



**DEPARTMENT OF INTERNATIONAL AND  
EUROPEAN ECONOMIC STUDIES**

**ATHENS UNIVERSITY OF ECONOMICS AND BUSINESS**

# **ON THE EVOLUTION OF COMPLIANCE AND REGULATION WITH TAX EVADING AGENTS**

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## **Working Paper Series**

**13-25**

**March 2013**

# On the Evolution of Compliance and Regulation with Tax Evading Agents

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December 2012

## Abstract

We study the evolution of compliance and regulation with tax-evading agents, allowing for imitation rather than rationality in the evolution of available strategies distribution in the population. The general framework of the approach combines a classical model for tax evasion where agents are imitators rather than rational optimizers and form an endogenized subjective probability of audit. A regulator chooses values to available policy instruments, either myopically or optimally - within an optimal control setup-, always with respect to the behavior of agents. A comparison is drawn between the evolutionary and rational case in order to evaluate the differences that occur.

**JEL classification:** H26, C73

**Keywords:** Tax Evasion, Replicator Dynamics, Optimization, Regulation

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# 1 Introduction

Tax evasion and tax avoidance, as well as ways to remedy such issues, are real, timely and universal facts and have always been the subject of research and investigation of tax authorities in the majority of countries. After all, by affecting a key sector of public finance, i.e. the fiscal policy, tax noncompliance renders any action to restrict the development and emergence of such phenomena worth it in terms of budget poise. However, the design and enforcement of such institutions is a rather awkward and costly task for any regulator, since monitoring costs and the existence of loopholes in the law system hinder desirable effectiveness. This issue has not been addressed successfully by the tax authorities, since one can easily observe not only tax evasion itself but also the immense economic literature concerning tax evasion and tax avoidance and the various ways by which it has been modeled throughout.

In the majority of modern literature taxpayers are treated as rational agents, i.e. making tax evasion decisions based on net expected gains and losses of utility, having perfect knowledge over other agents or even the problem faced by the government sector. The intrinsic problem with this approach is that it tends to disregard other aspects of human behavior, such as imitation processes, personal influence or tastes as well as many other realistic human characteristics. The omission of these particularities is often for the sake of result tractability, since their inclusion renders mathematical models highly non-linear. The ways modern literature models the interaction between tax authorities and taxpayers is reviewed by Andreoni *et al.* (1998), who categorize them in either principal-agent approaches (e.g. Reinganum & Wilde (1985), Slemrod (1985), Vasin & Vasina (2002), Chen & Chu (2005), etc.) and game theoretic approaches (e.g. Greenberg (1984), Reinganum & Wilde (1986), Bardsley (1997), Wane (2000), etc.). Adding to these approaches the most well-known models of tax evasion, i.e. the seminal paper of Allingham & Sandmo (1972), the contribution of Yitzhaki (1974), as well as Slemrod & Yitzhaki (2002), they all treat the agents as rational criminals, as proposed in the seminal paper of Becker (1968) without any interaction with each other, maximizing their expected payoffs of cheating.

More recent models try to capture these characteristics, namely the behavioral aspects of agents, using Game Theory or even Evolutionary Game Theory. The latter offers a sufficiently more realistic background treating the strategies of agents in the population as evolving over time, a process that allows for imitation and the emergence of a different concept of equilibrium. An approach towards the relaxation of the assumption that there is no social interaction among taxpayers is offered by Lipatov (2008). The agents are considered imitators in terms of learning each others strategy through the stages of a repeated game in which the probability of audit is held either fixed or is allowed to change. Our work starts from the same basis in the sense that we allow for imitation, but it differs as far as the steps of the analysis and regulation is concerned. We use the notion of the replicator dynamics equation as proposed by Schlag (1998) and we also allow for optimization with respect to that behavior. Nevertheless, we view both this work and the work of Lipatov (2008), as ways to expand the research on tax evasion by improving on assumptions and moving towards Evolutionary Game Theory. In contemporary literature, social imitation is considered among the fundamental motives in psychology regarding social norms and cultural evolution (e.g. Brosnan *et al.* (2009)), consequently contemporary research needs to allow for imitation rules and social interaction among agents.

Thus, our work incorporates the concepts of both classical and contemporary approaches in an attempt to unify regulation and enforcement mechanisms with evolutionary concepts of agent behavior. In particular, we consider a general model with a large population of agents and a public sector represented by a regulator whose main concern is to keep a balanced budget. Note that there is no tax avoidance in our model, since we are only interested in the *illegal* manifestations of this phenomenon. The players are the taxpayers and interact with each other by playing an evolutionary stage game in which each period, a random pair of players is formed and each player compares own payoff with the payoff of the other member of the pair. The mathematical equation explicitly showing the aforementioned procedure is the replicator dynamics equation, which is induced by imitation dynamics and describes the evolution of the different strategies of agents in the

population through time.<sup>1</sup>:

$$\dot{x}_i = x_i (\pi_i(x) - \bar{\pi}(x)),$$

where  $x_i$  is the share of strategy  $i$  in the population of strategies,  $\pi(\cdot)$  is the payoff function and  $\bar{\pi}(\cdot)$  is the average payoff in the population. The equation states that as long as the payoff of strategy  $i$  is greater than the average payoff in the population, strategy  $i$  will spread (thrive) in the population, as it is relatively fitter than its competitors. This is a result that is derived by imitation dynamics in the sense that there is a nonnegative probability that the agent following strategy  $j$  will switch to strategy  $i$  if the latter exhibits higher payoff. Indeed, if we consider a sufficiently large population and we restrict to two available strategies in the population, let these be  $i$  and  $j$ , then the replicator dynamics equation becomes:

$$\dot{x}_i = x_i (1 - x_i) (\pi_i(x) - \pi_j(x)).$$

This ongoing process determines which strategies (suppose  $i$  =tax evasion or  $j$  =compliance with the tax system) have a higher fitness relative to the other, rendering them more abundant in the population as the game is played for a long time. Thus, the agents in this doctrine are considered boundedly rational and not fully rational, having only subjective beliefs about the way the population of strategies work.

On the other hand, the regulator has the task to keep an intertemporal balanced budget path and enforce the tax regime by taking into account that the agents have evolving strategies governed by imitation dynamics. However, tax enforcement is a costly procedure by itself and thus the regulator needs to account for these costs as well, i.e. the cost of audit and the cost of the policy instruments she uses. The cost of audit incorporates the sum of operating costs of the fiscal sector as a whole, whereas the cost of policy instruments relates to the inelastic ability of the government to overact with the tax and penalty levels. Policy instrument costs have an inseparable relation to welfare and social stability and they should be normalized or even restricted to viable levels.

The novelty of this work relies primarily to the fact that the optimal control problem

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<sup>1</sup>For details see Weibull (1997) and Schlag (1998).

faced by the regulator is constrained by the boundedly rational behavior of the taxpayers. Consequently, the regulator aims to determine the optimal paths of the available costly policy instruments, i.e. monitoring effort, tax rate and penalty rate by taking into account the non-best-response imitating behavior of the tax base. Having found the optimal paths of the instruments, one can then evaluate fiscal policy in terms of equilibria, comparative statics and dynamical behavior. Thus our approach is distinct from the standard ‘rational tax-evader approach’, which constraints the regulator’s problem by the optimality conditions of the rational tax-evader, since in our approach the regulator is constrained by the imitation dynamics of the boundedly rational tax evaders.

There is a discussion whether this kind of models can capture stylized facts about the realized level of evasion, the underestimation of the subjective probability of audit etc. In our opinion, all these models including ours, can only deal with petty crimes, namely tax evasion performed by individuals or by small homogeneous firms, where decisions on whether to tax evade or not are taken by individuals as well, i.e. firm owners. In this context, ‘getting away with murder’ is easier and less costly in most cases and of course imitation plays a big role in all this decision-making. On the other hand, when we are dealing with large multinational corporations or conglomerates, the decision on whether to tax evade or not cannot be regarded as a result of imitation. It is a strictly rational choice and most of the times it dwells between tax evasion and tax avoidance, since these firms hire professionals tax experts in order to take advantage of loopholes present in the tax law. This is considered legal and is obviously a rather rational choice. Consequently, all these models can be said to act as an Occam’s razor in terms of modeling reality and providing explanation to stylized facts.

## 2 The Model

We consider an economy with a sufficiently<sup>2</sup> large population of agents, for which our primary objective is to investigate the evolution of compliance and tax evasion under certain assumptions. The model revolves, almost exclusively, around the interaction of agents

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<sup>2</sup>Sufficiently in the sense that the evolutionary stage game can effectively take place.

with each other and the implications that such an interaction has on tax enforcement policy and regulation. Restricting attention to tax evasion by excluding tax avoidance and other similar practices, we assume that agents have two available strategies, i.e. either to tax evade, by reporting a taxable income below its true value, or not, by legitimately reporting their whole taxable income. The decision of each agent on whether to tax evade or not depends on her inclination to imitate the behavior of the agent she comes across in the game that agents play, which is a standard  $2 \times 2$  symmetric evolutionary stage game according to which, each period, random pairs of agents are formed. Within each pair, each agent has a non-zero probability to switch to the opposing strategy, if her own payoff is lower than the perceived payoff of her match.

On the other side, we consider a regulator, whose role is to optimally control for the aforementioned agents' behavior by setting the level of the different policy instruments at her disposal accordingly, in order to attain a certain goal. We assume this goal to be a primary deficit/surplus level towards which she optimizes a corresponding social objective function. The regulator has perfect information over the agents' behavior and beliefs and is only limited by intrinsic characteristics of each policy instrument, e.g. the tax rate or penalty rate cannot surpass viable levels.

## 2.1 Agents Side

We assume that we have a sufficiently large population of agents who are subject to income taxation, with tax rate  $t$ . Each agent receives a fixed income  $Y$ , and decides whether to report it as a whole or a part of it. We consider the case where each agent has two available strategies, i.e. to tax evade or not. When the agent chooses to tax evade, her reported income,  $Y_R$ , is strictly below  $Y$ , i.e.

$$Y_R < Y,$$

where  $Y - Y_R$  is the evaded income.

Thus, we limit to two types of agents in the population, the complying and the non-

complying with the tax rule. At the end of each period, tax authorities conduct random audits in a part of the population and any tax understatement detected is subject to a proportional penalty  $\theta$ , applied over and above the evaded tax, as proposed by Yitzhaki (1974). The agents, therefore, face an audit probability, which is detrimental in the form of a penalty/fine, only to the non-complying type. Hence, the non-compliers' expected utility,  $EU_{NC}$ , incorporates a subjective probability of audit  $p$ , in order to properly weigh the two contingent outcomes, i.e. being caught or not, following Becker's (Becker (1968)) concept:

$$EU_{NC} = (1 - p)U_{NU} + pU_{NA} \quad (1)$$

where  $U_{NU}$ , depicts the utility of the non-complier that remains unaudited and  $U_{NA}$ , the utility of the non-complier that is audited and thus gets caught<sup>3</sup>. It follows from the standard von Neumann-Morgenstern utility function  $U(\cdot)$  properties that  $U_{NU} > U_{NA}$ . The expected utility of the complying agent,  $EU_C$ , is simply  $U_C$ , since she faces no threat as far as a penalty is concerned, i.e.:

$$EU_C \equiv U_C \quad (2)$$

We will come to the issue on how we define  $p$ , later on. For the time being, we consider it being fixed and exogenous, satisfying all the desirable properties of a probability measure.

To be more rigorous in notation, each agent is considered risk-averse and has a utility-based payoff depending on the strategy she follows and on the likelihood of being audited. Following the non-compliers first, each agent reports  $Y_R$ , and faces two contingencies; to be caught cheating or not. In the first case, the evader reported  $Y_R < Y$ , and has been taxed with  $tY_R$ . Since she has been caught, she will have to pay her full tax responsibilities, i.e. the remaining  $t(Y - Y_R)$  plus the penalty on evaded tax,  $\theta t(Y - Y_R)$ , where  $\theta$  is the penalty imposed. Her payoff,  $U_{NA}$ , is thus defined as  $U(Y - tY_R - t(Y - Y_R) - \theta t(Y - Y_R))$ , which simplifies to  $U(\Upsilon - t\theta(Y - Y_R))$ , where  $\Upsilon$  is the after-tax income  $Y(1 - t)$ . In the latter case, the respective payoff of the "lucky"

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<sup>3</sup>We assume that there is no corruption in the tax collection mechanism, namely when a non-complying agents is audited she will certainly be penalized.



non-complying agent who does not get audited, i.e.  $U_{NU}$ , turns out to be just  $U(Y - tY_R)$ , which in terms of after-tax income equals to  $U(\Upsilon + t(Y - Y_R))$ . Substituting into (1), we obtain the following expression for the expected utility of the non-complying agent:

$$EU_{NC} = (1 - p)U(\Upsilon + t(Y - Y_R)) + pU(\Upsilon - t\theta(Y - Y_R)). \quad (3)$$

On the other hand, the complying agent, i.e. the one who reports her whole income  $Y$ , gets a payoff,  $U_C$ , equal to  $U(Y - tY)$ , which simplifies to just  $U(\Upsilon)$ , since she fully paid her tax liabilities. Following, once again, the notion of the von Neumann-Morgenstern properties for utility ranking, it is always true that  $U_{NU} > U_C > U_{NA}$ . This is an intuitive and necessary condition, so that the individual rationality constraint, for the complying agents, is satisfied. Notice again in (2), that the expected utility of the complying agent is  $U_C$  itself, i.e.  $EU_C \equiv U_C$ , since she faces no uncertainty.

## 2.2 Tax Evasion and Replicator Dynamics

Let  $\mathbf{x}(t) = (x_{NC}(t), x_C(t))$  be the state vector denoting the share of the non-complying and complying agents in the population, respectively, at time  $t$ . As shown above, if an agent follows the complying strategy, she will have a payoff of  $U_C$ . On the other hand, the non-complying agent will have a payoff as shown in (1), i.e.  $(1 - p)U_{NU} + pU_{NA}$ . Thus, using the fact that  $x_{NC}(t) + x_C(t) = 1$ , for each  $t$ , the average expected payoff in terms of utility  $\overline{EU}$ , of the population will be as follows:

$$\begin{aligned} \overline{EU} &= x_{NC}(t)EU_{NC} + x_C(t)EU_C = \\ &= x_{NC}(t)[(1 - p)U_{NU} + pU_{NA}] + (1 - x_{NC}(t))U_C = \\ &= x_{NC}(t)[(1 - p)U(\Upsilon + t(Y - Y_R)) + pU(\Upsilon - t\theta(Y - Y_R))] + (1 - x_{NC}(t))U(\Upsilon). \end{aligned} \quad (4)$$

Consider now, the evolutionary game that the agents play, where each one follows one of two pure strategies  $s_i$  for  $i = NC, C$ , for "not comply" and "comply" respectively. The game is repeated in periods  $t = 1, 2, \dots$ . Since  $x_i^t$  is the fraction of players in the population playing  $s_i$  in period  $t$ , and the payoff to  $s_i$  is  $EU_i^t = EU_i(x^t)$ , where  $x^t = (x_{NC}^t, x_C^t)$ , we

look at a given time  $t$ , and we rank the strategies so that, e.g.  $EU_{NC}^t \leq EU_C^t$ . Suppose that in every time period  $dt$ , each agent following a certain strategy  $NC$  or  $C$ , learns the payoff and consequently the strategy, of another randomly chosen agent, i.e. her pair in the game, and with a probability  $\alpha dt > 0$  changes her strategy to her pair's strategy if she perceives that the other strategy's payoff is higher. The larger the difference in the expected payoffs, the higher the probability the agent will perceive it and change to the more profitable strategy. In this context we will regard strategy choices to be a result of imitation, in the sense that a specific strategy that leads to a payoff higher than the alternative will be imitated with a probability proportional to the payoff difference; see Schlag (1998, 1999). Thus, the probability  $p_{NC,C}^t$  that an agent playing strategy  $NC$  will shift to the strategy  $C$  is given by:

$$p_{NC,C}^t = \begin{cases} \beta (EU_C^t - EU_{NC}^t) & \text{for } EU_C^t > EU_{NC}^t \\ 0 & \text{for } EU_C^t \leq EU_{NC}^t \end{cases}$$

where  $\beta$  is sufficiently small that  $p_{NC,C}^t \leq 1$ . Then, the expected fraction of the population following the non-complying strategy  $NC$ , in period  $t + dt$  is defined as:

$$\begin{aligned} \mathcal{E}x_{NC}^{t+dt} &= x_{NC}^t + \alpha dt x_{NC}^t \sum_{s=NC,C} x_s^t \beta (EU_{NC}^t - EU_s^t) = \\ &= x_{NC}^t + \alpha dt x_{NC}^t [x_C^t \beta (EU_{NC}^t - EU_C^t)], \end{aligned} \tag{5}$$

where the second term in (5) reflects the expected proportion of the population that switches to the non-complying strategy. Using the definition of the average payoff as given in (4) and the fact that  $x_C^t = 1 - x_{NC}^t$ , we obtain that:

$$\mathcal{E}x_{NC}^{t+dt} = x_{NC}^t + \alpha dt x_{NC}^t \beta (EU_{NC}^t - \overline{EU}^t). \tag{6}$$

For a sufficiently large population, one can replace  $\mathcal{E}x_{NC}^{t+dt}$  with  $x_{NC}^{t+d}$ . Furthermore, subtracting  $x_{NC}^t$  from both sides of (6), dividing by  $dt$ , taking the limit as  $dt \rightarrow 0$ , and

setting without loss of generality  $\alpha\beta = 1$ , we get the replicator dynamics equation:<sup>4</sup>

$$\dot{x}_{NC}^t = x_{NC}^t \left( EU_{NC}^t - \overline{EU}^t \right). \quad (7)$$

The replicator dynamics equation describes the evolution of the share of the non-complying agents, and therefore the frequency of the evading strategy  $NC$ , in the population. More specifically, it states that its frequency increases exactly when the non-complying strategy has above average payoff and vice versa. Using the payoff definitions in (4), dropping  $t$  and renaming  $x_{NC}^t$  into just  $x$  for the sake of notation simplicity, we can rewrite the replicator dynamics equation as:

$$\dot{x} = x(1-x) \left[ \underbrace{(1-p)U_{NU} + pU_{NA}}_{\text{non-compliers' payoff}} - \underbrace{U_C}_{\text{compliers' payoff}} \right]. \quad (8)$$

In (8) we clearly notice that the ratio of non-compliers increases as long as the non-compliers' payoff is greater than the compliers' payoff in the population. Nonetheless, it is important to note that the replicator dynamics equation does not describe a best-reply dynamic; that is, the agents do not adopt a best reply to the overall frequency distribution of strategies in the previous period. On the contrary, agents have a localized and limited knowledge regarding the system as a whole and are thus considered to be "boundedly rational" as far as the distribution of information is concerned.

Until now, we have treated the subjective probability of audit  $p$ , formed by the non-complying agents, as fixed and exogenous. This means that the level of monitoring effort is taken as given, i.e. as if the regulator publicly announced the number of inspections to be exercised per period. We want to go beyond the case of a precommitment to a certain auditing probability and we will, therefore, endogenize  $p$  in a way that it makes our analysis more realistic inasmuch as tractability of results is attained, since there is a well known trade-off between these two. We assume that each agent elicits information in order to form the probability with which she thinks she will be audited, through two channels. The first channel consists of all publicly available information on the stringency

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<sup>4</sup>For this derivation see Schlag (1998) or Gintis (2000). For details on replicator dynamics see also, for example, Weibull (1997), Samuelson (1998) and Hofbauer & Sigmund (2003).

of the tax authorities, whereas the second channel incorporates all the knowledge that the agent can acquire through her interaction with the other agents, especially via the evolutionary game. Television, radio or printed news, political announcements, economic crises or booms directly affecting the real economy etc., perform as suitable examples for the first channel. People tend to shape their beliefs regarding the stringency of audits according to the aforementioned sources. As far as the second channel is concerned, human communication and interpersonal relationships provide plausible examples through which agents produce beliefs and most importantly imitate one another; and we have seen that the replicator dynamics is all about imitation. Thus, we assume that the subjective probability of audit is jointly defined firstly by the *overall perceived effort put into auditing* by the tax authorities and secondly by the *perceived magnitude of tax evasion* in the population.

As far as the former is concerned, let  $\varepsilon$  denote the effort put into auditing by the tax authorities. We can think of  $\varepsilon$  being the ratio of the spending channeled to audits over the total government spending towards tax authorities, i.e. the net expenditure ratio financing the audit mechanism. Thus, it holds that  $0 \leq \varepsilon \leq 1$ , since it constitutes a ratio, where the lower bound, 0, and the upper bound, 1, can never be truly achieved in reality, since they imply two extreme cases of either no audit spending or full audit spending, respectively, that can never be applied in reality. Although the two boundary values are limiting cases without further qualification, they serve so that effort has desirable probability-like characteristics and in the formation of the control variable constraints in the optimization problem below.

As far as the latter is concerned, the ratio of non-compliers in the population, i.e.  $x \in [0, 1]$ , represents the relative magnitude of tax evasion in the population and can easily be inferred by each agent through public relations and social interaction in general. Notice that the taxpayer has no knowledge of the true values of  $\varepsilon$  and  $x$ , instead she infers a value for each one of them; let these subjective values be denoted by  $(\tilde{\varepsilon}, \tilde{x})$ . These values are then entered as parameters in the class of probability distributions of audit, identifying the true function of subjective belief of being audited. In other words,

$\tilde{\varepsilon}$  and  $\tilde{x}$  must satisfy that:

$$\begin{aligned}\tilde{\varepsilon} &= \varepsilon(\cdot, x) + v_1(\omega) \\ \tilde{x} &= x(\cdot, \varepsilon) + v_2(\omega)\end{aligned}$$

The perceived values of  $\varepsilon$  and  $x$ , i.e.  $\tilde{\varepsilon}$  and  $\tilde{x}$ , are a function of the true values of  $\varepsilon$  and  $x$  plus random shocks  $v_1(\omega)$  and  $v_2(\omega)$  on the sample space, respectively. Note that, the true values of  $\varepsilon$  and  $x$  depend on certain factors as well as on each other, but this has no impact on the form of the subjective probability of audit whatsoever, since it incorporates only the parametric values  $\tilde{\varepsilon}$  and  $\tilde{x}$  in order to measure the audit chance. Both random shocks  $v_1(\omega)$  and  $v_2(\omega)$  are considered to be zero-mean stationary random variables. We can now define the endogenized subjective probability of audit to be a distribution function measuring the probability:

$$p(\text{"I get audited"}; \tilde{\varepsilon}, \tilde{x})$$

which will be denoted for simplicity by  $p(\varepsilon, x)$ , satisfying the property that:

$$0 \leq p(\varepsilon, x) \leq 1$$

Thus, we get a parameterized, by  $\tilde{\varepsilon}$  and  $\tilde{x}$ , specific probability distribution attributing values from 0 to 1 to the event that the agent is audited. Substituting the endogenized probability back to (8), we obtain the replicator dynamics equation with endogenized subjective probability, i.e.:

$$\dot{x} = x(1-x) [(1-p(\varepsilon, x))U_{NU} + p(\varepsilon, x)U_{NA} - U_C]. \quad (9)$$

Notice that with the endogenized probability we may obtain a feasible interior steady state solution for the replicator dynamics equation. In its original form, as depicted in (8), the only candidate solutions for a steady state ( $x; \dot{x} = 0$ ) were the corner solutions,

i.e.  $x = 0$  and  $x = 1$ , being interpreted as full compliance or full evasion in the population, respectively. This was due to the fact that all the terms in the bracketed section were independent of  $x$ .

### 2.2.1 Monomorphic vs. Polymorphic Behavior

The replicator dynamics equation as a differential equation, implies some dynamical behavior for  $x$ , i.e. the fraction of non-compliers in the population. Taking equation (8) as a benchmark, where probability is still considered exogenous, we can see that the bracketed section is independent of  $x$ . Thus, we can rewrite it as:

$$\dot{x} = x(1-x)K \quad (10)$$

where  $K = [(1-p)U_{NU} + pU_{NA} - U_C]$ , and can be viewed as a scaling constant, as far as the differential equation is concerned. It is clear that  $\dot{x}$  is a parabola, the convexity (curvature) of which depends on the sign of  $K$ . The critical points of (10), i.e. the ones for which it holds that  $\dot{x} = 0$ , are  $x_1^* = 0$ ,  $x_2^* = 1$ , or  $K = 0$ , which is trivial. Differentiating  $\dot{x}$  with respect to  $x$  we obtain that:

$$\frac{\partial \dot{x}}{\partial x} = K(1-2x). \quad (11)$$

Computing (11) in each steady state, i.e.  $x_1^* = 0$  and  $x_2^* = 1$ , we derive the dynamical behavior of the steady states of (10), which is summarized in Table 1:

$\frac{\partial \dot{x}}{\partial x} \Big _{x_1^*=0} = K,$	for $x_1^* = 0$ to be stable, $K < 0$
$\frac{\partial \dot{x}}{\partial x} \Big _{x_2^*=1} = -K,$	for $x_2^* = 1$ to be stable, $K > 0$
$\frac{\partial \dot{x}}{\partial x} \Big _{K=0} = 0,$	all levels of $x$ satisfy stability (not desirable)

-Table 1-

Notice that in the benchmark case with exogenous probability of audit we only get two corner solutions, i.e.  $x_1^* = 0$  and  $x_2^* = 1$ , and one trivial solution that consists of all interior  $x$ 's. Hence it describes a monomorphic behavior and lead to monomorphic phase

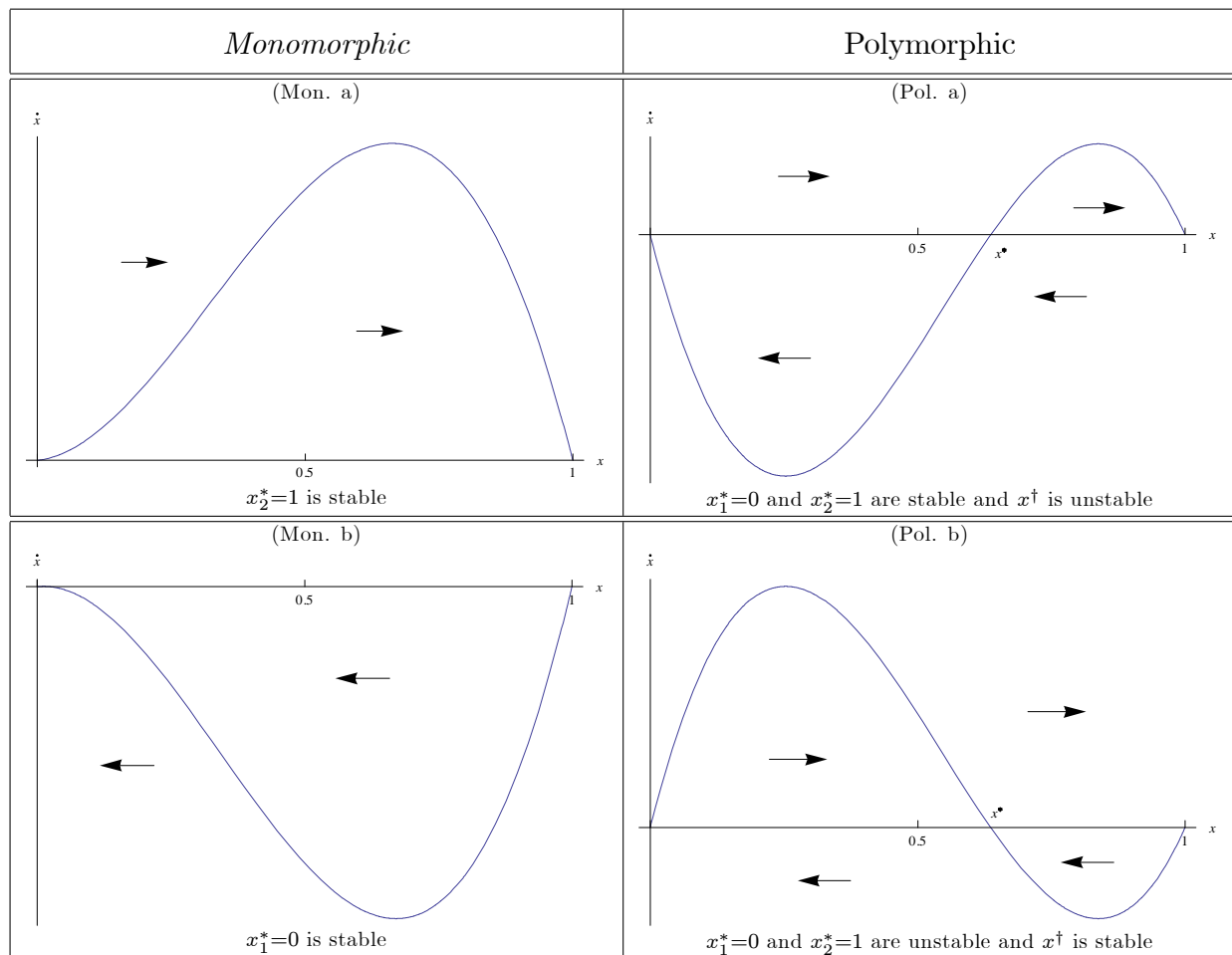
diagrams, in the sense that there are no transitional dynamics, but only a jump to the stable point. It is obvious that any tax authority would want to achieve a stable  $x_1^* = 0$  state, which means that all agents become complying and fully pay their tax responsibilities. In the benchmark environment this implies that  $K$  must be negative, which in turn means that  $U_C > (1 - p)U_{NU} + pU_{NA}$  at all times; a condition that makes sense, since switching to the "good" strategy means that it must score higher in terms of payoff than the "bad" strategy.

On the other hand, by endogenizing the subjective probability of audit we manage to transform  $K$  into a function of  $x$  through  $p$ , i.e.  $K : K(p(\varepsilon, x))$ . Thus, for the replicator dynamics equation (9), steady states, as before, are defined by  $\dot{x} = 0$  as  $x_1^* = 0$ ,  $x_2^* = 1$ , and, assuming that an interior  $x^\dagger$  exists, that satisfies the following relationship:

$$x^\dagger : (1 - p(\varepsilon, x^\dagger))U_{NU} + p(\varepsilon, x^\dagger)U_{NA} - U_C = 0. \quad (12)$$

To pinpoint the interior  $x^\dagger$  satisfying (12), one must assume a specific form for the joint cumulative distribution function  $p(\varepsilon, x)$ . Assuming that this interior  $x^*$  exists, we get a polymorphic behavior depending on the sign of the derivatives of  $\dot{x}$  calculated at each steady state  $x_i^* = x_1^*, x_2^*, x^\dagger$ , i.e.  $\frac{\partial \dot{x}}{\partial x} \Big|_{x=x_i^*} \leq 0$ . All the possible variations for both monomorphic and polymorphic (with *one* interior  $x^*$ ) phase diagrams are summarized in

Figure 1 below:



-Figure 1-

Notice that for the monomorphic case we can attain a stable full compliance level at once, if the dynamics permit it, whereas for the polymorphic case of one interior steady state, full compliance and full non-compliance are both present in the same contingency. In (Pol. a) a small perturbation from the unstable interior steady state  $x^\dagger$  leads to either total conformity or total non-compliance. The desirable polymorphic case is, therefore, the one that is characterized by unstable corner solutions and a steady interior  $x^\dagger$ , i.e. the one depicted in (Pol. b). This is also the most realistic of all four contingencies, not mentioning cases with multiple interior steady states that outperform all of the above.



## 2.3 Primary Deficit and Tax Evasion

We assume that the economy has a single regulating agent in lieu of the government, in the sense that she has no other responsibility than the planning, enforcement and optimization with respect to resource and policy instrument constraints, towards a certain goal. The regulator is aware of the behavior of the agents and has full knowledge over the whole system that describes it. The objective function towards which policy is set can vary depending on the ultimate goals of the regulator. These goals could regard spending, welfare, the monetary or fiscal sector, trade etc.; they could even concern idiosyncratic objectives, e.g. reelection. In our setup, the regulator will be a fiscal policy regulator, who sets the target for the primary deficit of the economy. The primary deficit consideration offers two major advantages.

Firstly, it incorporates, as we shall see below, the two channels that the agents use to form their subjective probability of audit, i.e. perceived effort of audit,  $\varepsilon$ , and the ratio of evaders in the population,  $x$ . That is mainly because the primary deficit is a publicly available information enabling agents to make a forecast about the stringency of the government, since any change in the primary deficit is mainly attributed to fluctuations in the tax revenues, which can be interpreted as a corresponding change in  $x$ . Secondly, it can be considered as a more straightforward approach than a generic social welfare target, since it is a measure that is also used in practice in order to assess the effectiveness of fiscal policy and determine necessary changes in policy instruments.

Let  $D(t) = G(t) - T^e(t)$  denote the primary deficit at time  $t$ , where  $G$  is the government spending and  $T^e$  are the expected tax revenues. Government spending can be regarded as constant over time, i.e.  $\dot{G} = 0$ , so that the rate of change of the primary deficit is fully attributed to the change of the tax revenues.

Notice that,  $T^e$  implies total *net* expected tax revenues of the government, since the effort put into audit is costly for the government. Let  $c(\varepsilon)$  be the cost function of effort, faced by the tax authorities, with properties

$$c' > 0 \text{ and } c'' \geq 0.$$

We also define a function  $\phi(\varepsilon)$ , denoting the ratio of the population to be randomly drawn and audited per period. Notice that  $\phi(\varepsilon)$  is deterministic as it is the regulator's choice, but it resembles to a probability measure, since it constitutes a ratio, and will be treated as such, with

$$\begin{aligned}\phi(\varepsilon) &\geq 0, \phi'(\varepsilon) > 0 \\ \phi(0) &= 0, \phi(\varepsilon^{\max}) = 1, \text{ where } \varepsilon^{\max} = 1 \\ \phi'(0) &> 0 \text{ and } \phi'(\varepsilon^{\max}) \simeq 0.\end{aligned}$$

For notation simplicity, let  $\phi \equiv \phi(\varepsilon)$  and  $\Delta Y \equiv Y - Y_R$ .

In order to compute  $T^e$ , it is wiser to split agents, once again, into evaders and compliers. Both groups face a probability to be audited equal to  $\phi$ , thus, a ratio  $\phi$  of the population is indeed audited and the remaining  $(1 - \phi)$  remains unaudited. For both audited and unaudited agents,  $x$  of them evade and  $(1 - x)$  of them comply. The table below, summarizes the payoffs for each contingency:

	<i>Audited, <math>\phi</math></i>	<i>Unaudited, <math>(1 - \phi)</math></i>
<i>Evaders, <math>x</math></i>	$tY_R + t\Delta Y + \theta t\Delta Y$	$tY_R$
<i>Compliers, <math>(1 - x)</math></i>	$tY$	$tY$

Thus, the *audited* population pays a total of

$$T_A = \phi [x (tY_R + t\Delta Y + \theta t\Delta Y) + (1 - x) tY],$$

whereas the *unaudited* population has a tax collection of

$$T_U = (1 - \phi) [xtY_R + (1 - x) tY].$$

Therefore, total net expected tax revenues  $T^e$ , is the sum of audited plus unaudited tax

collections, net of auditing costs:

$$\begin{aligned}
T^e &= T_A + T_U - c(\varepsilon) = \\
&= \phi [x(tY_R + t\Delta Y + \theta t\Delta Y) + (1-x)tY] + (1-\phi) [xtY_R + (1-x)tY] - c(\varepsilon) = \\
&= \underbrace{(1-x)tY}_{\mathcal{A}} + x \left( \underbrace{tY_R}_{\mathfrak{b}_1} + \underbrace{\phi(1+\theta)t\Delta Y}_{\mathfrak{b}_2} \right) - c(\varepsilon). \tag{13}
\end{aligned}$$

The intuition behind this result is that the government earns a part of the tax revenues from the complying agents, i.e. the  $(1-x)$  share of the population, who fully pay their tax responsibilities  $tY$ , as indicated by part  $\mathcal{A}$  of (13), and the rest comes from the non-complying agents, namely, the  $x$  share of the population, as in part  $\mathcal{B}$  of (13). Notice that all evading agents pay part  $\mathfrak{b}_1$  of  $\mathcal{B}$ , that is to say, the tax on understated income  $tY_R$ , while the remaining  $\mathfrak{b}_2$  part of  $\mathcal{B}$  is the  $\phi$  share of them that gets caught and is fined by the amount  $(1+\theta)t\Delta Y$ , which consists of the evaded tax  $t\Delta Y$  and the penalty on evaded tax  $\theta t\Delta Y$ . Subtracting the cost of effort  $c(\varepsilon)$  from the summation gives the total net expected tax revenues. Notice that the magnitude of the tax revenues  $T^e$  can only be known to the regulator and not the agents, since she is the only one to know the realized ratio of agents to be audited in the population,  $\phi$ .

### 3 Regulating Tax Evasion

In a standard optimization setup, the regulator would aim to control for tax evasion towards a certain goal, with respect to the behavior of agents as described by equation (9). Lets suppose, at first, and setting it as a benchmark case, that the regulator can perform an unconditional regulation, being able to tamper with the replicator dynamics equation without having to worry about costs or specific goals.

### 3.1 A Benchmark Case: Controlling Replicator Dynamics

Imagine that the regulator sees only equation (9), i.e. the replicator dynamics equation, and is indifferent towards other kinds of goals and/or restrictions:

$$\dot{x} = x(1-x)[(1-p(\varepsilon, x))U_{NU} + p(\varepsilon, x)U_{NA} - U_C].$$

Remember also, that:

$$\begin{aligned} U_{NU} &= U(\Upsilon + t\Delta Y), \\ U_{NA} &= U(\Upsilon - \theta t\Delta Y) \text{ and} \\ U_C &= U(\Upsilon), \end{aligned}$$

where  $\Upsilon = Y - tY$  is the true after tax income and  $\Delta Y = Y - Y_R$  is the evaded income.

The regulator's goal immediately becomes to make the fraction of non-compliers,  $x$ , in the population reach zero and make this point, i.e.  $x = 0$ , a stable steady state. With this being a first-best outcome in terms of efficiency of tax regulation, the second-best outcome would be to somehow drive the system to the lowest nonzero  $x$  possible, since  $x = 0$  is unattainable, and make that one a stable critical point. Notice that in the benchmark case, effort is costless, hence the regulator can put into as much as is required in order to attain the desirable result.

The channels through which the regulator can affect the replicator dynamics equation are the three policy instruments at her disposal, namely the effort  $\varepsilon$ , the tax rate  $t$  and the penalty rate  $\theta$ . We will see what can be achieved in terms of compliance when the regulator uses each policy instrument, unconditionally and to their full extent, implying that she uses one instrument at a time while the others remain fixed to a certain value (*ceteris paribus*). In order to be able to proceed, we must impose specific functional forms for the von Neumann–Morgenstern utility function describing the preferences of agents towards risk  $U(\cdot)$ , as well as the distribution function describing the subjective probability of audit,  $p(\varepsilon, x)$ . Since agents are treated as risk-averse, we choose one of

the most widely used utility functions exhibiting constant relative risk aversion, i.e. the isoelastic utility function:

$$U(w) = \frac{w^{1-\rho} - 1}{1-\rho},$$

with  $\rho > 0$  measuring the degree of risk aversion. Notice that, when  $\rho = 1$ , using l'Hôpital's rule, we obtain that  $\lim_{\rho \rightarrow 1} \frac{w^{1-\rho} - 1}{1-\rho} = \log(w)$ , as a special case. For all  $\rho \neq 1$ , the constant term "-1" in the numerator will be omitted, since optimal decisions are not affected by additive constant terms.

As far as the subjective probability of audit,  $p(\varepsilon, x)$  is concerned, since both  $\varepsilon$  and  $x$  take values in  $[0, 1]$ , we can use any relationship that satisfies that  $p(\varepsilon, x)$  will also range between  $[0, 1]$ . Examples may include, among others:

- 1)  $p(\varepsilon, x) = \varepsilon x$ ,
- 2)  $p(\varepsilon, x) = \varepsilon(1 - x)$ ,
- 3)  $p(\varepsilon, x) = z\varepsilon + (1 - z)x$ , with  $0 \leq z \leq 1$ ,
- 4)  $z\varepsilon + (1 - z)(1 - x)$ , with  $0 \leq z \leq 1$ ,

In all cases it is realistic to assume a positive relationship between  $p(\varepsilon, x)$  and  $\varepsilon$ , since higher perceived stringency causes a risk averse taxpayer to believe that the odds of being audited increase. This statement, though, is not always true as far as the relation between  $p(\varepsilon, x)$  and  $x$  is concerned. Observe that in cases 1) and 3) above, there is a positive relation between  $p(\varepsilon, x)$  and  $x$  which implies that a rise in the perceived ratio of evaders causes the agent to strengthen her belief that she might get audited, which in turn can be attributed to the fact that higher  $x$  is indeed the case and the tax authorities will increase effort to compensate for that rise in evasion. On the other hand, in cases 2) and 4) we have the exact opposite logic, implying that a perceived rise in  $x$  by the agent leads her to the conclusion that the tax authorities cannot deal with tax evasion efficiently and will not be able to do so in the near future as well, which in turn makes her infer that the probability of audit falls in the sense that it has become overwhelmingly costly for the government to increase audits. We will be using cases 1) and 2) and see the implications below. Incorporating the above in the replicator dynamics equation we end up with:

Case 1):

$$\dot{x}_1 = x(1-x) \left[ (1-\varepsilon x) \frac{(\Upsilon + t\Delta Y)^{1-\rho}}{1-\rho} + \varepsilon x \frac{(\Upsilon - \theta t\Delta Y)^{1-\rho}}{1-\rho} - \frac{(\Upsilon)^{1-\rho}}{1-\rho} \right] \quad (14)$$

Case 2):

$$\dot{x}_2 = x(1-x) \left[ (1-\varepsilon(1-x)) \frac{(\Upsilon + t\Delta Y)^{1-\rho}}{1-\rho} + \varepsilon(1-x) \frac{(\Upsilon - \theta t\Delta Y)^{1-\rho}}{1-\rho} - \frac{(\Upsilon)^{1-\rho}}{1-\rho} \right] \quad (15)$$

Note that the dynamical behavior of the special case where  $U(w) = \log(w)$ , i.e.  $\rho = 1$  is almost identical<sup>5</sup> to the general case and that all results hold for both types of utility functions.

The steady states of the replicator dynamics equations (14) and (15) are the solutions to  $\dot{x}_1 = 0$  and  $\dot{x}_2 = 0$  and include the corner solutions  $x_1^* = 0$  and  $x_2^* = 1$  as well as the interior  $x^\dagger$  which satisfies that the bracketed section of each equation is zero. For the first case, i.e. when  $p(\varepsilon, x) = \varepsilon x$ , and after making substitutions in order to avoid complex notation while providing a formula that applies to all cases, the interior  $x_1^\dagger$  is given by:

$$x_1^\dagger = \frac{U_{NU} - U_C}{\varepsilon(U_{NU} - U_{NA})}$$

By analogy, the interior steady state,  $x_2^\dagger$ , for the second case, i.e. when  $p(\varepsilon, x) = \varepsilon(1-x)$ , is given by:

$$x_2^\dagger = 1 - \frac{U_{NU} - U_C}{\varepsilon(U_{NU} - U_{NA})} = 1 - x_1^\dagger$$

While the fact that  $x_2^\dagger = 1 - x_1^\dagger$  implies a completely inverse trajectory of  $x_2^\dagger$  in comparison to  $x_1^\dagger$  they both share something in common. While effort level  $\varepsilon$  and penalty level  $\theta$  have a seemingly reverse effect on  $x_1^\dagger$  and  $x_2^\dagger$ , because they only appear positively in the denominator, the tax rate,  $t$ , has an ambiguous effect making the interior steady

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<sup>5</sup>The only thing that changes in the case where  $U(w) = \log(w)$  is that for the second case where  $p(\varepsilon, x) = \varepsilon(1-x)$ , the interior solution  $x_2^\dagger$  could be a complex number for some parametric values of the tax rate  $t$ .

states  $x_1^\dagger$  and  $x_2^\dagger$  behave very quirky for overly high levels of  $t$ . This is mainly attributed to the fact that as  $t$  reaches unsustainable levels ( $t \rightarrow 1$ ), the compliers' utility goes to zero, since the true after tax income reaches zero ( $\Upsilon \rightarrow 0$ ) and the evaders gain a complete advantage, rendering a strategy switch towards evasion perfectly profitable and spreading it in the population.

In order to characterize the steady states in terms of stability we need to compute the derivative with respect to  $x$  in each steady state  $x_i$ , i.e.:

$$\left. \frac{\partial \dot{x}}{\partial x} \right|_{x=x_i}$$

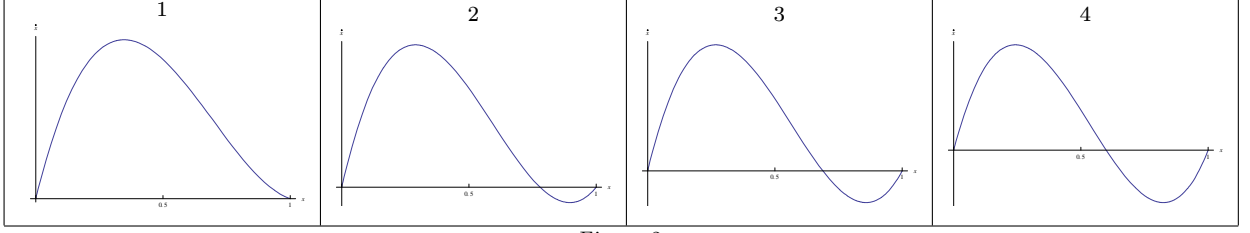
For the first case, where  $p(\varepsilon, x) = \varepsilon x$ , we have that:

$$\begin{aligned} \left. \frac{\partial \dot{x}_1}{\partial x} \right|_{x=0} &= U_{NU} - U_C &> 0 \\ \left. \frac{\partial \dot{x}_1}{\partial x} \right|_{x=1} &= U_C - U_{NA} - (1 - \varepsilon) U_{NU} &\leq 0, > 0 \text{ as } \varepsilon \text{ and/or } \theta \text{ rise} \\ \left. \frac{\partial \dot{x}_1}{\partial x} \right|_{x=x_1^\dagger} &= \frac{(U_{NU} - U_C)(\varepsilon U_{NA} - U_C + (1 - \varepsilon) U_{NU})}{\varepsilon(U_{NU} - U_{NA})} &\leq 0, < 0 \text{ as } \varepsilon \text{ and/or } \theta \text{ rise} \end{aligned}$$

This shows that as long as effort (or penalty) remains low and close to zero we have a phase diagram as depicted in Figure 1 (Mon. a), which implies full evasion, and as effort (or penalty) rises we get a smooth transition to a state as described by Figure 1 (Pol. b), with a stable interior  $x_1^\dagger$  which steadily decreases towards zero. Notice that we say nothing about the behavior of the phase diagram with respect to the tax rate because we get ambiguous results when it exceeds a specific level, which is by no means sustainable in an economy whatsoever ( $t > 0.7$ ). Up to that level, the transition is exactly the same as in  $\varepsilon$  and  $\theta$ , but when it passes beyond this threshold we get an inverse smooth yet rapid transition back to the initial monomorphic full evasion situation. This is attributed to the fact mentioned above, i.e. the evasion strategy gains in terms of payoffs.

Below, Figure 2 depicts frames [1-4] of this smooth transition of the phase diagram as  $\varepsilon$  (or  $\theta$ ) go from  $0 \rightarrow 1$ , ceteris paribus. For the case of the tax rate, as  $t$  grows, imagine moving from frame 1 through 4 as before, but after  $t$  reaches the critical threshold the

transition reverses following the path from frame 4 back to 1 in a more hasty pace.



-Figure 2-

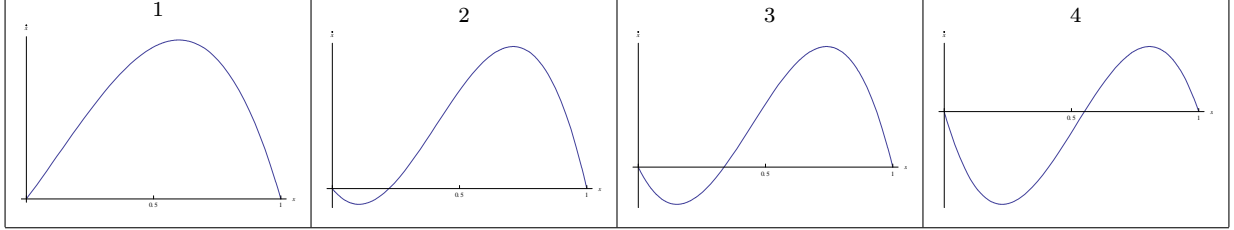
For the second case, where  $p(\varepsilon, x) = \varepsilon(1 - x)$ , we have that:

$$\begin{aligned} \frac{\partial \dot{x}_2}{\partial x} \Big|_{x=0} &= \varepsilon U_{NA} - U_C + (1 - \varepsilon) U_{NU} \leq 0, < 0 \text{ as } \varepsilon \text{ and/or } \theta \text{ rise} \\ \frac{\partial \dot{x}_2}{\partial x} \Big|_{x=1} &= U_C - U_{NU} < 0 \\ \frac{\partial \dot{x}_2}{\partial x} \Big|_{x=x_2^\dagger} &= \frac{(U_{NU} - U_C)(U_C - \varepsilon U_{NA} - (1 - \varepsilon) U_{NU})}{\varepsilon(U_{NU} - U_{NA})} \leq 0, > 0 \text{ as } \varepsilon \text{ and/or } \theta \text{ rise} \end{aligned}$$

This in turn, shows that as long as effort (or penalty) remains low and close to zero we have a phase diagram as depicted in Figure 1 (Mon. a), which implies full evasion, and as effort (or penalty) rises we get a smooth transition to a state as described by Figure 1 (Pol. a), described by an unstable interior  $x_2^\dagger$  which steadily increases towards one. Here we get exactly the opposite behavior than in the first case. Notice that this result is counter-intuitive since a rise in  $\varepsilon$  and  $\theta$  leads to higher levels of evasion. This, of course, is due to the fact that we have assumed that the taxpayers perceive large levels of evasion, i.e. large  $x$ , as a signal that audit chance lowers due to ineffective effort, and thus the system eventually makes evasion grow. Nevertheless, remember that in the (Pol. a) case of Figure 1, we have an unstable interior steady state  $x_2^\dagger$ , which means that  $x_2^\dagger$  will only hold in the short-run, as a small perturbation of this fragile (unstable) steady state will lead the economy to either full compliance ( $x_1^* = 0$ ) or full evasion ( $x_2^* = 1$ ). The regulator would surely want to be able to take advantage of this and lead the system to full compliance, which, after all, is the first-best outcome, but this is as likely as the full evasion scenario. The response of this case to changes in the policy instruments is depicted below, in Figure 3, following the same notion as in Figure 2. Moreover, the phase diagram's behavior to changes in the tax rate,  $t$ , follow the same pattern as in the first case, i.e. increasing tax rate until a certain level causes evasion to grow, as do



changes in  $\varepsilon$  and  $\theta$  [Frames 1  $\rightarrow$  4, Figure 3], but after that level is surpassed evasion falls back to zero [Frames 4  $\rightarrow$  1, Figure 3]; nevertheless this still means full evasion since Frame 1 describes a stable  $x_2^* = 1$  steady state.



-Figure 3-

One can easily conclude that the regulator would prefer to have a population of agents that form a subjective belief of audit similar to the first case, i.e. where  $p(\varepsilon, x) = \varepsilon x$ , so that accrements in the levels of policy instruments may have a reductional effect on evasion, rendering every interior  $x^\dagger$  stable and thus feasible. This means that the regulator expects that agents view a rise in evasion around them, not as an opportunity for them to evade as well, but rather as a sign of impending rise in the effort for audit by the tax authorities. Same results also hold for parameterization with weighted cases of behavior of agents, e.g.  $p(\varepsilon, x) = z\varepsilon x + (1 - z)\varepsilon(1 - x)$ , with  $0 \leq z \leq 1$ , depending on which behavior gets the biggest weight.

### 3.2 Optimal Regulation of Tax Evasion

We now suppose that the regulator wants to optimally control for the policy instruments with respect to the behavior of agents as described by the replicator dynamics equation. We consider only one policy instrument and more specifically effort,  $\varepsilon$ , as it generates tractable results. Recall that the regulator wants to set a goal for the primary deficit equation, i.e.

$$D(x, \varepsilon, t) = G(t) - T^e(x, \varepsilon, t).$$

More specifically, we will assume that her objective will be to minimize the square deviations from a target primary deficit,  $\bar{D}$ , plus any costs this minimization incurs. Moreover,

we consider this to be an infinite horizon problem<sup>6</sup>, so that it can be rendered autonomous and the time dimension enters only through the discount term,  $r$ .

Consider, where  $t$  here denotes time:

$$\min_{\varepsilon} \int_0^{\infty} e^{-rt} \left[ \frac{1}{2} (D(x, \varepsilon) - \bar{D})^2 + c(\varepsilon) \right] dt, \quad (16)$$

subject to the replicator dynamics equation

$$\dot{x} = x(1-x) [(1-p(\varepsilon, x))U_{NU} + p(\varepsilon, x)U_{NA} - U_C]. \quad (17)$$

The current value Hamiltonian for problem (16)-(17) is:

$$H(x, \varepsilon, \lambda) = f(x, \varepsilon) + \lambda g(x, \varepsilon). \quad (18)$$

where

$$f(x, \varepsilon) = \frac{1}{2} (D(x, \varepsilon) - \bar{D})^2 + c(\varepsilon),$$

$$g(x, \varepsilon) = \dot{x} = H_{\lambda},$$

$\lambda$  is the costate variable.

At this point it is worth noting that we also need to bound both the control variable as well as the state variable<sup>7</sup>, i.e. we need:

$$\begin{aligned} 0 \leq \varepsilon \leq 1 \quad \text{and} \\ 0 \leq x \leq 1. \end{aligned} \quad (19)$$

which constitutes another constraint. Thus, the problem becomes to minimize (18),

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<sup>6</sup>See Kamien & Schwartz (1991) and Hirsch *et al.* (1974)

<sup>7</sup>Notice that as far as the state variable  $x$  is concerned, the boundary values, i.e.  $x = 0$  and  $x = 1$  are always attained as steady states, for some corresponding levels of  $\lambda$ , due to the nature of the replicator dynamics equation, which involves the term  $x(1-x)$ .

subject to (19), i.e.:

$$\begin{aligned} \min_{\varepsilon} H(x, \varepsilon, \lambda) &= f(x, \varepsilon) + \lambda g(x, \varepsilon), \\ \text{subject to } 0 &\leq \varepsilon \leq 1 \text{ and } 0 \leq x \leq 1. \end{aligned} \quad (20)$$

By appending the constraints for the control and state variables to the objectives with multipliers  $w_1, w_2, \mu_1$  and  $\mu_2$ , we form the Lagrangian for (20):

$$L(x, \varepsilon, \lambda) = f(x, \varepsilon) + \lambda g(x, \varepsilon) + w_1(1 - \varepsilon) + w_2\varepsilon + \mu_1(1 - x) + \mu_2x. \quad (21)$$

The necessary conditions for a constrained minimum with respect to  $\varepsilon$  satisfy:

$$L_{\varepsilon} = f_{\varepsilon} + \lambda g_{\varepsilon} - w_1 + w_2 = 0, \quad (22)$$

$$w_1 \geq 0, \quad w_1(1 - \varepsilon) = 0, \quad (23)$$

$$w_2 \geq 0, \quad w_2\varepsilon = 0, \quad (24)$$

$$\dot{\lambda} = r\lambda - H_x = r\lambda - (f_x + \lambda g_x - \mu_1 + \mu_2), \quad (25)$$

$$\mu_1 \geq 0, \quad \mu_1(1 - x) = 0, \quad (26)$$

$$\mu_2 \geq 0, \quad \mu_2x = 0. \quad (27)$$

with conditions such as:

- 1) if  $\varepsilon^* = 0$ , then  $1 - \varepsilon^* > 0$ , so (23) requires  $w_1 = 0$ , thus:  
from (22),  $f_{\varepsilon} + \lambda g_{\varepsilon} \leq 0$  and  $f_{\varepsilon} + \lambda g_{\varepsilon} + w_2 = 0$ .
- 2) if  $\varepsilon^* = 1$ , then  $\varepsilon^* > 0$ , so (24) requires  $w_2 = 0$ , thus:  
from (22),  $f_{\varepsilon} + \lambda g_{\varepsilon} \geq 0$  and  $f_{\varepsilon} + \lambda g_{\varepsilon} - w_1 = 0$ .
- 3) if  $0 \leq \varepsilon^* \leq 1$ , the system reduces to the one in equation (18).

The logic for the state control conditions is analogue to that of the control variable, i.e.:

- 1) if  $x^* = 0$ , then  $1 - x^* > 0$ , so (26) requires  $\mu_1 = 0$ , thus:  
from (25),  $\dot{\lambda} = r\lambda - (f_x + \lambda g_x + \mu_2)$ .
- 2) if  $x^* = 1$ , then  $x^* > 0$ , so (27) requires  $\mu_2 = 0$ , thus:  
from (25),  $\dot{\lambda} = r\lambda - (f_x + \lambda g_x - \mu_1)$ .
- 3) if  $0 \leq x^* \leq 1$ , the system reduces to the one in equation (18).

Since  $\varepsilon$  minimizes (18), we have the first and second order necessary conditions:

$$H_\varepsilon = f_\varepsilon(x, \varepsilon) + \lambda g_\varepsilon(x, \varepsilon) = 0, \quad (28)$$

$$H_{\varepsilon\varepsilon} \leq 0. \quad (29)$$

Furthermore,

$$\dot{\lambda} = r\lambda - H_x = r\lambda - f_x - \lambda g_x. \quad (30)$$

Through (28) we can get the optimal  $\varepsilon^* \in (0, 1)$  for  $0 \leq x \leq 1$ , as a function of the state and costate variables and the other two policy instruments, i.e. the tax rate  $t$  and the penalty rate  $\theta$ , entering as parameters:

$$\varepsilon^* = E(x, \lambda; t, \theta). \quad (31)$$

For the properties of the control,  $\varepsilon$ , by totally differentiating (28) and (31) we get that:

$$dH_\varepsilon = H_{\varepsilon\varepsilon}d\varepsilon + H_{\varepsilon x}dx + g_\varepsilon d\lambda = 0. \quad (32)$$

From (32) we have:

$$d\varepsilon = -\frac{H_{\varepsilon x}}{H_{\varepsilon\varepsilon}}dx - \frac{g_\varepsilon}{H_{\varepsilon\varepsilon}}d\lambda, \quad (33)$$

while from (31):  $d\varepsilon = E_x dx + E_\lambda d\lambda$ , hence from (33):

$$E_x = -\frac{H_{\varepsilon x}}{H_{\varepsilon\varepsilon}}, \quad E_\lambda = -\frac{g_\varepsilon}{H_{\varepsilon\varepsilon}}. \quad (34)$$

Substituting from (31) into (17) and (30), we get the Hamiltonian system:

$$\begin{aligned}\dot{x} &= g(x, \varepsilon^*), \\ \dot{\lambda} &= r\lambda - H_x(x, \varepsilon^*, \lambda).\end{aligned}\tag{35}$$

A steady state  $(x_s, \lambda_s)$ , if it exists, satisfies:

$$g(x, \varepsilon^*) = 0,\tag{36}$$

$$r\lambda - H_x(x, \varepsilon^*, \lambda) = 0.\tag{37}$$

In order to describe the dynamical behavior of the steady states of the form  $(x_s, \lambda_s)$ , for which  $\varepsilon^* = E(x_s, \lambda_s; t, \theta)$  is the solution of (35), we study the linear differential equation system that approximates (36) and (37) at  $x_s, \lambda_s$ .

In our setup, we use the following simplifying assumptions in order to obtain analytical results:

- $p(\varepsilon, x) = \varepsilon x$ , since this type of reaction, as shown above, exhibits satisfying dynamical behavior. Note that the case where  $p(\varepsilon, x) = z\varepsilon + (1 - z)x$ , with  $0 \leq z \leq 1$ , also provides tractable results.
- $U(w) = \log(w)$ , as a special case. This is only done for tractability. The general case of the isoelastic function also provides tractable but more complex results.
- $c(\varepsilon) = c\varepsilon$ , a linear cost function. The case where  $c(\varepsilon) = \frac{1}{2}c\varepsilon^2$  is indisputably a better choice, but analytical solutions could not be attained.
- $\phi(\varepsilon) = \varepsilon$ , which means that the ratio of the population to be audited is proportional to the effort put into audit. This can be scaled by any constant, including the cost constant  $c$ , or take other plausible forms without affecting the general behavior of the system. The best  $\phi(\varepsilon)$  formulation would be a sigmoid type as in  $\phi(\varepsilon) = 1 - \exp(-\varepsilon)$ , but this renders the model unsolvable due to high non-linearity.

- $\bar{D} = 0$  without any loss of generality, i.e. setting a balanced budget goal from which to minimize deviations.

The solution to (18) for the optimal  $\varepsilon^*$  is given by:

$$\varepsilon^* = \frac{(A)(B) - c + (D)}{(A)^2}, \quad (38)$$

where

$$\begin{aligned} A &= xt(1 + \theta) \Delta Y - c, \\ B &= G - (1 - x)tY - xtY_R, \\ D &= \lambda x^2(1 - x)(U_{NU} - U_{NA}). \end{aligned}$$

The intuition behind  $\varepsilon^*$  is not so easy to describe, that is why we split it in parts to which an economic interpretation can be given:

- $A$ , for which it must hold that  $A \neq 0$ , denotes the *net* effective effort revenues, i.e. benefit of effort minus cost of effort. Notice that in  $A$ , the  $xt(1 + \theta) \Delta Y$  are the tax revenues from the audited evaders, whereas  $c$  is the cost for that audit.
- $B$  is the *certain* primary deficit. Certain in the sense that without any auditing at all, i.e.  $\phi(\varepsilon) = 0$ , the tax revenues would only come from the compliers, namely  $(1 - x)tY$  and from the underreporting evaders, i.e.  $xtY_R$ . Subtracting these two tax revenues from the social spending  $G$  gives us the certain primary deficit where no audits occurred.
- $D$  is a little bit more complicated. It can be said to describe the shadow price of remaining unaudited, measured in terms of a replicator dynamics equation inside evaders' strategy subpopulation.

Substituting  $\varepsilon^*$  from (38) into the Hamiltonian system (35) we can investigate for the dynamical behavior of the occurring steady states. To do so we must solve for the system

linearized around the steady state. The linearization matrix around any emerging steady state  $(x_s, \lambda_s)$ , i.e. the Jacobian of the Hamiltonian system, will be of the form:

$$J_{(x_s, \lambda_s)} = \left\{ \begin{array}{cc} \left. \frac{\partial \dot{x}(x, \varepsilon^*)}{\partial x} \right|_{(x_s, \lambda_s)} & \left. \frac{\partial \dot{x}(x, \varepsilon^*)}{\partial \lambda} \right|_{(x_s, \lambda_s)} \\ \left. \frac{\partial \dot{\lambda}(x, \varepsilon^*, \lambda)}{\partial x} \right|_{(x_s, \lambda_s)} & \left. \frac{\partial \dot{\lambda}(x, \varepsilon^*, \lambda)}{\partial \lambda} \right|_{(x_s, \lambda_s)} \end{array} \right\}$$

The dynamical behavior of each steady state depends on the properties of the characteristic roots (eigenvalues) of  $J_{(x_s, \lambda_s)}$ , denoted by  $r_1$  and  $r_2$ , for which it holds that:

$$tr(J) = r_1 + r_2 \text{ and}$$

$$|A| = r_1 r_2.$$

### 3.2.1 Numerical Approximation

Both the Hamiltonian system in (35), and its linearization matrix becomes very complicated to present and practically unintuitive. We give plausible values to all parameters in order to get an idea on how the system works.

The values used for parameterization attempt to simulate real relative magnitudes of data, some of which, for calculations facilitation are normalized to unity. This follows by the logic that the whole problem including the replicator dynamics equation, the net expected tax revenues equation and the objective function do not take into account the number of agents in the population, rather than a representative agent or, more rigorously, her mixed strategy of either to evade or not. Correspondingly, the level of public spending,  $G$ , will have a per-capita sense. Thus, global parameters will include:

$Y = 1$
$Y_R = 0.9$
$\bar{D} = 0$
$r = 0.02$

Remember that  $r$  is the rate of time preference, whereas  $\rho$  is the rate of risk aversion, if the isoelastic function is used. However, we will use the logarithmic utility case for  $\rho = 1^8$ . Tax evasion, if exhibited, is considered to be around 10% of total income in order to conform with data in petty crimes. Finally, we assume the regulator aims for a balanced budget, setting  $\bar{D} = 0$ . Before proceeding, it is worth noting that the results were very sensitive to the values of the cost of effort,  $c$ , which acts as a free parameter. However, it plays a crucial role in the numerical approximation, since it has to be set in a level that will make the regulation and the effort put in audits worth it, in the sense that the benefits of audits, i.e. the tax revenues, should always surpass the costs it implies. Since the cost of effort is linear,  $c(\varepsilon) = c\varepsilon$ , the constant  $c$  can be regarded as a measure of constant marginal (and average) cost of effort.

1. Benchmark case:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 1$ ,  $c = 0.014$

We set a plausible value  $G = 0.35$ , meaning that per-capita public spending is 35% the individual income,  $Y = 1$ . The per-capita public spending rate follows data concerning most OECD countries. Tax rate is 10% of income, following data on direct taxes as a percentage of GDP (Commission *et al.* (2009)), i.e. personal income taxes<sup>9</sup>. The penalty is set to 100% which is a typical penalty rate conditional on a 10% income evasion, without any loss of generality.

Optimization yields:

Steady State	Optimal Effort	Verdict
$x_1^* = 0$	$\varepsilon^* = -89.258$	constraint active: $\varepsilon^* = 0$
$x_2^* = 1$	$\varepsilon^* = -345.556$	constraint active: $\varepsilon^* = 0$
$x^\dagger = 0.698$	$\varepsilon^* = 0.711$	accepted

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<sup>8</sup>The reason we stick to the logarithmic utility is that it permits us to make a direct comparison of the results between the evolutionary and rational case, as shown in the next section. This is due to the fact that the general form of the isoelastic utility does not produce a closed form solution for  $Y_R$ , which is an indispensable element in the rational case. However, the logarithmic case does provide a closed form solution for the rational case and almost identical (both qualitatively and quantitatively) results for the evolutionary case.

<sup>9</sup>In general, GDP per capita is different than income per capita and should not be used as a proxy. However, data show that the ratio of income over GDP is very close to unity for the majority of OECD countries. The mean tax rate as a percentage of GDP is around 9% and since income/GDP is around 0.9 we settled with a *corrected* tax rate of 10%.



The interior steady state of  $x^\dagger = 0.698$ , i.e. 69.8% evasion is attained with an optimal level of effort of  $\varepsilon^* = 0.711$ , showing that a 71.1% of spending towards tax authorities will be channeled towards audits. Both corner solutions  $x_1^* = 0$  and  $x_2^* = 1$  activate the constraint on effort level setting it equal to zero, namely no effort will be put in audits at all when the system settles at the corner states. Remember that the planner's goal is to minimize the square deviations from a balanced budget, thus the effort exhibited in every point is optimal as far as spending is concerned. The intuition is that for both full evasion and full compliance the regulator understands that the best response is to stop funding the audit mechanism, since any effort will have no effect on tax revenues nor the level of evasion, at least local to the critical point attained by the economy.

Stability analysis implies that the steady state in  $x_1^* = 0$  behaves locally as an unstable node and both the steady state in  $x_2^* = 1$  and the interior steady state  $x^\dagger = 0.698$  behave as a saddle path<sup>10</sup>.

$$2. G = 0.35, t = 0.11, \theta = 1, c = 0.014$$

We change the tax rate to 11% in order to investigate a sensitivity on an increase in  $t$ . The new optimization yields:

Steady State	Optimal Effort	Verdict
$x_1^* = 0$	$\varepsilon^* = -88.571$	constraint active: $\varepsilon^* = 0$
$x_2^* = 1$	$\varepsilon^* = -187.375$	constraint active: $\varepsilon^* = 0$
$x^\dagger = 0.635$	$\varepsilon^* = 0.781$	accepted

A slight increase in the tax rate will lead to a fall in the level of evasion, as expected, and a slight increase in the effort put in audits, which can be attributed to the fact that an increased tax rate not only induces cost, through the increase of the evaded tax when the agent is a non-complier, but it also increases the penalty earned by the tax authorities if the agent is caught. This means that the increased tax rate has an ambiguous effect on optimal effort through two conflicting channels. The results here imply that the revenues

<sup>10</sup>As far as the corner solutions are concerned, their dynamical behavior remains the same in similar parameter sets, i.e. the  $x_1^* = 0$  always behaves as an unstable node and the  $x_2^* = 1$  always behaves as a saddle point. The interior steady state behaves as a saddle point as well, with some exceptions, in which it behaves as an unstable focus. See Appendix A for details.

from the increased tax rate make the rise in effort worth it in terms of revenues.

The stability properties of the steady states are the same as before.

3.  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 1.1$ ,  $c = 0.014$

We have reverted the tax rate back to 10% as in the benchmark case 1. and increased the penalty from 100% to 110% in order to test for the sensitivity on  $\theta$ .

The results are:

Steady State	Optimal Effort	Verdict
$x_1^* = 0$	$\varepsilon^* = -88.571$	constraint active: $\varepsilon^* = 0$
$x_2^* = 1$	$\varepsilon^* = -1010$	constraint active: $\varepsilon^* = 0$
$x^\dagger = 0.665$	$\varepsilon^* = 0.710$	accepted

An increase in the penalty rate leads to a fall in the level of evasion and in the level of effort put in audits as compared to the benchmark case. Both effects are expected and intuitively appealing, i.e. an increase in the penalty has no ambiguous effects on the preferences of the agent. It increases her cost of being or becoming a non-complier, thus leading to a fall in the level of evasion, and it also allows for a costless readjustment for the tax authorities, having managed to reduce evasion without changing the level of effort. This result is attributed by construction to the fact that the penalty rate affects only the evaders' utility and the revenue is earned only if the agent is an evader and gets caught.

The stability properties of the steady states still remain the same as in all of the above cases.

In all three cases, the corner solutions activated the control variable constraints. In the full compliance case, where  $x_1^* = 0$ , the optimal effort fluctuated around  $\varepsilon^* = -88$ , triggering the constraint which set it back to  $\varepsilon^* = 0$ . This means that when the system finds itself in no evasion environment, the tax authorities need not administer any funds in order to control for tax evasion, since it is not present at all. The same holds for the case of full non-compliance,  $x_2^* = 1$ , triggering again the constraint for  $\varepsilon^* = 0$ . This in turn means that in a full evasion scenario the tax authorities see that since the whole

population evades, the optimal policy is to earn the minimum tax revenues since any level of effort will not be cost effective<sup>11</sup>. Both an increase in the tax rate and the penalty rate lowered the ratio of evaders in the population, whereas only the tax rate seems to have an ambiguous effect on the change of the optimal effort level needed for the attainment of the interior steady state. Note that, respective reductions in the tax rate and penalty rate produce the expected converse results, which can be found in more detail in Appendix A.

## 4 Rational Case and Comparison

In this section we want to somehow be able to compare the results obtained in our model with a directly analogous model of the rational case, i.e. where the taxpayer acts as a rational optimizer. Following this approach imposes some changes in the model making it follow the methodology of the original articles of Allingham & Sandmo (1972) and Yitzhaki (1974). In these papers, there is a representative taxpayer who chooses the level of reported income,  $Y_R$ , that maximizes her expected utility:

$$EU = (1 - p)U(\Upsilon + t(Y - Y_R)) + pU(\Upsilon - t\theta(Y - Y_R)). \quad (39)$$

Notice that this is exactly the same as in (3), without any need to distinguish the taxpayers between compliers and non-compliers, since the outcome of the maximization process itself will characterize the taxpayer. The theoretical properties of this optimization process are fully described in the aforementioned articles, thus we shall not go through them again. However, since in our case we also have a second step, i.e. the optimal regulation, we hereby provide the alterations that have to be imposed in order to produce an analogous setting before we proceed with the comparison.

First, it is straightforward that we no longer need an evolutionary context for the

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<sup>11</sup>It is interesting to mention that in if we set a larger parameter value for public spending  $G$ , then for the steady state  $x_2^* = 1$ , the optimal effort is larger than unity, activating the constraint for  $\varepsilon^* = 1$ . This is because, the increased  $G$  increases the spread of the primary deficit, allowing for a more broad fiscal policy through taxation, thus leading the social planner to seek tax revenues even when evasion is at its peak.

behavior of agents, since we have a *rational* representative agent, rather than an imitating one. This implies that we have to withdraw the concept of the ratio of compliers and non-compliers, i.e.  $x$  and  $(1 - x)$ . In turn, this changes the way the expected tax revenues are formed and by extension the way the optimal regulation works. More specifically, remember that the expected tax revenues in the original model was described by equation (13):

$$T^e = (1 - x) tY + x (tY_R + \phi (1 + \theta) t\Delta Y) - c(\varepsilon). \quad (40)$$

In the rational case, the regulator knows that all agents will have the same reported income,  $Y_R$ , and thus it all comes down to how much she is able to audit. The level of audit performed is described by  $\phi(\varepsilon)$ , whereas its cost by  $c(\varepsilon)$ . To derive the expected tax revenues for the rational case we construct a new table, accounting for the fact that the reported income resulting from the maximization of (39) determines whether the taxpayer is an evader ( $Y_R < Y$ ) or a complier ( $Y_R = Y$ ):

	<i>Audited, <math>\phi</math></i>	<i>Unaudited, <math>(1 - \phi)</math></i>
<i>Evader, <math>Y_R &lt; Y</math></i>	$tY_R + t\Delta Y + \theta t\Delta Y$	$tY_R$
<i>Complier, <math>Y_R = Y</math></i>	$tY$	$tY$

Note that,  $\Delta Y = (Y - Y_R)$ ,  $\phi \equiv \phi(\varepsilon)$  and  $p \equiv p(\varepsilon)$ , with exactly the same properties as before. The tax revenues are derived by adding the elements of each row:

$$T^e = \begin{cases} \phi (tY_R + t\Delta Y + \theta t\Delta Y) + (1 - \phi) tY_R - c(\varepsilon) & \text{for evader} \\ \phi tY + (1 - \phi) tY - c(\varepsilon) & \text{for complier} \end{cases}$$

The tax revenues for the complier case simplify to just  $T^e = tY - c(\varepsilon)$ , rendering the case of compliance almost trivial, because the cost of effort  $c(\varepsilon)$ , is not a necessary expenditure for the regulator to make, since everyone fully pays their tax responsibilities. On the other hand, the tax revenues for any level of reported income below  $Y$ , simplify to  $T^e = tY_R + \phi (t\Delta Y + \theta t\Delta Y) - c(\varepsilon)$ , which is the general formula for the tax revenues

for the rational case.

Second, we need to find the optimal level of reported income  $Y_R$ . In order to produce a closed form solution, we impose the logarithmic utility function,  $U(w) = \log(w)$ , without any loss of generality. The first order condition for a maximum will produce a  $Y_R^*$  such that:

$$Y_R^* = \arg \max_{Y_R} (EU).$$

The closed form solution for  $Y_R^*$  turns out to be:

$$Y_R^*(\varepsilon) = \frac{Y [t + t\theta - 1 + p(\varepsilon)(1+t)(1+\theta)]}{t\theta}.$$

It is obvious that  $Y_R^*$  can also be negative for certain parameter values. The conditions for a positive interior solution are thoroughly described in Allingham & Sandmo (1972), and will not be covered here.

After having set the framework for the rational case, we can now proceed to the part of the regulation. The regulator takes  $Y_R^*$  as given and acts as a Stackelberg leader choosing the optimal level of effort  $\varepsilon$ , that will minimize the square deviation from a target primary deficit, as seen before. The major difference relies on the fact that in the evolutionary case, i.e. when the agents were imitators, the level of reported income  $Y_R$  was a given and the regulator controlled for the compliance in the population through the effort put on audits. In the rational case, we have a variable reported income that changes with effort perceived making all taxpayers acting as one. Therefore, the problem becomes static and for each level of the fixed instruments, i.e. the tax rate  $t$ , and the penalty rate  $\theta$ , the regulator finds the optimal effort that will drive the system to a level of controlled evasion, that is a specific level of  $Y_R^*$ .

The objective function for the regulator remains the same as far as the target is concerned. The only difference is that the problem has become static due to the nature of the constraints. Note that  $D(\varepsilon) = G - T^e$  is the primary deficit, incorporating the

newly formulated tax revenues as in (40):

$$\varepsilon^* = \arg \min_{\varepsilon} \left[ \frac{1}{2} (D(\varepsilon) - \bar{D})^2 + c(\varepsilon) \right], \text{ subject to } Y_R^* = \arg \max_{Y_R} (EU).$$

The problem is solved by simple substitution of the constraint inside the objective function. Analytical solution is not attainable, because the effort level appears in  $c(\varepsilon)$ . Only numerical solutions can be obtained given appropriate parameter sets.

In order to be able to make the comparison as direct as possible we impose the same global parameter values with the benchmark case of the original model:

$Y = 1$
$\bar{D} = 0$
$r = 0.02$

Notice that now we do not parameterize  $Y_R$ , as it's value will be a result of optimization of the taxpayer.

1. Benchmark case:  $G = 0.35, t = 0.1, \theta = 1, c = 0.014$

In the benchmark case, using the same values as in the benchmark for the evolutionary case, we get a reported income  $Y_R^* = -8 + 18p(\varepsilon)$ , which of course is a function of the effort level,  $\varepsilon$ . Solving for the optimal  $\varepsilon$ , we get  $\varepsilon^* = 0.490$  as the real root and two other complex roots. The level of effort is accepted in terms of intuition since it lies within  $[0, 1]$ . Substituting  $\varepsilon^*$  back at  $Y_R^*$ , we get  $Y_R^* = 0.828$ , i.e. ceteris paribus, the taxpayers will report 82.8% of their income and the authorities will channel 49% of the available resources of the Ministry of Finance towards audits.

2.  $G = 0.35, t = 0.11, \theta = 1, c = 0.014$

Following the same practice as before, we raise the tax rate by 1 percentage point. The optimal reported income by the taxpayers, after noticing and adapting to this change becomes  $Y_R^* = -7.09 + 16.181p(\varepsilon)$ . The real root for optimal  $\varepsilon$ , in turn, becomes  $\varepsilon^* = 0.490$ , almost staying at the same level. Substituting it back in the reported income equation, we get  $Y_R^* = 0.839$ , which is in turn slightly greater than  $Y_R^*$  in the benchmark

case. This means that a rise in the tax rate causes a rise in reported income and a trivial fall in the effort put in by the tax authorities. In the evolutionary case, this change led to a higher effort level of the authorities and a smaller ratio of evaders. One can say that the agents that are allowed to imitate, rather than behave as rational criminals, show more propensity to comply with the new regime, whereas the regulator is highly penalized when facing imitators, as shown by the increase in effort in the evolutionary case counter to the decrease in the rational case.

$$3. G = 0.35, t = 0.1, \theta = 1.1, c = 0.014$$

We set the tax rate back to the benchmark level and increase the penalty from 100% to 110%. The optimal reported income becomes  $Y_R^* = -7.181 + 17.181p(\varepsilon)$ . The real root for optimal  $\varepsilon$  becomes  $\varepsilon^* = 0.466$ , exhibiting a notable fall. Substituting it back in the reported income equation, we get  $Y_R^* = 0.836$ , which is greater than the reported income in the benchmark case. In the evolutionary case, an increase in the penalty caused both a fall in effort and evasion, confirming the monotonic effect of a change in the penalty rate in contrast to the ambiguity of a change in the tax rate.

## 5 Conclusion and Further Research

In this paper we wanted to merge both classical and contemporary approaches into an evolutionary game theoretical setting that is characterized by simultaneously incorporating imitation within agents, an endogenized probability of audit and regulation with either non-optimizing (myopic) behavior of the regulator, or an optimizing behavior with respect to the boundedly rational maximizing behavior of imitating agents. The resulting model was characterized by a highly non-linear structure, which is by definition unable to provide analytical results. Nevertheless, the main results prove to be particularly descriptive as far as the behavior of the agents is concerned. We have seen that if the agents are imitators and their choices over the two available strategies (evade or comply with the tax rule) are governed by the replicator dynamics equation, leads to different types of dynamics depending on the way the regulator acts. These dynamics lead to either

monomorphic or polymorphic evasion or compliance in the population of taxpayers, a fact which the regulator wishes to manipulate by means of the policy instruments at her disposal, i.e. the tax rate, the penalty rate and the effort put in auditing. We have seen two ways that the regulator wishes to intervene. The first was a direct interference with the replicator dynamics equation, the results of which were highly dependent on the form of the subjective probability of audit and led to polymorphic steady states, whereas the second involved an optimal control of effort put into audit subject to the constraint that agents are imitators. The optimal control problem led to a more complex result which if approached numerically behaved in a way that was intuitively appealing, meaning that the system led us to multiple equilibria, of which the interior steady state, where a fraction of the population evades was a saddle point. The dynamics of the system have shown very intense flows over the manifolds of each steady state. Nevertheless, saddle path stability is achieved for both the interior steady state and the state of full non-compliance.

It is interesting to note when comparing the case of imitating agents with optimizing agents, that when agents are imitators a proportion of them complies, around 30% in our benchmark numerical example, while 70% evades. On the other hand when agents are fully rational in choosing reported income, everybody evades in the corresponding numerical example. Thus imitating behavior seems to support the idea of polymorphic behavior regarding compliance and evasion.

Further research and refinement of the model is necessary, in order to somehow be able to overcome the high nonlinearity problems. We have yet to cover the part of the optimal control problem concerning the optimal choice of the tax rate and penalty level, which exhibited more complex behavior and did not provide analytical solutions. Further research could also address the case where imitating agents evade by considering more than one levels of reported income. This will increase the dimensionality of the replicator dynamics system, but could provide some more insights into a case where we want to distinguish between large and small tax evasion. Finally another possible area of extending the model is to study localized interaction regarding the decision to evade



or not. That is agents imitating their neighbors, where the concept of neighborhood can be defined in terms of geographical or socioeconomic characteristics.

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# Appendix A

We present tables for the numerical examples provided in the text along with some more extreme examples which have interesting implications. Note that the optimal effort presented in the tables is the effort resulting after the implementation of the constraint in the control variable.

1. Benchmark case:  $G = 0.35, t = 0.1, \theta = 1, c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(7.710, -7.690)	Saddle Path
$x^\dagger = 0.698$	$\varepsilon^* = 0.711$	(23.122, -23.102)	Saddle Path

- 2a. Tax rate increase:  $G = 0.35, t = 0.11, \theta = 1, c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.012, 0.007)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(4.664, -4.644)	Saddle Path
$x^\dagger = 0.635$	$\varepsilon^* = 0.781$	(138.989, -138.969)	Saddle Path

- 2b. Tax rate decrease:  $G = 0.35, t = 0.09, \theta = 1, c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.010, 0.009)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(16.007, -15.987)	Saddle Path
$x^\dagger = 0.776$	$\varepsilon^* = 0.641$	(0.01 + 31.035i, 0.01 - 31.035i)	Unstable Focus

3a. Penalty rate increase:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 1.1$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(5.834, -5.814)	Saddle Path
$x^\dagger = 0.665$	$\varepsilon^* = 0.710$	(64.829, -64.809)	Saddle Path

3b. Penalty rate decrease:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 0.9$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(10.749, -10.729)	Saddle Path
$x^\dagger = 0.735$	$\varepsilon^* = 0.711$	(0.01 + 18.736i, 0.01 - 18.736i)	Unstable Focus

## Extreme cases

We study some cases using extreme changes in parameter values. We first present the benchmark case for the sake of comparison.

1. Benchmark case:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 1$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(7.710, -7.690)	Saddle Path
$x^\dagger = 0.698$	$\varepsilon^* = 0.711$	(23.122, -23.102)	Saddle Path

2a. Extreme tax rate increase:  $G = 0.35$ ,  $t = 0.3$ ,  $\theta = 1$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.041, -0.021)	Saddle Path
$x_2^* = 1$	$\varepsilon^* = 0$	(0.480, -0.460)	Saddle Path
$x^\dagger = 0.233$	$\varepsilon^* = 1$	(1082.57, -1082.55)	Saddle Path

A very interesting result since, first of all, the optimal effort level of the interior steady state  $x^\dagger = 0.233$  is  $\varepsilon^* = 2.097$ , activating the constraint, which renders it equal to unity. Moreover, the type of all three steady states has changed to saddle paths, which means that with the social planner can drive the economy to full compliance if specific initial conditions are met.

2b. Extreme tax rate decrease:  $G = 0.35$ ,  $t = 0.03$ ,  $\theta = 1$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.016, 0.003)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(1.625, -1.605)	Saddle Path
$x^\dagger = 0.999$	$\varepsilon^* = 0.499$	(0.01 + 0.178i, 0.01 - 0.178i)	Unstable Focus

3a. Extreme penalty rate increase:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 3$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable
$x_2^* = 1$	$\varepsilon^* = 0$	(0.512, -0.492)	Saddle Path
$x^\dagger = 0.349$	$\varepsilon^* = 0.702$	(489.602, -489.582)	Saddle Path

An extreme increase in the penalty rate causes a serious drop in the evasion level similar to the one cause by an extreme increase to the tax rate. However, notice that the increase in the penalty rate keeps the optimal effort level within viable levels, lower than the one in the benchmark case. Moreover, it renders the interior steady state a saddle point.

3b. Extreme penalty rate decrease:  $G = 0.35$ ,  $t = 0.1$ ,  $\theta = 0.05$ ,  $c = 0.014$

Steady State	Optimal Effort	Eigenvalues	Type
$x_1^* = 0$	$\varepsilon^* = 0$	(0.011, 0.008)	Unstable Node
$x_2^* = 1$	$\varepsilon^* = 0$	(14.156, -14.136)	Saddle Path
$x^\dagger = 0.999$	$\varepsilon^* = 0.952$	(0.01 + 0.531i, 0.01 - 0.531i)	Unstable Focus