DEPARTMENT OF INTERNATIONAL AND EUROPEAN ECONOMIC STUDIES

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Environmental Policy: The Coevolution of Pollution and Compliance

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Working Paper Series

15-19

December 2015

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July 2015

Abstract

We study the evolution of compliance of firms in a setup of taxable emissions. Firms can either choose to comply with the emissions rule or violate it. Violation is considered either as a single option or is let to vary between low and high emissions, inducing a different level of fine if the firm gets caught. The firms can switch between strategies according to an evolutionary proportional rule and the conditions for stability are investigated accounting for two distinct types of probability of inspection.

Keywords: Emission taxes, compliance, replicator dynamics.

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1 Introduction

This work focuses on setting the problem of environmental policy on emissions on an evolutionary perspective. No matter its form, environmental policy imposes extra costs on any regulated sector. The firms of the sector will thus have an incentive to deviate from the rules set by the regulator and find ways to exploit legislation and regulation mechanisms. Admittedly, the two main instruments used for environmental policy is taxation on emissions and emissions trading. In the present, we focus on emission taxation alone, without disregarding the fact that emissions trading should be analyzed as well in future research. We investigate the behavior of emitting firms that adapt to the stringency of inspections and choose the strategy that has above average payoff according to imitative dynamics.

Original research on the field of environmental policy on emissions has produced many well known results but has based itself on the fact that there was full compliance, or at least that full compliance is achieved through certain conditions, e.g. that the cost of compliance is smaller or equal to the marginal expected penalty. A very interesting synopsis for this can be found in Lappi (2013) where the main findings of the relative literature are exposed. In Lappi (2015), a comparison is drawn between the welfare effects of emissions trading and emission taxes in the presence of market imperfections. The main finding is that given the same enforcement level, emission taxes are superior in terms of social welfare than emissions trading in the presence of non-compliance. Air pollution from mobile sources and violation issues will draw much attention with the Volkswagen emissions scandal. However, the emissions taxes or permits used for stationary sources of pollution such as industrial plants, are not considered feasible for mobile sources of pollution. Indicatively, in Fullerton & West (2002) it is shown that vehicle-specific or mileage-specific taxes prove to be more efficient than flat rates when heterogeneous consumers are introduced. A great review of monitoring and enforcement effectiveness with empirical evidence can be found in Gray & Shimshack (2011). Facts show that environmental monitoring can successfully deter future violations of both targeted and non-targeted firms, as well as lead to a reduction in emissions.

Our work focuses on the choice of the firm if we assume that firms adapt over time choosing strategies that are fittest in the sense that they have above average expected profit. This can be perceived as a way of imitation, and follows the idea of Schlag (1998) using the replicator dynamics equation as a proportional rule describing the evolution of the strategy switch in the whole population of strategies. Firms are allowed to either choose between two available strategies, i.e. either to comply or not with the emission standard or between three strategies, by choosing either to comply or to produce low or high emissions. As far as pollution is concerned, we provide an accumulation equation that will serve as a competing dynamic for our model. The simultaneous analysis of the behavior of firms and pollution accumulation will provide insight about how the system evolves through time. The evolutionary rest points attained by the system signify the equilibrium levels of pollution level and emission choice, in which the system can be expected to arrive at, given the policy instrument set by the regulator and parameter values. The nature of the steady states may be monomorphic, implying either full compliance or full non-compliance states, or polymorphic, where compliance and non-compliance can coexist in equilibrium. The latter case describes a partial compliance steady state where further regulation adjustments can be made to drive the system to more desirable compliance levels. This work does not involve optimal control solutions but provides a simple model for the evolution of emission choice and pollution levels in an evolutionary framework.

In the first section, we propose the model and distinguish between the two cases, i.e. the two strategy case and the three strategy case. In the second section we briefly provide some specifications about how the emission rule is set and about the different formulations of the subjective probability of audition that are used. The last section provides analytical solutions and conditions for the polymorphic steady states, their stability properties as well as policy implications.

2 The Model

Consider there are N homogeneous risk neutral firms indexed by $i = 1, ..., n$, which produce a homogeneous product q and a negative externality e as emissions, e.g. air pollution. Without loss of generality, we can assume that one unit of output produces one unit of emissions. Individual firm unconstrained profits will be given by

$$\pi_i = pq_i - c(q_i, e_i) \tag{2.1}$$

where $c(q_i, e_i)$ is the cost of the firm with properties $c_q > 0$, $c_e < 0$, so that each firm can reduce own costs by producing less, for the same level of emissions, or by increasing emissions for the same level of output. The market price p is taken as given by all firms in the economic sector.

The benefit function from emissions for each firm can be computed by maximizing individual unconstrained profits in (2.1) with respect to $q_i(e_i)$ and express them in terms of emissions, i.e. $\pi_i(e_i)$. The benefit function has the properties $\pi_i'(e_i) > 0$ and $\pi_i''(e_i) < 0$.

The emission level is a choice variable for the firms which determine its type. The complying type of firm chooses the emission level e_c which is the emission rule imposed by the government and is regarded both legitimate and environmentally sustainable. The non-complying firm, depending on the scenario that we analyze in the following sections, will always make an emission choice e_{nc} that is over and above the compliance level, rendering the firm a violator. The characteristics and properties of the complying and the non-complying emission levels will be analyzed in the next section.

In our context, regulation for ensuring that firms comply with emission standards is achieved through an emissions tax and random audits to firms that induce a proportional tax-based fine if a violation is encountered. The expected payoff of firm i in the presence of an emissions tax and an audit probability will be given by

$$\Pi_i = \pi_i(e_i) - \tau e_c - \sigma F \tau (e_i - e_c) \tag{2.2}$$

where τ is the tax rate, F is the fine rate and σ is the subjective probability of audition, i.e. the likelihood of being audited as perceived by the firm. Notice that the tax is imposed on the complying level of emissions denoted by e_c since every firm will choose to report e_c regardless of its type. Firms that do not choose to comply with the rule will emit more and, as long as they do not get caught, will be more profitable. In this non-cooperative equilibrium, the highest non-complying emissions rate at each point in time will be determined as

$$e_h = \arg\max_{e_i} \pi_i, \text{ subject to } e_i \leq \bar{e}$$

Notice that $\bar{e}$ is the upper bound in each firm's emissions, due to technology and infrastructure limitations.

The individual expected payoff function under compliant emissions is determined as

$$\begin{aligned} \Pi_c &= \pi_i(e_c) - \tau e_c \\ &= \pi_c - \tau e_c \end{aligned}$$

and the respective expected payoff function for the non-complying firm will be give by

$$\begin{aligned} \Pi_{nc} &= \pi_i(e_{nc}) - \tau e_c - \sigma F \tau (e_{nc} - e_c) \\ &= \pi_{nc} - \tau e_c - \sigma F \tau (e_{nc} - e_c) \end{aligned}$$

The accumulation of pollution by the total emissions of firms is described by the pollution dynamics equation

$$\dot{P} = \sum_{i=1}^{n} e_i - \delta P \tag{2.3}$$

where P is the pollution, $\dot{P}$ is the time derivative and δ is the pollution depreciation factor, e.g. due to atmosphere's self cleansing capabilities.

In this work, we consider two distinct scenarios regarding the available rules for emissions. In the first scenario, we restrict firms to be able to choose only between two possible emission strategies $\{e_c, e_h\}$, i.e. the firm can either comply with the emission standard or violate it, producing the non-cooperative level of emissions. In the second scenario, the rules available will become three $\{e_c, e_l, e_h\}$, allowing for two levels of violation, i.e. low emissions violation e_l, and high emissions violation e_h. It is essential that in both scenarios the following holds as a strict inequality: $e_c < e_l < e_h$.

2.1 First Scenario - 2 Strategies

In the first scenario, individual firms can either choose to comply with the emission standard by emitting e_c and have a payoff of Π_c or choose not to comply with regulation, emit e_h and have a payoff of $\Pi_h = \pi_h - \tau e_c - \sigma F \tau (e_h - e_c)$. In the case of a non-complying firm, if audited, the firm is liable to a fine, which amounts to a fine on the evaded tax, $F\tau (e_h - e_c)$. Let the ratio of non-complying firms in the total population of firms is denoted by $x_{nc} \equiv x$; consequently the rest $(1-x)$ will be the ratio of compliers. The average payoff flow for the population will then be given by

$$\bar{\pi} = x\Pi_h + (1-x)\Pi_c = x(\pi_h - \tau e_c - \sigma F \tau (e_h - e_c)) + (1-x)(\pi_c - \tau e_c) \tag{2.4}$$

We want to investigate the evolution of the two strategies in the population, given that the firms are allowed to change between available strategies according to the proportional rule, as described by the replicator dynamics equation. This means that in every period, each firm learns the profit and consequently the emissions strategy of another randomly chosen firm. There is a probability that the firm will change its strategy to the strategy of the opponent firm, if the rival strategy is more profitable in terms of payoff. The greater the difference between payoffs, the greater the tendency to make a strategy switch; for more information see e.g. Hofbauer & Sigmund (1998). The limiting equation that describes this evolutionary game is called replicator dynamics equation and it is a proportional rule that can describe imitating processes. Given a large population of firms the replicator dynamics equation is given by

$$\dot{x} = x(\Pi_h - \bar{\pi}) \tag{2.5}$$

The replicator dynamics in (2.5) indicates that the ratio of firms of the non-complying type increases in the population of firms as long as it has above average payoff and vice versa. The greater the difference, the faster the switch, i.e. the non-complying type will spread faster in the population of strategies. Substituting the average profit from (2.4) into (2.5) the replicator dynamics can be written as

$$\dot{x} = x\left(1-x\right)\left(\pi_h - \sigma F \tau \left(e_h - e_c\right) - \pi_c\right) \tag{2.6}$$

The replicator dynamics in (2.6) is the equivalent equation when there are only two competing strategies in the population. It describes the evolution of the non-complying strategy of firms as a result of a payoff comparison with its adversary which is the complying strategy. As long as the expected payoff of the non-complying strategy is greater than the payoff of the complying strategy, i.e. $\pi_h - \sigma F \tau \left(e_h - e_c\right) - \pi_c > 0$, the non-complying strategy's ratio increases in the population and vice versa. Notice that the term τe_c has cancelled out in the process. The pollution dynamics equation is also affected by the firm's choice of emissions in the following way

$$\dot{P} = n\left[x e_h + \left(1-x\right) e_c\right] - \delta P \tag{2.7}$$

The dynamical system (2.6), (2.7) can be used to analyze the coevolution of emission choice and pollution stock towards an equilibrium determining the ratio of complying and non-complying types of firms and the respective accumulated pollution.

2.2 Second Scenario - 3 Strategies

In the second scenario we consider the case in which each firm can choose either to comply with the emission rule e_c, or violate by choosing between high or low emissions, i.e. e_h or e_l respectively. The choice between high and low emissions can be thought as a step to make the firm's strategy vector more realistic. In a real-life scenario, one could think of an infinite dimension strategy vector in which each firm of the sector could choose either to comply or deviate from the rule following its own emission strategy and therefore output control. For result tractability we need to reduce the strategy vector to either binary choice as in the first scenario, or go one step further and increase dimensions by one, as in the second scenario.

Consider again that each strategy $\{e_c, e_l, e_h\}$ has a respective ratio in the population of strategies $\{x_c, x_l, x_h\}$ for which it holds that $x_c + x_l + x_h = 1$. The average profit flow will now be determined as

$$\bar{\pi} = x_h \Pi_h + x_l \Pi_l + x_c \Pi_c \tag{2.8}$$

where $\Pi_l = \pi_i(e_l) - \tau e_c - \sigma F \tau (e_l - e_c) = \pi_l - \tau e_c - \sigma F \tau (e_l - e_c)$. The payoff functions of Π_h and Π_c remain the same as in the first scenario.

The replicator dynamics equations will be of the form

$$\dot{x}_h = x_h (\Pi_h - \bar{\pi}) \tag{2.9}$$

$$\dot{x}_l = x_l (\Pi_l - \bar{\pi}) \tag{2.10}$$

$$\dot{x}_c = 1 - \dot{x}_h - \dot{x}_l, \text{ since } x_c = 1 - x_h - x_l \tag{2.11}$$

After some simple algebraic substitutions the replicator dynamics equations that describe the evolution of the ratio of high and low emitting firms are described by

$$\dot{x}_h = x_h [(1 - x_h)(\pi_h - \sigma F \tau (e_h - e_c) - \pi_c) - x_l (\pi_l - \sigma F \tau (e_l - e_c) - \pi_c)] \tag{2.12}$$

and

$$\dot{x}_l = x_l [(1 - x_l)(\pi_l - \sigma F \tau (e_l - e_c) - \pi_c) - x_h (\pi_h - \sigma F \tau (e_h - e_c) - \pi_c)] \tag{2.13}$$

The replicator dynamics equation describing $\dot{x}_c$ is redundant since it is a linear combination of (2.12) and (2.13) as shown in equation (2.11).

The existence of three available strategies also affects the pollution dynamics equation, which in this scenario will be determined as

$$\begin{aligned} \dot{P} &= n [x_h e_h + x_l e_l + x_c e_c] - \delta P \\ &= n [x_h e_h + x_l e_l + (1 - x_h - x_l) e_c] - \delta P \end{aligned} \tag{2.14}$$

The dynamical system described by the differential equations (2.12), (2.13) and (2.14) will be used to describe the coevolution of emission choice and pollution accumulation in the three strategy scenario.

3 Model Properties

In this section we describe characteristics and properties of the model as far as timing, and specific variables and functional forms are concerned. In this work the regulator has no authority or direct control over the behavior of firms besides setting the emission rule, and thus does not proceed to any kind of regulation after the rule has been announced. Consequently, the regulator moves first setting the rules of the game, namely the emissions rule e_c directly, or indirectly through the tax rate τ, and the level of the fine F.

More specifically, the complying emissions rule e_c, acts as a standard that has been set in an exogenous step by the regulator, prior to anything described in the stage game that follows. The emissions rule can be thought of as a response to the non-cooperative emission levels e_h and e_l, depending on the scenario. The direct way of enforcement could be through an infinite horizon optimal control problem defined by a general pollution control function of the form $E_t^* = Q(P_t)$. The socially optimum total emission level E_t^* is computed and under symmetry of firms, the socially optimal quota for each firm will be $e_c = \frac{E_t^*}{n}$. As an alternative, considered the indirect way of enforcement, we can think of a respective optimal setup in which the regulator chooses an optimal tax τ^* that will act as a Pigouvian tax for individual complying firm in a decentralized setup, as described

$$e_c = \arg\max_{e_i} \ [\pi_i(e_i) - \tau^* e_i], \text{ for } \tau^* = \arg\max_{\tau} \ W(P_t)$$

This means that the complying firm indirectly chooses e_c as a response to optimal taxation policy, where $W(P_t)$ is again some welfare function for pollution control. The direct and the indirect levels of compliance need not necessarily be identical, but serve the role of an emission rule that is publicly announced and known to all firms.

Both the complying rule e_c, and the non-cooperative rules $e_{nc} = \{e_h\}$ or $e_{nc} = \{e_h, e_l\}$, depending on the scenario, are strategies that are being adopted by firms prior to the stage game described by the replicator dynamics. It can be thought of as a situation in which each firm is hard-wired to follow an emission strategy, with a propensity to change to another strategy only if the firm they are randomly matched with in every period of the stage game has sufficiently greater profits.

The subjective probability of audition, σ, is the perceived level of regulatory stringency from the point of view of the firm. A general form can be defined by $\sigma(k)$, where k is a vector of parameters. If the regulator were to announce a fixed number of inspections per period, then we have the simplest case of fixed audit probability $\sigma(k) = \bar{\sigma}$, describing a fixed monitoring effort. On the other hand, in the case of variable monitoring effort, the vector of parameters k can vary depending on the model. We will consider two distinct cases of the subjective probability of audition.

In the first case, we assume that firms connect the level of total atmospheric pollution P, with the likelihood of being audited, i.e. $\sigma(k) \equiv \sigma(P)$. This could be rationalized through environmental sustainability goals and pollution control objectives of the authorities as expressed by the regulator's policy measures. Firms could be made aware of total pollution through public announcements after emission measurements performed by the respective competent authority and perceive that if say the total environmental pollution has increased, more effort will be exercised towards inspections, i.e. $\sigma'(P) > 0$.

In the second case, we consider the case in which firms connect monitoring effort with the share of violators, i.e. $\sigma(k) \equiv \sigma(x_{nc})$. The level of violation can be computed through various techniques such as difference in expected and actual tax revenues or tax loss, through the loss in sector profits, etc. It can also be approximated by the regulator using the share of violators detected in the previous period of inspections. A rising share of violators will lead the regulator and consequently firms to increase their subjective belief that they might get inspected in the next period, i.e. $\sigma'(x_{nc}) > 0$, where $x_{nc} = \{x\}$ or $x_{nc} = \{x_h, x_l\}$, depending on the scenario.

In both cases described above, the subjective probability of audition can be directly linked with the monitoring effort of the regulator. Assume that the regulator is announcing measurements of either pollution level, or corruption level respectively, and her goals and objectives are made clear through public announcements and the social media. In that sense, one can assume that the subjective probability of audition fully reflects the monitoring effort scheme of the regulator. It is obvious that variable monitoring effort using mixed parameters, such as $\sigma(P, x)$, or other considerations including more state variables of the model, can also be considered increasing the dimensions of the problem. The choice made in this work is to present some of the most intuitive yet simple forms and provide tractable results in later sections.

The following section provides the solution of the model, in which we only make an assumption about the functional form of the subjective probability of audition. In the case of $\sigma(P)$, due to the fact that atmospheric pollution, if measured in some multiple of tons, can theoretically take any positive value, i.e. $P \in [0, +\infty)$, we use $\sigma(P) = \frac{P}{P+1}$, in order for it to serve as a proper probability function. For the case of $\sigma(x)$, we use the simple form $\sigma(x) = x$, since by its construction as a ratio it holds that $x \in [0, 1]$. For the second scenario, where $x_{nc} = \{x_h, x_l\}$, we use $\sigma(x_{nc}) = x_h + x_l$, which meets the requirements for a probability function, since $x_h, x_l, x_c \in [0, 1]$ and $x_h + x_l = 1 - x_c \leq 1$.

4 Model Solution

In this section we solve the model and find the steady states for $\{x, P\}$ as they evolve together, starting by the simple case where each firm has two strategies available, i.e. $e = \{e_c, e_h\}$. We proceed with both the endogenized subjective probability functions of $\sigma(P)$ and $\sigma(x)$ to compare results. Next, we solve the model when the firm has a three strategy option, i.e. $e = \{e_c, e_l, e_h\}$. The same procedure concerning the subjective probability is followed, and comparisons are drawn.

4.1 First Scenario - 2 Strategies

4.1.1 Pollution based inspection probability

The dynamical system describing the model of two available strategies is described by equations (2.6) and (2.7), rewritten using $\sigma(k) \equiv \sigma(P)$.

$$\dot{x} = x(1-x)(\pi_h - \sigma(P) F\tau(e_h - e_c) - \pi_c) \tag{4.1}$$

$$\dot{P} = n[xe_h + (1-x)e_c] - \delta P \tag{4.2}$$

The replicator dynamics equation in (4.1) has steady states defined by $\dot{x} = 0$ for $x_1^* = 0$, $x_2^* = 1$, and assuming its existence, a P^* satisfying

$$P^* : \pi_h - \sigma(P^*) F\tau(e_h - e_c) - \pi_c = 0 \tag{4.3}$$

For $\sigma(P) = \frac{P}{P+1}$, the pollution level of the isocline $\dot{x} = 0$ is given by

$$P^* = \frac{\pi_c - \pi_h}{\pi_h - F\tau(e_h - e_c) - \pi_c} \tag{4.4}$$

For P^* to be well-defined, it should hold that $\pi_h \neq F\tau(e_h - e_c) - \pi_c$ and $P^* > 0$. Since $\pi_c - \pi_h < 0$, for the non-negativity property, we need

$$\pi_c > \pi_h - F\tau(e_h - e_c) \tag{4.5}$$

Notice that P^* is also non-zero, since $\pi_h > \pi_c$. The condition for the non-negativity of P^* is a critical and plausible assumption that resembles to a participation constraint, suggesting that the expected profit of the complying firm π_c, should be greater than expected profit of the non-complying firm $\pi_h - F\tau(e_h - e_c)$, when the probability of audition is equal to one. In other words, when there is an absolute certainty of inspection, there should be a clear incentive for the non-complying firm to switch to the complying strategy, as the latter offers an unambiguous profit advantage.

From the pollution accumulation equation in (4.1), we derive the steady state relationship between pollution and the ratio of non-complying firms, by solving in terms of P in $\dot{P} = 0$, yielding

$$P(x) = \frac{n\left[xe_h + (1-x)\,e_c\right]}{\delta} \tag{4.6}$$

For $x_1^* = 0$, i.e. in the case of full sector compliance, the total steady state pollution becomes

$$P_1^* = P(0) = \frac{ne_c}{\delta} \tag{4.7}$$

For $x_2^* = 1$, i.e. in the case where all firm violate, the total steady state pollution becomes

$$P_2^* = P(1) = \frac{ne_h}{\delta} \tag{4.8}$$

Notice that pollution in full non-compliance is greater than the pollution level when all comply, i.e. $P(1) > P(0)$, following $e_h > e_c$; a desirable and intuitively necessary result.

The pollution level P^* corresponds to an interior level of compliance x_3^*, that can be identified by equating (4.4) with (4.6), i.e. $P^* = P(x)$, and solving for $x \equiv x_3^*$, yielding

$$x_3^* = \frac{1}{n}\left(\underbrace{\frac{\delta + ne_c}{e_c - e_h}}_{(-)} - \underbrace{\frac{\delta F\tau}{\pi_h - F\tau\,(e_h - e_c) - \pi_c}}_{(-)}\right) \tag{4.9}$$

We need $0 < x_3^* < 1$, in order for the third level of compliance to be an "interior" and well-defined steady state. More specifically, we need the following to hold:

$$0 < x_3^* < 1 \tag{4.10}$$

$$0 < \frac{1}{n}\left(\frac{\delta + ne_c}{e_c - e_h} - \frac{\delta F\tau}{\pi_h - F\tau\,(e_h - e_c) - \pi_c}\right) < 1 \tag{4.11}$$

$$0 < \left(\frac{\delta + ne_c}{e_c - e_h} - \frac{\delta F\tau}{\pi_h - F\tau\,(e_h - e_c) - \pi_c}\right) < n \tag{4.12}$$

For the non-negativity condition, notice that the first term in the parenthesis is negative, since $e_c - e_h < 0$, whereas $\delta + ne_c > 0$. This implies that the second part must be negative as well, so that their difference can have a chance of yielding positive result. The second term, however has a positive nominator, $\delta F\tau > 0$ and the denominator is the participation constraint as in equation (4.5), which is negative.

Moreover, we also need $x_3^* < 1$, and that is achieved when the term in parenthesis is less than the total number of firms n, since it constitutes a ratio of the total population. Solving the double inequality in (4.12), we end up with the following necessary conditions which ensure that $0 < x_3^* < 1$, namely

$$\pi_c > \pi_h - F\tau\left(e_h - e_c\right) \tag{4.13}$$

and

$$\frac{\delta\left(\pi_h - \pi_c\right)}{e_h\left(-\pi_h + F\tau\left(e_h - e_c\right) + \pi_c\right)} < n < \frac{\delta\left(-\pi_h + \pi_c\right)}{e_c\left(\pi_h - F\tau\left(e_h - e_c\right) - \pi_c\right)} \tag{4.14}$$

Notice that the first condition (4.13) is again the participation constraint as in (4.5). The second condition (4.14) is the range of the population of firms, between which x_3^* behaves as a proper ratio, i.e. $0 < x_3^* < 1$. Both sides of inequality (4.14) are non-negative if and only if the first condition (4.13) holds.

As we have seen, the level of pollution that corresponds a level of violation of x_3^*, is the one given in equation (4.4), rewritten here as P_3^*, for notation compatibility:

$$P_3^* = P^* = \frac{\pi_c - \pi_h}{\pi_h - F\tau\left(e_h - e_c\right) - \pi_c}$$

To sum up, the three steady states of the form $\{x^*, P^*\}$ are the following

$$\{x_1^*, P_1^*\} = \left\{0, \frac{ne_c}{\delta}\right\} \tag{4.15}$$

$$\{x_2^*, P_2^*\} = \left\{1, \frac{ne_h}{\delta}\right\} \tag{4.16}$$

$$\{x_3^*, P_3^*\} = \left\{\frac{\left(\frac{\delta + ne_c}{e_c - e_h} - \frac{\delta F\tau}{\pi_h - F\tau(e_h - e_c) - \pi_c}\right)}{n}, \frac{\pi_c - \pi_h}{\pi_h - F\tau\left(e_h - e_c\right) - \pi_c}\right\} \tag{4.17}$$

4.1.2 Stability Properties

The stability properties of the steady states described above, will be given by the analysis of the Jacobian matrix of the system in (4.1) and (4.2). The Jacobian matrix of the system is given by

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial P} \\ \frac{\partial \dot{P}}{\partial x} & \frac{\partial \dot{P}}{\partial P} \end{bmatrix} = \begin{bmatrix} \frac{(2x-1)[F\tau(e_h-e_c)P-(1+P)(\pi_h-\pi_c)]}{1+P} & -\frac{(e_h-e_c)F\tau(1-x)x}{(1+P)^2} \\ n\left(e_h - e_c\right) & -\delta \end{bmatrix}$$

The sign of each element of the Jacobian of the system is as follows

$$sign\left(J\right) = \begin{bmatrix} ? & \leq 0 \\ > 0 & < 0 \end{bmatrix}$$

Notice that the only ambiguity concerns the partial derivative $\partial\dot{x}/\partial x$, whether it is positive, negative, or zero. All other elements have a fixed sign, or are non-zero under the assumptions of the model.

Below we compute the Jacobian for each steady state, in order to characterize the dynamical behavior of each point.

- For the steady state in $\{x_1^*, P_1^*\}$, the Jacobian will be

  $$J|_{\{x_1^*,P_1^*\}} = \begin{bmatrix} \pi_h - \pi_c - \frac{ne_cF\tau(e_h-e_c)}{\delta+ne_c} & 0 \\ n\left(e_h - e_c\right) & -\delta \end{bmatrix}$$

  with eigenvalues

  $$\{\lambda_1, \lambda_2\} = \left\{-\delta, \frac{\delta\left(\pi_h - \pi_c\right) + ne_c\left(\pi_h - F\tau\left(e_h - e_c\right) - \pi_c\right)}{\delta + ne_c}\right\}$$

  The first eigenvalue, λ_1 is always negative. The second eigenvalue, λ_2 has an ambiguous sign. The conditions for its sign are given below

$$\lambda_2 > 0, \text{ iff } n < \frac{\delta\left(-\pi_h+\pi_c\right)}{e_c\left(\pi_h - F\tau\left(e_h-e_c\right)-\pi_c\right)} \text{ or } \pi_c \leq \pi_h - F\tau\left(e_h-e_c\right)$$

$$\lambda_2 < 0, \text{ iff } n > \frac{\delta\left(-\pi_h+\pi_c\right)}{e_c\left(\pi_h - F\tau\left(e_h-e_c\right)-\pi_c\right)} \text{ and } \pi_c > \pi_h - F\tau\left(e_h-e_c\right)$$

The first condition for $\lambda_2 > 0$, is only admissible for $n < \frac{\delta(-\pi_h+\pi_c)}{e_c(\pi_h-F\tau(e_h-e_c)-\pi_c)}$, rendering the full compliance steady state $\{x_1^*, P_1^*\}$ a saddle point. The condition for $\lambda_2 < 0$, is always admissible and renders the full compliance steady state $\{x_1^*, P_1^*\}$ is stable. Notice that $n > \frac{\delta(-\pi_h+\pi_c)}{e_c(\pi_h-F\tau(e_h-e_c)-\pi_c)} > 0$, when the participation constraint in (4.5) holds, therefore we cannot rule out the case where $\lambda_2 < 0$. It can be easily verified that the discriminant, i.e. $tr^2\left(J|_{\{x_1^*,P_1^*\}}\right) - 4Det\left(J|_{\{x_1^*,P_1^*\}}\right)$ is non-negative, implying that $\{x_1^*, P_1^*\}$ is either a saddle path or a stable node, depending on the sign of λ_2.

- For the steady state in $\{x_2^*, P_2^*\}$, the Jacobian will be

$$J|_{\{x_2^*,P_2^*\}} = \begin{bmatrix} -\pi_h + \pi_c - \frac{ne_hF\tau(e_h-e_c)}{\delta+ne_h} & 0 \\ n\left(e_h-e_c\right) & -\delta \end{bmatrix}$$

with eigenvalues

$$\{\lambda_1, \lambda_2\} = \left\{-\delta, -\frac{\delta\left(\pi_h-\pi_c\right) + ne_h\left(\pi_h - F\tau\left(e_h-e_c\right)-\pi_c\right)}{\delta+ne_h}\right\}$$

While $\lambda_1 < 0$, the second eigenvalue, λ_2 has an ambiguous sign and the conditions for its sign are given below

$$\lambda_2 > 0, \text{ iff } n > \frac{\delta\left(\pi_h-\pi_c\right)}{e_h\left(-\pi_h + F\tau\left(e_h-e_c\right)+\pi_c\right)} \text{ and } \pi_c > \pi_h - F\tau\left(e_h-e_c\right)$$

$$\lambda_2 < 0, \text{ iff } n < \frac{\delta\left(\pi_h-\pi_c\right)}{e_h\left(-\pi_h + F\tau\left(e_h-e_c\right)+\pi_c\right)} \text{ or } \pi_c \leq \pi_h - F\tau\left(e_h-e_c\right)$$

The first condition for $\lambda_2 > 0$ is always admissible and renders the full non-compliance steady state $\{x_2^*, P_2^*\}$ a saddle point, whereas the condition for $\lambda_2 < 0$, which renders the steady state stable, is only admissible for $n < \frac{\delta(\pi_h - \pi_c)}{e_h(-\pi_h + F\tau(e_h - e_c) + \pi_c)}$. The latter cannot be ruled out since $0 < n < \frac{\delta(\pi_h - \pi_c)}{e_h(-\pi_h + F\tau(e_h - e_c) + \pi_c)}$ as long as the participation constraint in (4.5) holds. It can be easily verified that the discriminant, i.e. $tr^2\left(J|_{\{x_2^*, P_2^*\}}\right) - 4Det\left(J|_{\{x_2^*, P_2^*\}}\right)$ is non-negative. Thus, depending on the sign of λ_2, $\{x_2^*, P_2^*\}$ is either a saddle point or a stable node.

- For the possible interior steady state $\{x_3^*, P_3^*\}$, the Jacobian will be

$$J|_{\{x_3^*, P_3^*\}} = \begin{bmatrix} 0 & \frac{[\delta(\pi_h - \pi_c) + ne_c(\pi_h - F\tau(e_h - e_c) - \pi_c)][\delta(\pi_h - \pi_c) + ne_h(\pi_h - F\tau(e_h - e_c) - \pi_c)]}{n^2 F\tau(e_h - e_c)^3} \\ n(e_h - e_c) & -\delta \end{bmatrix}$$

The conditions for the sign of the determinant of $J|_{\{x_3^*, P_3^*\}}$ are

$$\begin{aligned} Det\left(J|_{\{x_3^*, P_3^*\}}\right) &> 0, \text{ iff } \pi_c > \pi_h - F\tau(e_h - e_c), \\ &\text{and } \frac{\delta(\pi_h - \pi_c)}{e_h(-\pi_h + F\tau(e_h - e_c) + \pi_c)} < n < \frac{\delta(-\pi_h + \pi_c)}{e_l(\pi_h - F\tau(e_h - e_c) - \pi_c)}. \end{aligned}$$

$$\begin{aligned} Det\left(J|_{\{x_3^*, P_3^*\}}\right) &< 0, \text{ iff } \pi_c < \pi_h - F\tau(e_h - e_c), \\ &\text{or } n < \frac{\delta(\pi_h - \pi_c)}{e_h(-\pi_h + F\tau(e_h - e_c) + \pi_c)}, \\ &\text{or } n > \frac{\delta(-\pi_h + \pi_c)}{e_l(\pi_h - F\tau(e_h - e_c) - \pi_c)}. \end{aligned}$$

The only admissible sign for the determinant of the Jacobian of steady state $\{x_3^*, P_3^*\}$ is being positive. For the trace it holds that $tr\left(J|_{\{x_3^*, P_3^*\}}\right) < 0$, implying that there are two real negative eigenvalues and the steady state in $\{x_3^*, P_3^*\}$ will be stable. The negative determinant is ruled out because of the fact that the conditions for the number of firms n would imply that $x_3^* \notin (0, 1)$ as described in (4.14), which is contradictory with the nature of x_3^* itself. The sign the discriminant $tr^2\left(J|_{\{x_3^*, P_3^*\}}\right) - 4Det\left(J|_{\{x_3^*, P_3^*\}}\right)$ is ambiguous, implying that the stable steady state can either be a node or a focus.

4.1.3 Violation based inspection probability

We now proceed with the same formulation, as far as strategies are concerned, but using a probability of audition that is a function of the ratio of non-complying firms, i.e. $\sigma(k) \equiv \sigma(x)$. The system of differential equations becomes

$$\dot{x} = x(1-x)(\pi_h - \sigma(x) F\tau(e_h - e_c) - \pi_c) \quad (4.18)$$

$$\dot{P} = n[xe_h + (1-x)e_c] - \delta P \quad (4.19)$$

The steady states of the replicator dynamics equation in (4.18) are the solutions to $\dot{x} = 0$, namely $x_1^* = 0$, $x_2^* = 1$, and a x_3^* satisfying

$$x_3^* : (\pi_h - \sigma(x_3^*) F\tau(e_h - e_c) - \pi_c) = 0 \quad (4.20)$$

Using the simplest functional form $\sigma(x) = x$, we can identify x_3^*, which will be given by

$$x_3^* = \frac{\pi_h - \pi_c}{F\tau(e_h - e_c)} \quad (4.21)$$

For x_3^* in (4.21) to be a proper interior steady state we need $0 < x_3^* < 1$. The necessary conditions for the double inequality constraint to hold, are

$$\begin{aligned} \pi_h &> \pi_c, \\ e_h &> e_c, \text{ and} \\ \pi_c &> \pi_h - F\tau(e_h - e_c). \end{aligned}$$

The first two conditions hold according to the model's initial assumptions and the third condition is the participation constraint.

Notice that in this formulation where the subjective probability is not anymore a function of pollution, the only interdependence of the two differential equations is only through the existence of x in $\dot{P}$. The pollution steady states are the solutions to $\dot{P} = 0$, defined by

$$P^*(x) = \frac{n\left[xe_h + (1-x)\,e_c\right]}{\delta} \tag{4.22}$$

For $x_1^* = 0$, i.e. in the case of full sector compliance, the steady state pollution becomes

$$P_1^* = P(0) = \frac{ne_c}{\delta} \tag{4.23}$$

For $x_2^* = 1$, i.e. in the case where all firm violate, the steady state pollution becomes

$$P_2^* = P(1) = \frac{ne_h}{\delta} \tag{4.24}$$

For $x_3^* = \frac{\pi_h - \pi_c}{F\tau(e_h - e_c)}$, i.e. in the case of a polymorphic compliance level, the steady state pollution is

$$P_3^* = \frac{n\left(\pi_h + F\tau e_c - \pi_c\right)}{\delta F\tau} \tag{4.25}$$

which is always positive.

Thus, the three steady states are pairs of the form $\{x^*, P^*\}$ as follows

$$\{x_1^*, P_1^*\} = \left\{0, \frac{ne_c}{\delta}\right\} \tag{4.26}$$

$$\{x_2^*, P_2^*\} = \left\{1, \frac{ne_h}{\delta}\right\} \tag{4.27}$$

$$\{x_3^*, P_3^*\} = \left\{\frac{\pi_h - \pi_c}{F\tau\left(e_h - e_c\right)}, \frac{n\left(\pi_h + F\tau e_c - \pi_c\right)}{\delta F\tau}\right\} \tag{4.28}$$

4.1.4 Stability Properties

The Jacobian for the new formulation is the following

$$J = \begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial P} \\ \frac{\partial \dot{P}}{\partial x} & \frac{\partial \dot{P}}{\partial P} \end{bmatrix} = \begin{bmatrix} x\left[F\tau\left(3x-2\right)\left(e_h - e_c\right) + 2\left(-\pi_h + \pi_c\right)\right] + \pi_h - \pi_c & 0 \\ n\left(e_h - e_c\right) & -\delta \end{bmatrix}$$

The ambiguity of the sign of each element of the Jacobian of the system derives again only by the element $\partial \dot{x}/\partial x$

$$sign\left(J\right)=\left[\begin{array}{cc} ? & 0 \\ >0 & <0 \end{array}\right]$$

Below we compute the Jacobian in every steady state

- For the steady state in $\{x_1^*, P_1^*\}$, the Jacobian will be

$$J|_{\{x_1^*,P_1^*\}}=\left[\begin{array}{cc} \pi_h-\pi_c & 0 \\ n\left(e_h-e_c\right) & -\delta \end{array}\right]$$

  with eigenvalues

$$\{\lambda_1,\lambda_2\}=\{-\delta,\pi_h-\pi_c\}$$

  We can see that $\lambda_1<0<\lambda_2$, implying that the steady state in $\{x_1^*, P_1^*\}$ is a saddle path.

- For the steady state in $\{x_2^*, P_2^*\}$, the Jacobian will be

$$J|_{\{x_2^*,P_2^*\}}=\left[\begin{array}{cc} -\pi_h+F\tau\left(e_h-e_c\right)+\pi_c & 0 \\ n\left(e_h-e_c\right) & -\delta \end{array}\right]$$

  with eigenvalues

$$\{\lambda_1,\lambda_2\}=\{-\delta,-\pi_h+F\tau\left(e_h-e_c\right)+\pi_c\}$$

  The first eigenvalue is negative, i.e. $\lambda_1<0$, while the second eigenvalue, λ_2 has an ambiguous sign. The conditions for its sign are given below

$$\lambda_2 \;>\; 0,\text{ iff } \pi_c>\pi_h-F\tau\left(e_h-e_c\right)$$

$$\lambda_2 \;<\; 0,\text{ iff } \pi_c<\pi_h-F\tau\left(e_h-e_c\right)$$

The only admissible condition is for $\lambda_2 > 0$, since it is the participation constraint. Thus the steady in $\{x_2^*, P_2^*\}$ is also a saddle path.

- For the polymorphic steady state in $\{x_3^*, P_3^*\}$, the Jacobian will be

$$J|_{\{x_3^*, P_3^*\}} = \begin{bmatrix} \frac{(\pi_h - \pi_c)(\pi_h - F\tau(e_h - e_c) - \pi_c)}{F\tau(e_h - e_c)} & 0 \\ n\left(e_h - e_c\right) & -\delta \end{bmatrix}$$

with eigenvalues

$$\{\lambda_1, \lambda_2\} = \left\{-\delta, \frac{\left(\pi_h - \pi_c\right)\left(\pi_h - F\tau\left(e_h - e_c\right) - \pi_c\right)}{F\tau\left(e_h - e_c\right)}\right\}$$

The first eigenvalue is negative, i.e. $\lambda_1 < 0$, while the second eigenvalue, λ_2 has an ambiguous sign. The conditions for its sign are given below

$$\lambda_2 \;>\; 0, \text{ iff } \pi_c < \pi_h - F\tau\left(e_h - e_c\right)$$

$$\lambda_2 \;<\; 0, \text{ iff } \pi_c > \pi_h - F\tau\left(e_h - e_c\right)$$

The only admissible condition is for $\lambda_2 < 0$, since it is the participation constraint. Thus, the steady in $\{x_3^*, P_3^*\}$ is a stable node, since the discriminant $tr\left(J|_{\{x_3^*, P_3^*\}}]\right)^2 - 4Det\left(J|_{\{x_3^*, P_3^*\}}\right)$ is non-negative. Notice that $Det\left(J|_{\{x_3^*, P_3^*\}}\right) > 0$, iff $\pi_c > \pi_h - F\tau\left(e_h - e_c\right)$.

4.1.5 Policy Implications

The difference between the two distinct formulations used for the subjective probability of audition, is that when it becomes a function of the ratio of non-complying firms, $\sigma\left(x\right)$, then the replicator dynamics equation is fully decoupled from the pollution level. This could offer an opportunity for the regulator to intervene by manipulating this to her advantage in order to achieve a specific pollution goal. For example, if we assume that the emission rule has been set indirectly, i.e. through the use of optimal taxation in the frame of an infinite horizon pollution control problem, then the tax rate τ^* is not a policy instrument anymore and can be considered to be fixed to the original level that leads complying firms to emit e_c. The only available policy instrument for the regulator is the fine F that can now be used for any adjustments of the system back to any pollution goal she would set. Suppose that there is a desirable pollution level, denoted by P^d, that needs to be attained due to the fact that the existence of violators has perturbed the original system from the pollution goal. The regulator could use equation (4.22) separately from the replicator dynamics in order to find the required violators ratio, that would lead pollution to the desired level, as follows

$$x^* : P^d(x) = \frac{n\left[x^* e_h + (1 - x^*)\, e_c\right]}{\delta} \tag{4.29}$$

The solution for x^* will be

$$x^* = \frac{n e_c - \delta P^d}{n\left(-e_h + e_c\right)} \tag{4.30}$$

For x^* to be well-defined we need $0 < x^* < 1$. Solving the double inequality we end up with the following condition

$$\frac{n e_c}{\delta} < P^d < \frac{n e_h}{\delta}$$

Notice that the sides of the inequality are the steady state levels of pollution in the case of full and no compliance respectively as seen in (4.23) and (4.24), i.e.

$$P^*|_{x=0} < P^d < P^*|_{x=1}$$

This condition is quite intuitive since it states the obvious, i.e. the target pollution has no need to be below the socially optimum $P^*|_{x=0}$ which is the one if all firms emit according to the rule e_c, nor above the socially worst case scenario $P^*|_{x=1}$ which is the one if all firms violate and emit e_h.

Having found the required ratio of violation x^*, the regulator can now obtain the level of fine that will lead the pollution through the behavior of firms described by the replicator dynamics equation to the desirable level. This is done by substituting x^* back into (4.20) and solving for F^* as follows

$$F^* : (\pi_h - \sigma(x^*) F^* \tau (e_h - e_c) - \pi_c) = 0$$

yielding

$$F^* = \frac{n(\pi_h - \pi_c)}{\tau(\delta P^d - n e_c)}$$

which will be positive iff $P^d > \frac{n e_c}{\delta}$, or equivalently when $P^*|_{x=0} < P^d$ a plausible and necessary requirement, as shown above. This means that unless the sector achieves a first best, i.e. a pollution level where every firm complies, and pollution level is the least possible $P^*|_{x=0} = \frac{n e_c}{\delta}$, the regulator can set a desirable second best target just above that level and find the required fine level to achieve it.

In the case of a subjective probability as a function of pollution level $\sigma(P)$, the regulator cannot proceed with such an intervention due to the nature of the problem. The pollution level is present in the replicator dynamics equation, and thus the regulator can only intervene after the system has settled to a steady state. For example if the system settles in the interior steady state described by

$$\{x_3^*, P_3^*\} = \left\{ \frac{\left(\frac{\delta + n e_c}{e_c - e_h} - \frac{\delta F \tau}{\pi_h - F\tau(e_h - e_c) - \pi_c}\right)}{n}, \frac{\pi_c - \pi_h}{\pi_h - F\tau(e_h - e_c) - \pi_c} \right\}$$

then the regulator can change the level of fine, in order to drive the system to lower levels. The first order derivatives with respect to the fine are

$$\frac{\partial x_3^*}{\partial F} = -\frac{\delta\tau(\pi_h - \pi_c)}{n(\pi_h - F\tau(e_h - e_c) - \pi_c)^2} < 0$$

$$\frac{\partial P_3^*}{\partial F} = -\frac{\tau(\pi_h - \pi_c)(e_h - e_c)}{(\pi_h - F\tau(e_h - e_c) - \pi_c)^2} < 0$$

Both derivatives can only be negative, with only $\pi_c \neq \pi_h - F\tau(e_h - e_c)$ as a necessary condition, which means that any policy change concerning the fine rate would move the system towards the desirable direction.

4.2 Second Scenario - 3 Strategies

4.2.1 Pollution based inspection probability

The dynamical system describing the model of three available strategies is described by equations (2.12), (2.13), and (2.14), rewritten using $\sigma(k) \equiv \sigma(P)$

$$
\begin{aligned}
\dot{x}_h &= x_h(1-x_h)(\pi_h - \sigma(P)F\tau(e_h - e_c) - \pi_c) \\
&\quad - x_h x_l(\pi_l - \sigma(P)F\tau(e_l - e_c) - \pi_c)
\end{aligned} \tag{4.31}
$$

$$
\begin{aligned}
\dot{x}_l &= x_l(1-x_l)(\pi_l - \sigma(P)F\tau(e_l - e_c) - \pi_c) \\
&\quad - x_l x_h(\pi_h - \sigma(P)F\tau(e_h - e_c) - \pi_c)
\end{aligned} \tag{4.32}
$$

$$
\dot{P} = n[x_h e_h + x_l e_l + (1 - x_h - x_l)e_c] - \delta P \tag{4.33}
$$

For $\sigma(P) = \frac{P}{P+1}$, the system has six steady states of the form $\{x_h^*, x_l^*, P^*\}$ that are defined as solutions to $\dot{x}_h = \dot{x}_l = \dot{P} = 0$ and distinguished according to their type as:

Monomorphic steady states, which imply states described by a mixture of full compliance or full non-compliance in the population

$$
\begin{aligned}
\{x_{h1}^*, x_{l1}^*, P_1^*\} &= \left\{0, 0, \frac{ne_c}{\delta}\right\} \\
\{x_{h2}^*, x_{l2}^*, P_2^*\} &= \left\{0, 1, \frac{ne_l}{\delta}\right\} \\
\{x_{h3}^*, x_{l3}^*, P_3^*\} &= \left\{1, 0, \frac{ne_h}{\delta}\right\}
\end{aligned}
$$

and polymorphic steady states, which imply states of polymorphic compliance or non-compliance in more than one of the types of firms

$$
\{x_{h4}^*, x_{l4}^*, P_4^*\} = \left\{0, \frac{\left(\frac{\delta + ne_c}{e_c - e_l} - \frac{\delta F\tau}{\pi_l - F\tau(e_l - e_c) - \pi_c}\right)}{n}, \frac{\pi_c - \pi_l}{\pi_l - F\tau(e_l - e_c) - \pi_c}\right\}
$$

$$\{x_{h5}^*, x_{l5}^*, P_5^*\} = \left\{ \frac{\left(\frac{\delta + n e_c}{e_c - e_h} - \frac{\delta F \tau}{\pi_h - F\tau(e_h - e_c) - \pi_c} \right)}{n}, 0, \frac{\pi_c - \pi_h}{\pi_h - F\tau\left(e_h - e_c\right) - \pi_c} \right\}$$

$$\{x_{h6}^*, x_{l6}^*, P_6^*\} = \left\{ \frac{\left(-\frac{\delta + n e_l}{e_h - e_l} + \frac{\delta F \tau}{\pi_l + F\tau(e_h - e_l) - \pi_h} \right)}{n}, \frac{\left(\frac{\delta + n e_h}{e_h - e_l} - \frac{\delta F \tau}{\pi_l + F\tau(e_h - e_l) - \pi_h} \right)}{n}, \frac{\pi_h - \pi_l}{\pi_l + F\tau\left(e_h - e_l\right) - \pi_h} \right\}$$

Notice that in the polymorphic case, all three steady states describe a situation in which two of the three emission levels attract a share of the population and one of them is always left with a zero share. Since by construction it holds that $x_c^* = 1 - x_h^* - x_l^*$, it is easy to verify that there will be no steady state where all three shares, x^*, will be non-zero at the same time.[1]

4.2.2 Stability Properties

In this section we provide the stability properties of the steady states described above. It is important to note that due to the increased dimensionality of the problem the Jacobian matrices cannot be presented due to their size. Furthermore, ambiguities arise concerning some signs of eigenvalues and determinants, therefore we present the general stability properties and the main conditions for which they hold.

We start with the monomorphic steady states, which bear much resemblance to the steady states of the two-strategy case. The first steady state, $\{x_{h1}^*, x_{l1}^*, P_1^*\} = \left\{0, 0, \frac{n e_c}{\delta}\right\}$, is the best case scenario since we have a state of full compliance, i.e. $x_c^* = 1$. The second steady state, $\{x_{h2}^*, x_{l2}^*, P_2^*\} = \left\{0, 1, \frac{n e_l}{\delta}\right\}$, implies that all firms violate, but choose the low emissions level, and the third steady state $\{x_{h3}^*, x_{l3}^*, P_3^*\} = \left\{1, 0, \frac{n e_h}{\delta}\right\}$, is the worst case scenario, where all firms violate and choose the high emissions rule. The eigenvalues of these three steady states respectively are given below

[1] It can be easily verified that $x_{c6}^* = 0$.

Full Compliance

$$\{\lambda_1, \lambda_2, \lambda_3\} =$$

$$\left\{-\delta, \frac{\delta\left(\pi_h-\pi_c\right)+ne_c\left(\pi_h-F\tau\left(e_h-e_c\right)-\pi_c\right)}{\delta+ne_c}, \frac{\delta\left(\pi_l-\pi_c\right)+ne_c\left(\pi_l-F\tau\left(e_l-e_c\right)-\pi_c\right)}{\delta+ne_c}\right\}$$

Low emissions monomorphic rule

$$\{\lambda_1, \lambda_2, \lambda_3\} =$$

$$\left\{-\delta, \frac{\delta\left(\pi_h-\pi_l\right)+ne_l\left(\pi_h-F\tau\left(e_h-e_l\right)-\pi_l\right)}{\delta+ne_l}, \frac{-\delta\left(\pi_l-\pi_c\right)+ne_l\left(-\pi_l+F\tau\left(e_l-e_c\right)+\pi_c\right)}{\delta+ne_l}\right\}$$

High emissions monomorphic rule

$$\{\lambda_1, \lambda_2, \lambda_3\} =$$

$$\left\{-\delta, \frac{-\delta\left(\pi_h-\pi_c\right)+ne_h\left(-\pi_h+F\tau\left(e_h-e_c\right)+\pi_c\right)}{\delta+ne_h}, \frac{-\delta\left(\pi_h-\pi_l\right)+ne_h\left(-\pi_h+F\tau\left(e_h-e_l\right)+\pi_l\right)}{\delta+ne_h}\right\}$$

In all cases, the first eigenvalue is always negative, thus there is always a stable subspace around these steady states, implying that the saddle path stability cannot be ruled out. The sign of the rest eigenvalues varies depending mainly on two new participation constraints and double inequalities concerning the population of firms n. The participation constraints are given below

$$\pi_c > \pi_h - F\tau\left(e_h-e_c\right) \tag{4.34}$$

$$\pi_c > \pi_l - F\tau\left(e_l-e_c\right) \tag{4.35}$$

$$\pi_l > \pi_h - F\tau\left(e_h-e_l\right) \tag{4.36}$$

which imply that there must be an unambiguous motive for a strategy switch, i.e. from higher to lower emission levels, when inspection is certain. The case of equality must be eliminated in order to have hyperbolic steady states. The specific signs of each eigenvalue is beyond the scope of this part, since the conditions for all eigenvalues to be negative are way too complicated, and require conditions that cannot be supported by intuition alone, but have to do with the magnitude of the parameters. The only thing worth noting is that all three steady states that describe monomorphic behavior have real valued eigenvalues and that at least one is negative, implying a saddle path stability.

As far as the polymorphic steady states are concerned, things get even more complicated, since the eigenvalues contain square roots. Nevertheless, for each steady state there exists one real-valued eigenvalue the sign of which depends on the relative magnitude of the parameters. The real-valued eigenvalues of each polymorphic steady state are shown below

For the steady state $\{x_{h4}^*, x_{l4}^*, P_4^*\}$:

$$\lambda_{\text{real}} = \frac{e_h (\pi_l - \pi_c) + e_l (\pi_c - \pi_h) + e_c (\pi_h - \pi_l)}{e_c - e_l}$$

For the steady state $\{x_{h5}^*, x_{l5}^*, P_5^*\}$:

$$\lambda_{\text{real}} = \frac{e_h (\pi_c - \pi_l) + e_l (\pi_h - \pi_c) + e_c (\pi_l - \pi_h)}{e_c - e_h}$$

For the steady state $\{x_{h6}^*, x_{l6}^*, P_6^*\}$:

$$\lambda_{\text{real}} = \frac{e_h (\pi_c - \pi_l) + e_l (\pi_h - \pi_c) + e_c (\pi_l - \pi_h)}{e_h - e_l}$$

The rest eigenvalues for each polymorphic steady state can either be real-valued or complex, thus the steady states can be one of the following:

- Saddle: if all eigenvalues are real and at least one of them is positive and at least one is negative.

- Focus-Node: if it has one real eigenvalue and a pair of complex-conjugate eigenvalues, and all eigenvalues have real parts of the same sign. The equilibrium will be stable when the sign is negative.

- Saddle-Focus: if it has one real eigenvalue with the sign opposite to the sign of the real part of a pair of complex-conjugate eigenvalues.

4.2.3 Violation based inspection probability

We now proceed with the same formulation, as far as strategies are concerned, but using a probability of audition that is a function of the ratio of non-complying firms, i.e. $\sigma(k) \equiv \sigma(x)$. The system of differential equations becomes

$$\begin{aligned} \dot{x}_h &= x_h(1-x_h)(\pi_h - \sigma(x)F\tau(e_h - e_c) - \pi_c) \\ &\quad -x_h x_l(\pi_l - \sigma(x)F\tau(e_l - e_c) - \pi_c) \end{aligned} \tag{4.37}$$

$$\begin{aligned} \dot{x}_l &= x_l(1-x_l)(\pi_l - \sigma(x)F\tau(e_l - e_c) - \pi_c) \\ &\quad -x_l x_h(\pi_h - \sigma(x)F\tau(e_h - e_c) - \pi_c) \end{aligned} \tag{4.38}$$

$$\dot{P} = n[x_h e_h + x_l e_l + (1 - x_h - x_l)e_c] - \delta P \tag{4.39}$$

We will be using a simple form for the probability of audition, i.e.

$$\sigma(x) = x_h + x_l \tag{4.40}$$

The main advantages of this formulation is that it provides a realistic counterpart of the two-strategy case and yields tractable results. Due to the nature of this specific formulation, we end up with one less steady state due to simplifications. To tackle this, one can transform the probability to a weighted sum as follows

$$\sigma(x) = \alpha x_h + (1-\alpha)x_l$$

with $\alpha \in [0,1]$ and $\alpha \neq \frac{1}{2}$. Then we get all six steady states as in the previous case of $\sigma(P)$. Without any loss of generality and intending to keep the model as simple as possible, we will be using the original formulation as in (4.40) and end up with five steady states. The steady states can again be distinguished in monomorphic and polymorphic as follows:

Monomorphic steady states imply the existence of only one dominant strategy in the population

$$
\begin{aligned}
\{x_{h1}^*, x_{l1}^*, P_1^*\} &= \left\{0, 0, \frac{ne_c}{\delta}\right\} \\
\{x_{h2}^*, x_{l2}^*, P_2^*\} &= \left\{0, 1, \frac{ne_l}{\delta}\right\} \\
\{x_{h3}^*, x_{l3}^*, P_3^*\} &= \left\{1, 0, \frac{ne_h}{\delta}\right\}
\end{aligned}
$$

Polymorphic steady states imply the existence of more than one strategy shares in the population

$$
\begin{aligned}
\{x_{h4}^*, x_{l4}^*, P_4^*\} &= \left\{0, \frac{\pi_l - \pi_c}{F\tau\left(e_l - e_c\right)}, \frac{n\left(\pi_l + F\tau e_c - \pi_c\right)}{\delta F\tau}\right\} \\
\{x_{h5}^*, x_{l5}^*, P_5^*\} &= \left\{\frac{\pi_h - \pi_c}{F\tau\left(e_h - e_c\right)}, 0, \frac{n\left(\pi_h + F\tau e_c - \pi_c\right)}{\delta F\tau}\right\}
\end{aligned}
$$

Notice that the monomorphic steady states are exactly the same as in the previous probability formulation. The polymorphic steady states imply that when the probability of audition depends on the shares of violators as a whole, there will be no equilibrium in which both violating strategies, x_h^* and x_l^* coexist.

4.2.4 Stability Properties

The eigenvalues of the monomorphic steady states are given below:

Full Compliance

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{-\delta, \pi_h - \pi_c, \pi_l - \pi_c\}$$

Low emissions monomorphic rule

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{-\delta, -\pi_l + F\tau\left(e_l - e_c\right) + \pi_c, \pi_h - F\tau\left(e_h - e_c\right) - \pi_l\}$$

High emissions monomorphic rule

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{-\delta, -\pi_h + F\tau (e_h - e_c) + \pi_c, -\pi_h + F\tau (e_h - e_l) + \pi_l\}$$

In all cases the first eigenvalue is negative, and the only plausible assumption for all the rest real-valued eigenvalues is to be positive if the participation constraints (4.34), (4.35) and (4.36) hold. We have seen these conditions throughout this work, and the logic remains the same here as well. The case of equality must be eliminated in order to have hyperbolic steady states. Therefore, all monomorphic steady states behave as saddle paths.

The eigenvalues of the polymorphic steady states are given below:

For the steady state $\{x^*_{h4}, x^*_{l4}, P^*_4\}$:

$$\{\lambda_1, \lambda_2, \lambda_3\} = \left\{-\delta, \frac{(\pi_l - \pi_c)(\pi_l - F\tau (e_l - e_c) - \pi_c)}{F\tau (e_l - e_c)}, \frac{e_h (\pi_l - \pi_c) + e_l (\pi_c - \pi_h) + e_c (\pi_h - \pi_l)}{e_c - e_l}\right\}$$

For the steady state $\{x^*_{h5}, x^*_{l5}, P^*_5\}$:

$$\{\lambda_1, \lambda_2, \lambda_3\} = \left\{-\delta, \frac{(\pi_h - \pi_c)(\pi_h - F\tau (e_h - e_c) - \pi_c)}{F\tau (e_h - e_c)}, \frac{e_h (\pi_c - \pi_l) + e_l (\pi_h - \pi_c) + e_c (\pi_l - \pi_h)}{e_h - e_l}\right\}$$

In all cases, the first eigenvalue is negative. The second one is negative as well, given that the participation constraints (4.34) and (4.35) hold, and the third eigenvalue has ambiguous sign depending on the relative magnitude of parameters. Depending on the sign of the last eigenvalue, we have a stable node if $\lambda_3 < 0$ or a saddle path if $\lambda_3 > 0$.

4.2.5 Policy Implications

As in the previous section, let us assume that the only available instrument for the regulator is the fine rate, F, and that the tax rate has been optimally set at a time prior to what is described in this work. The setting in which $\sigma(k) \equiv \sigma(x)$ decouples the replicator dynamics equations from the pollution accumulation equation, offering an opportunity for intervention. The difference in this setup is that the regulator can no longer indirectly compute all desired levels of violation. Notice that the first step will be to solve the pollution accumulation equation when $\dot{P} = 0$, set desired level of pollution $P^* = P^d$ and solve for one of the levels of violation, say x_l. More specifically, we set equation (4.39) equal to zero and try to indirectly find the level of violation x_l^* that will lead to the desired pollution level P^d as:

$$x_l^* : P^d(x_l) = \frac{n\left[x_h e_h + x_l e_l + (1 - x_h - x_l)e_c\right]}{\delta} \tag{4.41}$$

The solution for x_l^* will be

$$x_l^*(x_h) = \frac{ne_c - \delta P^d + nx_h(e_h - e_c)}{n(-e_l + e_c)} \tag{4.42}$$

For x_l^* to be well-defined we need $0 < x_l^* < 1$. Solving the double inequality we end up with the following condition

$$\frac{n(e_c + x_h(e_h - e_c))}{\delta} < P^d < \frac{n(e_l + +x_h(e_h - e_c))}{\delta}$$

Notice that both sides of the inequality are positive.

The regulator can now substitute $x_l^*(x_h)$ back into the replicator dynamics equations (4.37) and (4.38), rendering them functions of x_h and the fine. Thus, solving the system of the two equations when $\dot{x}_h = \dot{x}_l = 0$, the regulator can find the steady states for $\{x_h^*, F^*\}$. In this way, she can indirectly manipulate two of the three ratios of compliance and since the third is a linear combination, she can choose the best combination of ratios, having also set a pollution goal. More specifically, the solution to the system after the substitution yields three steady states.

The first steady state:

$$\{x_{h1}^*, x_{l1}^*, x_{c1}^*\} = \left\{0, \frac{ne_c - \delta P^d}{n(-e_l + e_c)}, \frac{-ne_l + \delta P^d}{n(-e_l + e_c)}\right\}$$

$$F_1^* = \frac{n(-\pi_l + \pi_c)}{\tau(ne_c - \delta P^d)}$$

which is a polymorphic steady state where no high violation is present. The conditions for well-defined x^*_{l1} and x^*_{c1} and a fine that is positive are synopsized in the following

$$\frac{ne_c}{\delta} < P^d < \frac{ne_l}{\delta}$$

or

$$P^*|_{x_c=1} < P^d < P^*|_{x_l=1}$$

The condition states that the desired pollution goal is bounded by the levels of pollution of full compliance and full low violation. Notice that as the pollution goal approaches to the socially optimum, i.e. $P^*|_{x_c=1} = \frac{ne_c}{\delta}$, then we arrive at a monomorphic steady state $x^*_{c1} \to 1$. The same holds for the case that the regulator chooses a very lax policy, choosing a high pollution goal. As the goal approaches the level of $P^*|_{x_l=1} = \frac{ne_l}{\delta}$, which is the full low violation state, then we arrive close to a situation of a monomorphic steady state $x^*_{l1} \to 1$. It is obvious that the regulator will have an incentive to keep the goal as low as possible to the level of full compliance.

The second steady state:

$$\{x^*_{h2}, x^*_{l2}, x^*_{c2}\} = \left\{ \frac{ne_c - \delta P^d}{n\left(-e_h + e_c\right)}, 0, \frac{-ne_h + \delta P^d}{n\left(-e_h + e_c\right)} \right\}$$

$$F^*_2 = \frac{n\left(-\pi_h + \pi_c\right)}{\tau\left(ne_c - \delta P^d\right)}$$

which is again a polymorphic steady state where no low violation is present. The conditions for well-defined x^*_{h2} and x^*_{c2} and a fine that is positive are synopsized in the following

$$\frac{ne_c}{\delta} < P^d < \frac{ne_h}{\delta}$$

or

$$P^*|_{x_c=1} < P^d < P^*|_{x_h=1}$$

The same logic holds here as well, and although it might seem inferior to the first steady state, due to the fact that it allows for high violation, it is all a matter of policy prioritization. After all, the pollution goal is what matters, no matter how contradictory the results may seem, and as we can see from the inequality, it can approach the level of the socially optimum $P^*|_{x_c=1} = \frac{ne_c}{\delta}$. Therefore, in terms of total pollution, this steady state is equally desirable.

The third and final steady state:

$$\{x_{h3}^*, x_{l3}^*, x_{c3}^*\} = \left\{ \frac{-ne_l + \delta P^d}{n(e_h - e_l)}, \frac{ne_h - \delta P^d}{n(e_h - e_l)}, 0 \right\}$$

$$F_3^* = \frac{\pi_h - \pi_l}{\tau(e_h - e_l)}$$

which is again a polymorphic steady state where no compliance is present. The conditions for well-defined x_{h3}^* and x_{l3}^* and a fine that is positive are synopsized in the following

$$\frac{ne_l}{\delta} < P^d < \frac{ne_h}{\delta}$$

or

$$P^*|_{x_l=1} < P^d < P^*|_{x_h=1}$$

This steady state, is unambiguously not desirable, since the lowest desirable pollution level that it can achieve is greater than the one where all firms produce low emissions. It is thus dominated by the previous two and will never be chosen by the regulator.

5 Discussion and further research

In this work we have investigated a simple evolutionary model of compliance of firms under the emissions tax regime. All firms had an incentive to report the least produced emissions and taxed accordingly. However, they also had the incentive to deviate and violate by polluting more. Inspections guaranteed that violating firms are liable to a fine based on the taxable emission violation. The firms could choose between the available strategies, including compliance and violation with the emission rule, by following the strategy that would offer the highest expected payoff. Firms formed beliefs about the auditing stringency depending on the announced level of pollution or the announced level of violation. Depending on the case of available strategies and the subjective probability, we have seen the steady states achieved by the coevolving system of violating firms and air pollution. We have seen that in our context, if the firm expects to be inspected as the ratio of violators increases, i.e. when the subjective probability of audition is a function of announced violation rate, the work of the regulator becomes more flexible. In terms of efficiency, it is a matter of prioritization and parameter values that would shed light to whether the signal about stringency should be the ratio of non-compliance or the level of pollution. The same holds for the steady state stability, since all monomorphic steady states can be achieved depending on initial conditions.

Future research would definitely need to address the optimal control problem with respect to the behavior of firms described here. In that way, one can use an appropriate objective function that would serve as the goal for the regulation authorities, and investigate the implications for the policy instruments and as well as which subjective probability works best. There can also be a generalization on the population of strategies, using either a discrete number or infinite dimension replicator dynamics to allow for a more general and realistic treatment for that subject.

References

Fullerton, Don, & West, Sarah E. 2002. Can taxes on cars and on gasoline mimic an unavailable tax on emissions? *Journal of Environmental Economics and Management*, **43**(1), 135–157. (Page 2).

Gray, Wayne B, & Shimshack, Jay P. 2011. The effectiveness of environmental monitoring and enforcement: A review of the empirical evidence. *Review of Environmental Economics and Policy*, req017. (Page 2).

Hofbauer, Josef, & Sigmund, Karl. 1998. *Evolutionary games and population dynamics*. Cambridge University Press. (Page 6).

Lappi, Pauli. 2013. Emissions trading under market imperfections. *Ph.D. dissertation, Department of Economics and Management, University of Helsinki.* (Page 2).

Lappi, Pauli. 2015. The welfare ranking of prices and quantities under noncompliance. *International Tax and Public Finance*, 1–20. (Page 2).

Schlag, Karl H. 1998. Why imitate, and if so, how?: A boundedly rational approach to multi-armed bandits. *Journal of Economic Theory*, **78**(1), 130–156. (Page 3).