as

$$\frac{\partial v}{\partial t} = -d_1\Delta v + a_1 v + a_2 p + A_1\mathcal{T}S$$

so that substituting the expansions (24) yields

$$\sum_{n\in\mathbb{N}_0} v_n'\phi_n = d_1\sum_{n\in\mathbb{N}_0}\lambda_n v_n\phi_n + a_1\sum_{n\in\mathbb{N}_0} v_n\phi_n + a_2\sum_{n\in\mathbb{N}_0} p_n\phi_n + A_1\mu_0 S_0\phi_0,$$

where we used (56) for the $\mathcal{T}S$ term. Now take the inner product of this equation with ϕ_m for every $m \in \mathbb{N}_0$ and use the orthogonality. For $m = 0$ only the terms proportional to ϕ_0 will contribute therefore

$$v_0' = a_1 v_0 + a_2 p_0 + A_1\mu_0 S_0.$$

In all the other terms for $m \neq 0$, the last term will not contribute so

$$v_m' = -d_1\lambda_m v_m + a_1 v_m + a_2 p_m, \;\; m \neq 0.$$

The second and the third one are done similarly giving

$$S_0' = a_3 S_0 + a_4 q_0,$$
$$S_m' = -d_2\lambda_m S_m + a_3 S_m + a_4 q_m, \;\; m \neq 0,$$

and

$$p_0' = a_5 v_0 + a_6 p_0 + A_2\mu_0 S_0,$$
$$p_m' = d_1\lambda_m p_m + a_5 v_m + a_6 p_m, \;\; m \neq 0.$$

We now consider the fourth equation in (23) which is more complicated. This reads,

$$\frac{\partial q}{\partial t} = -d_2\Delta q + a_7 S + a_8 q + A_3\mathcal{T}^*v + A_4\mathcal{T}^*\mathcal{T}S + A_5\mathcal{T}^*p,$$

and substituting the expansions in (24) and noting that

$$\mathcal{T}^*v = \mu_0 v_0\phi_0,$$
$$\mathcal{T}^*p = \mu_0 p_0\phi_0,$$
$$\mathcal{T}^*\mathcal{T}S = \mu_0^2 S_0\phi_0,$$

performing the necessary projections we see that

$$\begin{aligned} q_0' &= a_7 S_0 + a_8 q_0 + A_3 \mu_0 v_0 + A_4 \mu_0^2 S_0 + A_5 \mu_0 p_0, \\ q_m' &= d_2 \lambda_m q_m + a_7 S_m + a_8 q_m, \;\; m \neq 0. \end{aligned}$$

We therefore reduce the linearized system (18) to the countable set of ODEs,

$$\begin{aligned} v_0' &= a_1 v_0 + a_2 p_0 + A_1 \mu_0 S_0, \\ S_0' &= a_3 S_0 + a_4 q_0, \\ p_0' &= a_5 v_0 + a_6 p_0 + A_2 \mu_0 S_0, \\ q_0' &= a_7 S_0 + a_8 q_0 + \left\{ A_3 \mu_0 v_0 + A_4 \mu_0^2 S_0 + A_5 \mu_0 p_0 \right\} \end{aligned}$$

and

$$\begin{aligned} v_n' &= -d_1 \lambda_n v_n + a_1 v_n + a_2 p_n, \\ S_n' &= -d_2 \lambda_n S_n + a_3 S_n + a_4 q_n, \\ p_n' &= d_1 \lambda_n p_n + a_5 v_n + a_6 p_n, \\ q_n' &= d_2 \lambda_n q_n + a_7 S_n + a_8 q_n \end{aligned} \tag{57}$$

Note that this is a system of uncoupled ODEs that may be solved for each n separately to provide the full solution for (v_n, S_n, p_n, q_n), $n \in \mathbb{N}_0$ and then from that we may obtain by summing the representations (24) the full spatiotemporal dependence.

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