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Time Scale Externalities and the Management of Renewable Resources

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Abstract

The evolution of renewable resources is characterized in many cases by different time scales where some state variables such as biomass, may evolve relatively faster than other state variables such as carrying capacity. Ignoring this time scale separation means that a slowly changing variable is treated as constant over time. Management rules designed without accounting for time scale separation will result in inefficiencies in resource management. We call this inefficiency time scale externality and we analyze renewable resource harvesting when carrying capacity evolves slowly, either in response to exogenous forcing or in response to emissions generated by the industrial sector of the economy. We study cooperative and non-cooperative solutions under time scale separation. Using singular perturbation reduction methods (Fenichel 1979), we examine the role of different time scales in environmental management and the potential errors in optimal regulation when time scale separation is ignored.

Keywords: optimal resource harvesting, fast slow dynamics, singular perturbation, regulation, open loop, closed loop..

JEL Classification: D81, Q20

1 Introduction

The study of fast and slow processes and slow/fast interactions is an integral part of ecosystem analysis and an important factor in understanding ecosystem dynamics and management (e.g., Carpenter et al. 2001, Gunderson and

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Pritchard 2002). To account for this time scale separation the dynamical system which is used to model the ecosystem should include both slow and fast evolving state variables. For example coevolutionary processes are analyzed in separate time scales where population dynamics evolve rapidly while evolution generally takes place more slowly. Similar time separation appears in models of antagonistic species coevolution in which population (or biomass) dynamics interact with mutation (or trait) dynamics leading to the so-called Red Queen cycles. Modeling economic/ecological systems as fast/slow systems has been associated with issues like biological resource management, water management and pest control (e.g. Milik et al., 1996; Brock and Xepapadeas, 2004; Grimsrud and Huffaker, 2006; Huffaker and Hotchkiss, 2006; Crepin et al., 2011; Walker et al., 2012; Levin et al., 2013).

In terms of the mathematical approach, dynamical systems evolving in a fast/slow time framework can be analyzed using singular perturbation analysis (e.g., Wasow, 1965; Fenichel, 1979; Berglund, 1998; Berglund and Gentz, 2003). Two cases are of main interest in problems with the state variables evolving in different time scales, the case of an adiabatic system and the case of a fast/slow system. Consider a dynamical system with state variables characterized by different relaxation time to equilibrium (or characteristic time) T_S for a slow state variable and T_F for a fast state with $T_S >> T_F$. The so-called adiabatic system emerges when the evolution of the slow state variable is imposed exogenously and acts on the fast state variable as a slowly changing time-dependent parameter. On the other hand a slow/fast system emerges when the slow variable evolves endogenously and is coupled to and influenced by the fast state variable.

In environmental and resource economics there have been a few attempts to study ecosystems in separate time scales. In particular Huffaker and Hotchkiss (2006) apply singular equations of motion to accommodate the separate time scale and analyze the economic dynamics of reservoir sedimentation management using the hydrosuction-dredging sediment-removal system. Grimsrud and Huffaker (2006) apply singular perturbation methods, in a bio-economic model, to investigate the optimal management of pest resistance to pesticide crops, and Rinaldi and Scheffer (2000) use a range of examples from natural and terrestrial ecosystems to study the effects of slow and fast variables to ecosystems. Crepin (2007) presents a general framework to handle systems with fast and slow variables, and illustrates the method using a model of coral reefs subject to fishing pressure. Crepin et al. (2011) explore how non-convexities and slow fast dynamics affect coupled human nature systems, adopting a specific system where they link changes in the number and diversity of birds to the abundance of a pest (insects) that causes damages to goods and services. Milik et al., (1996) considers a simple model of demographic, economic and environmental interactions to illustrate the use of geometric singular perturbation theory in environmental economics.

When time scale separation exists with state variables evolving in different time scales ignoring this separation and treating everything in one time scale, the fast time scale, introduces an externality. This is because agents' actions are taken by regarding a certain state variable, say carrying capacity in a renewable resource problem, as fixed, which implies that the potential impact of their actions on the evolution of this state variable is ignored. These actions, however, will potentially affect the utility or the profits of the same or other economic agents slowly without been internalized. This is a source of externality which we will call time scale externality. If the underlying system is nonlinear, unregulated slowly changing state variables may cause the system to cross thresholds or tipping points which may induce transitions to non desirable basins of attraction. It should be noted that even if the agents' actions generate a well defined externality, such as emissions which is regulated by conventional policy instruments (e.g., emissions taxes or tradable emission permits), but time scale separation is ignored then regulation is inefficient because it does not internalize all the external effects.

The contribution of the present paper is the analysis of externalities related to time scale separation, and the potential inefficiencies in regulation when this separation is ignored. We analyze the problem by incorporating fast and slow evolving variables in a one species renewable resource harvesting model. Initially we formulate a basic model under the assumption that the carrying capacity is a slow variable which evolves adiabatically, in response to exogenous forcing. This case may be realistic and might give useful results in some cases of periodic changes in carrying capacity due to climate forcing. We extend this model by allowing for the carrying capacity to evolve endogenously in response to emissions generated by an industrial sector of the economy.[1] We analyze ecosystem management via the application of singular perturbation reduction methods (Fenichel, 1979) and we compare the solutions to those emerging from conventional models where the carrying capacity is regarded as fixed. Since the analysis leads to quite complex dynamical systems tractability makes necessary the use, after a certain stage, of numerical simulations obtained by calibrating our models using plausible parameters. It should be noted that ignoring a slowly evolving state variable may affect the efficiency of regulation. For example in coral reefs, algae and fish populations often evolve more rapidly than the coral. If coral dynamics change very slowly, they may appear constant, which could cause mistakes in management. Slow coral dynamics also imply that it can take a long time before all the impacts of a particular regulation take place (Crépin, 2007). Our comparisons provide, therefore, insights into potential

[1] We use the term industrial sector to include both conventional industry and services, such as tourism, and industrial agriculture. These sectors may influence the carrying capacity of the renewable resource through their emissions.

losses in terms of regulatory efficiency when we ignore slow variables and adjustments necessary so that a regulatory scheme such as landing tax, or quota will internalize the time scale externality.

2 Resource harvesting with carrying capacity evolving slowly in response to exogenous forcing

We consider the case of one renewable resource growing according to:

$$x' = f(x, K) - h, \; x' = \frac{dx}{dt}$$

where x is the biomass of resource, K is carrying capacity and h is harvesting. Harvesting could be undertaken by a finite number of agents $j = 1, ..., J$, thus

$$h = \sum_{j=1}^{J} h_j \; , \; \mathbf{h} = (h_1, ..., h_J) \, .$$

The growth function f is assumed to be logistic, that is

$$f(x, K) = \rho x \left(1 - \frac{x}{K}\right)$$

where ρ is the intrinsic growth rate and K is carrying capacity. Harvesting can be expressed in terms of a generalized production function in terms of biomass and effort, or

$$\begin{aligned} h_j &= qx^{\alpha} E_j^{\beta} \; , \alpha > 0, 0 < \beta < 1, j = 1, ..J. \\ \mathbf{E} &= (E_1, ..., E_J) \end{aligned}$$

where q is the catchability coefficient and E_j is fishing effort. If p is the exogenous price (e.g. a world price) of the harvested resource and w is cost per unit effort then individual profits are defined as $\pi_j(x, E_j) = pqx^{\alpha} E_j^{\beta} - wE_j$

We consider the case where biomass and carrying capacity evolve in two distinct and separated time scales a fast time scale and a slow time scale. The separation of time scales is measured by ε, an arbitrarily small positive parameter, where the limit $\varepsilon \to 0$ corresponds to infinite separation. We denote fast time by (t) and slow time by (τ) so that

$$\tau = \varepsilon t \;\; , \;\; \frac{d\tau}{dt} = \varepsilon.$$

Thus if the time scale of biomass is months and the time scale of the carrying capacity is decades, $\varepsilon = \frac{1}{120}$.

As a starting point we assume that the carrying capacity evolves slowly in response to exogenous (e.g. climatic) forcing. This is a simplified case that may be realistic in some cases of periodic changes in climatic forcing. For example oscillatory patterns of climate variability like El Nino or the Pacific-Decadal Oscillation (PDO) have significant impacts on local ecosystems and the fishing industry such as the Peruvian anchoveta (El Nino), or salmon production in the northeast Pacific Ocean (PDO).[2] In this case the carrying capacity is an adiabatic variable, which could be specified as:

$$K(\tau) = K_0 \left(1 + \alpha_0 \tau + \alpha_1 \cos(\beta_0 \tau)\right)$$

with $\alpha_0, \alpha_1, \beta_0$ positive parameters.

The model in slow time can be written as

$$\begin{aligned} \varepsilon x'(\tau) &= \rho x(\tau)\left(1 - \frac{x(\tau)}{K(\tau)}\right) - \sum_{j=1}^{J} h_j && (1) \\ K(\tau) &= K_0 \left(1 + \alpha_0 \tau + \alpha_1 \cos(\beta_0 \tau)\right). && (2) \end{aligned}$$

At the adiabatic limit $\varepsilon \to 0$ and the model becomes in slow time

$$\begin{aligned} 0 &= \rho x(\tau)\left(1 - \frac{x(\tau)}{K(\tau)}\right) - h(\tau) \\ K(\tau) &= K_0 \left(1 + \alpha_0 \tau + \alpha_1 \cos(\beta_0 \tau)\right). \end{aligned}$$

In fast time, the slow variation of carrying capacity is described by a function $K(\varepsilon t)$ and the dynamics are:

$$\begin{aligned} x'(t) &= \rho x(t)\left(1 - \frac{x(t)}{K(\varepsilon t)}\right) - \sum_{j=1}^{J} h_j && (3) \\ K(\varepsilon t) &= K_0 \left(1 + \alpha_0 \varepsilon t + \alpha_1 \cos(\beta_0 \varepsilon t)\right). && (4) \end{aligned}$$

It is clear that the model taken at the adiabatic limit ($\varepsilon \to 0$) in slow time provides the long term fluctuations of the resource's biomass, while in fast time provides the short term evolution of biomass under the assumption that carrying capacity is fixed since (4) implies that $K(0)$ is fixed independent of time. The large majority of resource management models correspond to that case where carrying capacity is fixed over the long run. If this is not the case however then management rules are derived from a misspecified model and it will be worth examining the structure of management rules which are derived from a properly specified model that takes into account separation of time scales.

[2] See for example Trenberth (1997), Philander (1983), Hare and. Francis (1994), Mantua (2002).

2.1 A Cooperative solution: Optimal regulation

Since harvesting which is the variable of interest, takes place in fast time it is natural to study these problems in fast time and allow for the impact of the slow variable through ε. Consider a regulator seeking to derive a harvesting rule to maximize the sum of harvesters' profits by taking into account fast and slow dynamics. The regulator's problem is:

$$\max_{\mathbf{E}} \int_0^{\infty} \mathrm{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j (x, E_j) \right] dt \tag{5}$$
$$subject \;\; to \;\; (3), (4), x(0) = x_0$$

with

$$\pi_j (x, E_j) = pqx^{\alpha} E_j^{\beta} - wE_j = \pi(x, \mathbf{E}),$$
$$h_j = qx^{\alpha} E_j^{\beta}, \alpha > 0, 0 < \beta < 1, j = 1, ..J.$$

Defining the associated current value Hamiltonian H, as:

$$\begin{aligned} H &= \pi(x, \mathbf{E}) + mg(x, \mathbf{E}), \\ g(x, \mathbf{E}) &= \rho x (t) \left(1 - \frac{x (t)}{K (\varepsilon t)}\right) - \sum_{j=1}^{J} h_j \end{aligned}$$

with $m = m(t)$ the associated costate variable we obtain, using the maximum principle, the following optimality conditions for problem[3] (5):

$$\begin{aligned} H_{E_j} &= 0, j = 1, ..J \\ m' &= rm - H_x \\ x' &= g(x, \mathbf{E}) \\ K (\varepsilon t) &= K_0 (1 + \alpha_0 \varepsilon t + \alpha_1 \cos (\beta_0 \varepsilon t)) . \end{aligned} \tag{6}$$

From (6) the control $\mathbf{E}$ is a function of the state and the costate variable and is given by the feedback rule:

$$E_j = (\frac{p - m}{w} qx^{\alpha} \beta)^{\frac{1}{1-\beta}}, j = 1, ..J \tag{7}$$

and thus :

$$\begin{aligned} m' &= rm - m\rho \left(1 - \frac{2x}{K}\right) - J\alpha(\frac{\beta}{w})^{\frac{\beta}{1-\beta}} ((p - m)q)^{\frac{1}{1-\beta}} x^{\frac{\alpha+\beta-1}{1-\beta}} \\ x' &= \rho x \left(1 - \frac{x}{K}\right) - J(\frac{\beta}{w})^{\frac{\beta}{1-\beta}} (p - m)^{\frac{\beta}{1-\beta}} q^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}}, \\ K &= K (\varepsilon t) . \end{aligned} \tag{8}$$

[3] Subscripts associated with a function indicate partial derivatives. Thus $H_{E_j} = \frac{\partial H}{\partial E_j}$.

2.1.1 Numerical simulations

System (8) is nonlinear and nonautonomous so in order to obtain a clear picture of the solution we resort to numerical simulations using the following parametrization:[4]

$$\beta = \alpha = 1/2,\ J = 2,\ p = 10,\ w = 5,\ q = 0.048,\ \rho = 0.45,\ r = 0.05 \quad (9)$$

Thus system (8) can be written as:

$$\begin{aligned} m^{'} &= rm - m\rho\left(1 - \frac{2x}{K}\right) - J\frac{1}{4w}((p-m)q)^2 \\ x^{'} &= \rho x\left(1 - \frac{x}{K}\right) - J\frac{1}{2w}(p-m)xq^2, \\ K &= K(\varepsilon t). \end{aligned} \quad (10)$$

System (10) can be rewritten in slow time $\tau = \varepsilon t$ as

$$\begin{aligned} \varepsilon m^{'} &= rm - m\rho\left(1 - \frac{2x}{K(\tau)}\right) - J\frac{1}{4w}((p-m)q)^2 = f_1 \\ \varepsilon x^{'} &= \rho x\left(1 - \frac{x}{K(\tau)}\right) - J\frac{1}{2w}(p-m)xq^2 = f_2, \\ K(\tau) &= K_0\left(1 + \alpha_0\tau + \alpha_1\cos(\beta_0\tau)\right). \end{aligned} \quad (11)$$

We define as an equilibrium branch for this system a pair of solutions $(m^*(K(\tau)), x^*(K(\tau)))$ such that $(\varepsilon m'(K(\tau))), \varepsilon x'(K(\tau))) = (0,0)$. Assuming that such a branch exists then its structure can be characterized by the Jacobian matrix $A = \begin{bmatrix} (f_1)_m & (f_1)_x \\ (f_2)_m & (f_2)_x \end{bmatrix}$ evaluated at any equilibrium branch $(m^*(K(\tau)), x^*(K(\tau)))$. If this matrix contains no eigenvalues with zero real parts then an adiabatic solution, associated with any of the equilibrium branches, which is the form $(\bar{m}(\tau), \bar{x}(\tau) = (m^*(\tau), x^*(\tau) + \bigcirc(\varepsilon)$, exists.[5]

In particular system (10) has four pairs of solutions for m^* and x^* as functions of $K(\tau)$. Two solutions lead to resource extinction and two result in positive biomass along the equilibrium branch. We further study the two solutions leading to positive biomass. One solution has a saddle point structure, while the other has the structure of an unstable spiral. Since matrix A contains no eigenvalues with zero real part all equilibrium branches $(m^*(K(\tau)), x^*(K(\tau)))$ of (11) admit an adiabatic solution $(\overline{m}(\tau), \overline{x}(\tau))$. We concentrate on the saddle-point solution which is compatible with the optimal control structure of the problem.

[4] The values of the parameters are similar to those used by Da-Rocha et al. (2014).

[5] For more details see Berglund (1998).chapters 4 and 5.

Following Berglund (1998) an adiabatic approximation of order one which obtained as

$$
\begin{aligned}
(\overline{m}(\tau),\overline{x}(\tau))^T &= (m^*(\tau),x^*(\tau))^T+\varepsilon u(\tau,\varepsilon)+\bigcirc(\varepsilon^2), \qquad (12)\\
u(\tau,\varepsilon) &= -A^{-1}w(\tau,\varepsilon),\quad w(\tau,\varepsilon)=-[((m^*(\tau))',(x^*(\tau))')^T],
\end{aligned}
$$

is associated with this solution, where matrix A is evaluated at the saddle-point solution. This adiabatic approximating solution (12) describes how the stock of biomass evolves under optimal regulation given the slow evolution of carrying capacity. Taking $\varepsilon = 0.04$, so that the fast time unit is 1/25 of the slow time unit, biomass dynamics and the corresponding path for the biomass shadow value (i.e., the costate variable), at the regulators equilibrium with the saddle point structure satisfy the following:

$$
\begin{aligned}
S_4 &: \quad \bar{x}(\tau)=0.989808K(\tau)-0.0888812K'(\tau)+\bigcirc(\varepsilon^2) \qquad (13)\\
S_4 &: \quad m_4^*(\tau)=0.0465056+0.00750913K'(\tau)/K(\tau)+\bigcirc(\varepsilon^2) \qquad (14)\\
K(\tau) &= K_0\left(1+\alpha_0\tau+\alpha_1\cos\left(\beta_0\tau\right)\right) \qquad (15)\\
K'(\tau) &= K_0\left(1+\alpha_0-\beta_0\alpha_1\sin\left(\beta_0\tau\right)\right) \qquad (16)
\end{aligned}
$$

Using (7) we derive the corresponding adiabatic solutions for the optimal effort E. In fast time the optimal paths for biomass and effort will be determined as

$$
\begin{aligned}
\overline{x}(t) &= 0.989808K(\varepsilon t)-0.0888812K'(\varepsilon t)+\bigcirc(\varepsilon^2) \qquad (17)\\
\bar{E}(t) &= 0.00002304(10-\bar{m}(t))^2\bar{x}(t), \qquad (18)\\
\bar{m}(t) &= 0.0465056+0.00750913K'(\varepsilon t)/K(\varepsilon t)+\bigcirc(\varepsilon^2) \qquad (19)
\end{aligned}
$$

where $K(\varepsilon t), K'(\varepsilon t)$, are determined by replacing into (15), (16) εt for τ.

Figure (1) depicts $\bar{x}(t)$, $\bar{m}(t)$, $\bar{E}(t)$ and $J\bar{h}(t)$ given by (17)-(19) for

$$K_0=7000, \alpha_0=0.15, \alpha_1=0.4, \beta_0=0.5, \varepsilon=0.04,\ t\epsilon[0,50\pi]$$

It should be noted that if the regulator ignores the slow variable, then $\hat{x} = 0.989808K$ and $\hat{E} = (\frac{p-m}{w}q\hat{x}^{\alpha}\beta)^{\frac{1}{1-\beta}} = \hat{x}(\frac{p-m}{w}q\beta)^2 = (\frac{p-m}{w}q\beta)^2 * 0.989808K = 0.0225936K$, $m = 0.0465056$, for any given value of K assigned by the regulator. However, the actual long run evolution of the system, when the misspecified model that ignores the slow variable is used to determine the optimal effort, will be:

$$
\begin{aligned}
x'(t) &= \rho x\left(1-\frac{x}{K}\right)-J\widehat{h}\ ,or \qquad (20)\\
&= \rho x\left(1-\frac{x}{K}\right)-0.00453982K\\
K &= K(\varepsilon t)=K_0\left(1+\alpha_0\varepsilon t+\alpha_1\cos\left(\beta_0\varepsilon t\right)\right).
\end{aligned}
$$

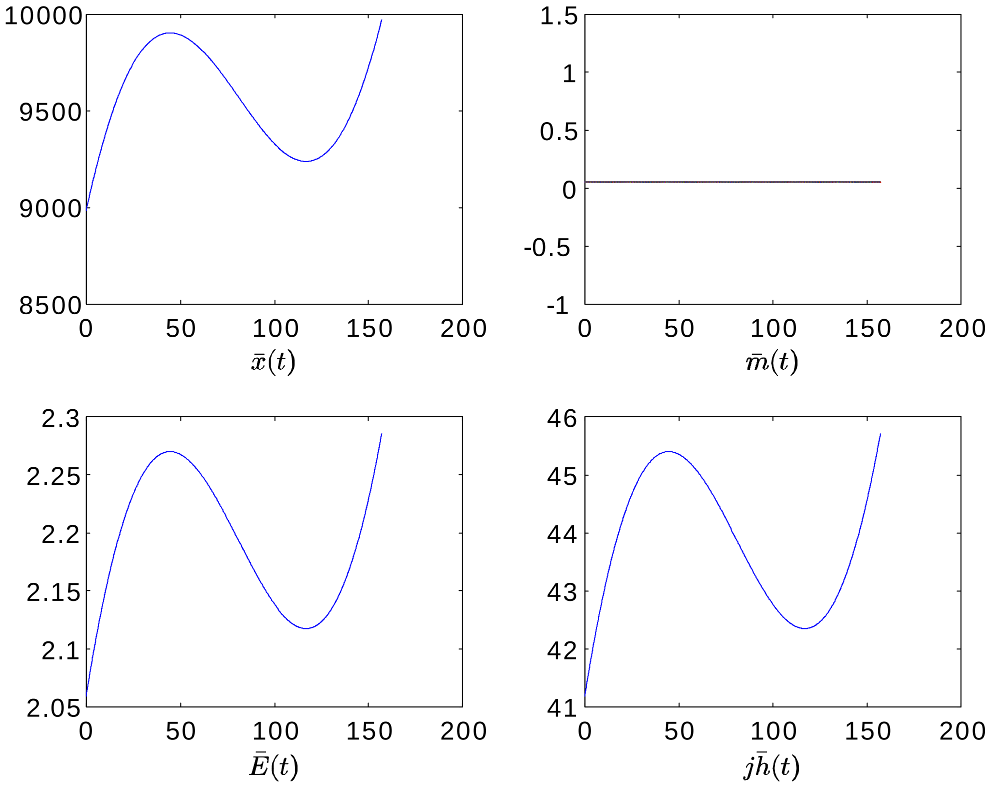

Figure 1: Slowly fluctuating carrying capacity

The path for $x(t)$ resulting from (20) is shown in Figure (2), with $x(0) = 6500$ and parameters as in (9), relative to the path $\bar{x}\,(t)$ resulting from (17). The comparison of the two biomass paths shows the deviation between them, when the regulator ignores the slow variable. Note that ignoring the slow variable for this specific parametrization implies that the regulator is more conservative, since $x(t)$ is above $\bar{x}\,(t)$ for most of the time horizon. This is due to the fact that by ignoring the slow variable the regulator ignores the slow positive linear trend $\alpha_0\tau$ on the carrying capacity that allows for more biomass. It is interesting to note that if the trend was negative then ignoring the slow variable would have implied excessive use of the resource. This is a potentially important observation because if climate change, for example, results in a negative slow linear trend on the carrying capacity, then ignoring the slow dynamics implies regulation leading to excess use of the resource.

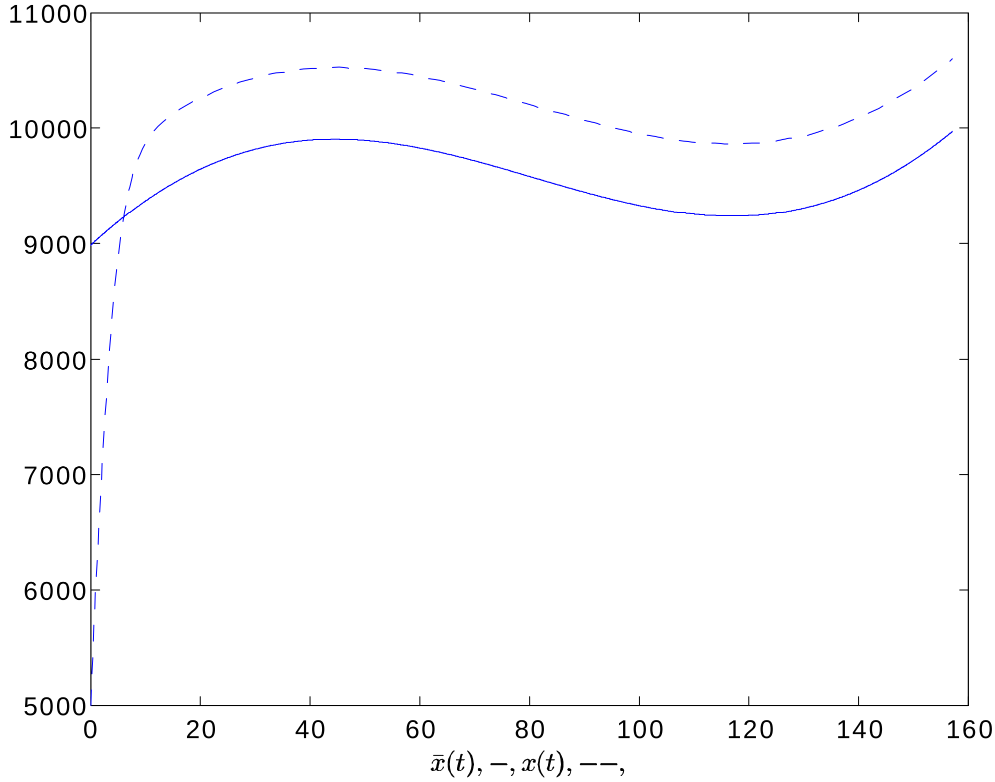

Figure 2: Ignoring the slow variable

3 Non-cooperative solutions

We consider now non-cooperative harvesting where each agent maximizes individual profits. We study, as usual, open loop and closed loop Nash equilibrium solutions.

3.1 Open loop Nash equilibrium

Each agent takes the harvesting effort of his competitors as given and solves:

$$\begin{aligned} & \max_{E_j(t)} \int_0^{\infty} \mathrm{e}^{-rt} \left[\pi_j \left(x, E_j \right) \right] dt, \qquad (21) \\ & s.t. \\ x'(t) &= \rho x(t) \left(1 - \frac{x(t)}{K(\varepsilon t)} \right) - h_j(t) - \sum_{l \neq j} h_l(t) \\ K(\varepsilon t) &= K_0 \left(1 + \alpha_0 \varepsilon t + \alpha_1 \cos(\beta \varepsilon t) \right), x(0) = x_0. \end{aligned}$$

Defining the associated current value Hamiltonian H, for a specific $j = 1, .., J$, as:

$$
\begin{aligned}
H &= \pi_j(x, \mathbf{E}) + mg(x, \mathbf{E}), \\
g(x, \mathbf{E}) &= \rho x(t)\left(1 - \frac{x(t)}{K(\varepsilon t)}\right) - h_j(t) - \sum_{l \neq j} h_l(t)
\end{aligned}
$$

with $m = m(t)$ the costate variable, we obtain the following optimality conditions for problem (21):

$$
\begin{aligned}
H_{E_j} &= 0, \\
m' &= rm - H_x \\
x' &= g(x, \mathbf{E}) \\
K(\varepsilon t) &= K_0\left(1 + \alpha_0 \varepsilon t + \alpha_1 \cos(\beta \varepsilon t)\right).
\end{aligned}
$$

The open loop effort is obtained as:

$$
E_j = \left(\frac{p - m}{w} q x^{\alpha} \beta\right)^{\frac{1}{1-\beta}}, j = 1, .., J.
$$

By symmetry the Hamiltonian system describing the open loop Nash equilibrium is:

$$
\begin{aligned}
m' &= \begin{array}{c} rm - m\rho\left(1 - \frac{2x}{K}\right) - \alpha\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}}\left((p-m)q\right)^{\frac{1}{1-\beta}} x^{\frac{\alpha+\beta-1}{1-\beta}} \\ +m(J-1)\alpha\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}}(p-m)^{\frac{\beta}{1-\beta}} q^{\frac{1}{1-\beta}} x^{\frac{\alpha+\beta-1}{1-\beta}} \end{array} \\
x' &= \rho x\left(1 - \frac{x}{K}\right) - J\left(\frac{\beta}{w}\right)^{\frac{\beta}{1-\beta}}(p-m)^{\frac{\beta}{1-\beta}} q^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}} \\
K &= K(\varepsilon t) \qquad (22)
\end{aligned}
$$

Numerical simulations Substituting, as before, $\beta = \alpha = 1/2$ in (22) and rescaling in slow time, we obtain:

$$
\begin{aligned}
\varepsilon m' &= rm - m\rho\left(1 - \frac{2x}{K}\right) - \frac{1}{4w}\left((p-m)q\right)^2 + m(J-1)\frac{1}{4w}(p-m)q^2 \\
\varepsilon x' &= \rho x\left(1 - \frac{x}{K}\right) - \frac{J}{2w}(p-m)xq^2. \qquad (23) \\
K &= K(\varepsilon t) = K(\tau) = K_0\left(1 + \alpha_0 \tau + \alpha_1 \cos(\beta_0 \tau)\right).
\end{aligned}
$$

The system of equation (23) can be regarded as a Hamiltonian system similar to system (11) in the previous section. For a more specific analysis, we adopt the parameter setting of the cooperative solution with two players. Concentrating on the equilibrium branch with positive biomass and saddle point structure, we derive as in the previous section the adiabatic approximations for the biomass and its shadow value, and the corresponding optimal effort for the open loop Nash equilibrium. In fast time the optimal paths for biomass, its shadow value, and effort will be determined as

$$\begin{aligned}
\overline{x}_{OL}(t) &= 0.989784K(\varepsilon t) - -0.088885K'(\varepsilon t) + \bigcirc(\varepsilon^2) && (24)\\
\bar{E}_{OL}(t) &= 0.00002304(10 - \overline{m}(t))^2\overline{x}(t), \\
\overline{m}_{OL}(t) &= 0.0233078 + 0.00377246K'(\varepsilon t)/K(\varepsilon t) + \bigcirc(\varepsilon^2) \\
K(\varepsilon t) &= K_0(1 + \alpha_0\tau + \alpha_1\cos(\beta_0\varepsilon t)) && (25)\\
K'(\varepsilon t) &= K_0(1 + \alpha_0 - \beta_0\alpha_1\sin(\beta_0\varepsilon t)). && (26)
\end{aligned}$$

Comparing with the cooperative case, the above solution is similar to the solution (17)-(19).

For the same parameter setting used for the cooperative solution, that is,

$$K_0 = 7000, \alpha_0 = 0.15, \alpha_1 = 0.4, \beta_0 = 0.5, \varepsilon = 0.04,\ t \in [0, 50\pi],$$

the solution corresponding to the open loop Nash equilibrium is shown in Figure 3 below. It is close to the cooperative solution but, as expected, corresponds to relatively higher harvesting and resource use and lower shadow value for the biomass.

3.1.1 Feedback Nash Equilibrium

A strong time consistent feedback Nash equilibrium (FBNE) is obtained by assuming that each agent conditions his/her effort on existing biomass $x(t)$. Assuming time stationary symmetric feedback strategies $E_i(t) = e(x(t))$, each agent solves the problem:

$$\begin{aligned}
& \max_{E_j(t)} \int_0^\infty \mathsf{e}^{-rt}\left[\pi_j(x, E_j, e(x(t)))\right] dt, && (27)\\
& s.t. \\
x'(t) &= \rho x(t)\left(1 - \frac{x(t)}{K(\varepsilon t)}\right) - h_j(t) - \sum_{l \neq j} h_l(t) \\
K(\varepsilon t) &= K_0(1 + \alpha_0\varepsilon t + \alpha_1\cos(\beta\varepsilon t)), x(0) = x_0.
\end{aligned}$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with this problem can be written for $j = 1, ..J$ as

$$\begin{aligned}
rV(x) &= \max_{E_j}\{pqx^\alpha E_j^\beta - wE_j + V'(x)[\rho x(t)(1 - \frac{x(t)}{K(\varepsilon t)}) - && (28)\\
& qx^\alpha E_j^\beta - (J-1)qx^\alpha e(x)^\beta]\}
\end{aligned}$$

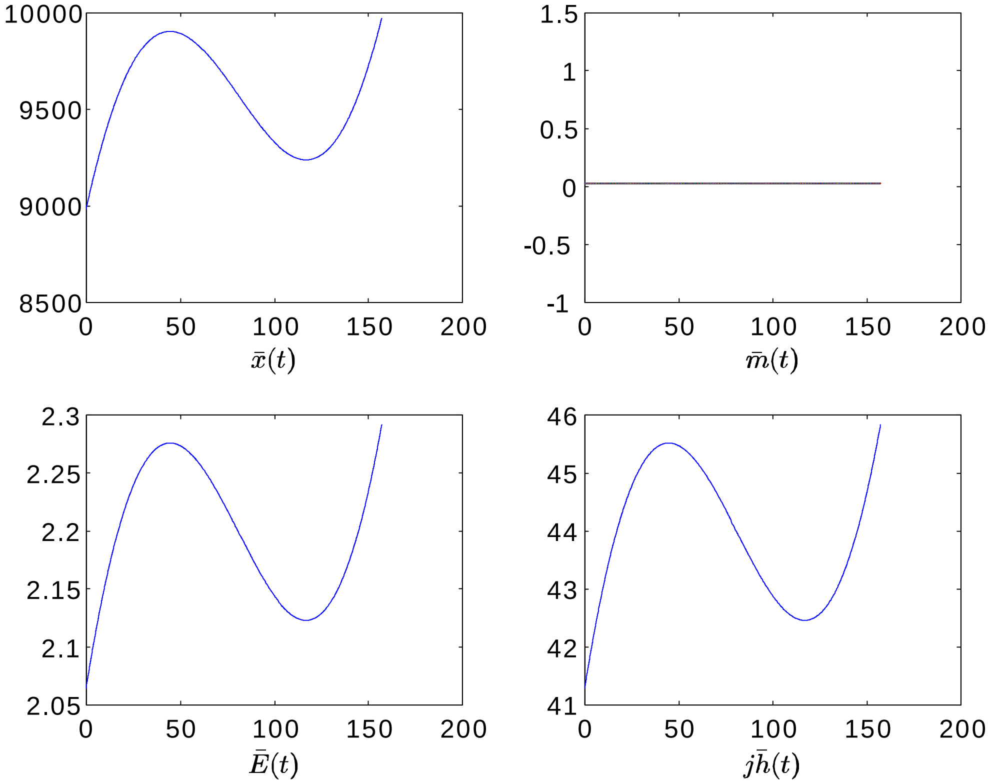

Figure 3: Open loop Nash equilibrium with a slowly changing variable.

where $V(x)$ is the value function for the problem. The optimality condition becomes:

$$V^{'}(x) = p - \frac{w}{qx^{\alpha}\beta E_j^{\beta-1}}. \tag{29}$$

Taking into account that in equilibrium $E_j = e(x)$, assuming that the value function is differentiable, using the envelope condition to take the derivative in (28) and then using (29), we obtain after some manipulations that the equilibrium feedback rule $e = e(x)$ satisfies the nonlinear ordinary differential equation:

$$e'\{pq^2\beta^2(J-1)e^{2\beta-1}x^{2\alpha+1}+e^{\beta}x^{\alpha+1}(wq\beta(1-J)+Jq(\beta-1))$$
$$+\rho x^2\left(1-\frac{x}{K}\right)(1-\beta)\}$$
$$=(1-J)pq^2\beta\alpha e^{2\beta}x^{2\alpha}+wex(\alpha\rho\left(1-\frac{x}{K}\right)+r-\rho\left(1-\frac{2x}{K}\right))$$
$$+(\rho\left(1-\frac{2x}{K}\right)-r)pq\beta e^{\beta}x^{\alpha+1}$$
$$K=K_0(1+\alpha_0\varepsilon t+\alpha_1\cos(\beta_0\varepsilon t)) \tag{30}$$

with boundary condition

$$ne(x_f)=\rho x_f\left(1-\frac{x_f}{K(\varepsilon t)}\right) \tag{31}$$

where x_f corresponds to an equilibrium branch. Since x_f is not known a priori (30) and (31) indicate the well known multiplicity of feedback equilibrium strategies at the FBNE.

A numerical approximation Using $\beta=\alpha=1/2$ in (30) the feedback rule satisfies

$$e'\{\frac{1}{4}pq^2(J-1)x^2+\frac{1}{2}e^{1/2}x^{3/2}(wq(1-J)-Jq)+\frac{1}{2}\rho x^2\left(1-\frac{x}{K}\right)\}=$$
$$\frac{1}{4}(1-J)pq^2ex+\frac{1}{2}wex(\rho\left(\frac{3x}{K}-1\right)+2r)+\frac{1}{2}(\rho\left(1-\frac{2x}{K}\right)-r)pqe^{1/2}x^{3/2}$$
$$K=K_0(1+\alpha_0\varepsilon t+\alpha_1\cos(\beta\varepsilon t))$$

where $E_j=e(x)=e$, $e'=e'(x)$ to ease notation. After rearranging we obtain:

$$e'\{(pq^2(J-1)+2\rho)x^2+2e^{1/2}x^{3/2}(wq(1-J)-Jq)-\frac{2\rho x^3}{K}\} \tag{32}$$
$$=\quad((1-J)pq^2+4rw)ex+2wex\rho\left(\frac{3x}{K}-1\right)+2(\rho\left(1-\frac{2x}{K}\right)-r)pqe^{1/2}x^{3/2}$$

and

$$e'\{0,00576x^2-0,168e^{1/2}x+0,225x^2\left(1-\frac{x}{K}\right)\}=-0,00576ex \tag{33}$$
$$+5ex(0.05-0,225\left(1-\frac{2x}{K}\right))+(0.45\left(1-\frac{2x}{K}\right)-0,05)0.24e^{1/2}x^{3/2},$$

for the parametrization

$$J\ =\ 2,p=10,w=5,q=0.048,\rho=0.45,r=0.05,$$
$$\alpha_0\ =\ 0.15,\alpha_1=0.4,\beta_0=0.5,\varepsilon=0.04$$

with boundary conditions (31). A unique FBNE, in the sense of an equilibrium which is the best for all agents, can be obtained by following the numerical method proposed by Kossioris et al. (2008, 2011). Although the full characterization of the FBNE is beyond the scope of the present paper, our results suggest that both the OLNE and FBNE can be analyzed in the context of adiabatic dynamical systems.

3.1.2 Regulation and time-scale externality

Comparison of the cooperative solution with the open loop solution will provide the basis for regulation. For example deviations $\bar{m}(t) - \bar{m}_{OL}(t)$, or $\bar{h}(t) - \bar{h}_{OL}(t)$, between the adiabatic approximation at the cooperative and the open loop solution can provide the basis for price or quantity regulation, under the assumption that non-cooperative agents follow open loop strategies but take into account the slow evolution of carrying capacity.

If agents do not take into account slow dynamics, then a fixed-carrying-capacity open loop Nash equilibrium can be easily obtained. Let $\hat{m}_{OL}(t), \hat{h}_{OL}(t)$ denote the biomass shadow value and harvesting at the open loop Nash equilibrium when $K(\tau) = K$ fixed. In this case, the basis for designing price or quantity regulation will be the deviations $\bar{m}(t) - \hat{m}_{OL}(t)$ and $\bar{h}(t) - \hat{h}_{OL}(t)$. Since the regulator takes into account slow dynamics, this regulation will internalize the time scale externality.

A similar type of approach, although much more complex, applies when non-cooperating agents follow feedback strategies. In this case, the shadow value of the biomass, m_{OL}, should be replaced by $V'(x)$, the derivative of the value function associated with feedback Nash equilibrium, and the open loop harvesting rule h_{OL} should be replaced by the corresponding feedback rule h_{FB}. Since the regulator accounts for slow dynamics, the time-scale externality is again internalized.

4 Optimal regulation when emissions cause a slowly varying carrying capacity

At this point we introduce a link between emissions in the industrial sector of the economy and the evolution of carrying capacity by assuming that the industrial sector of the economy generates emissions through production processes. Emissions are generated by a finite number of homogeneous agents $i = 1, ..I$, and generate benefits according to a strictly concave benefit function

$$B_i(s_i), B_i' \geq 0, B_i'' < 0$$

with aggregate emissions defined by

$$s = \sum_{i=1}^{I} s_i.$$

Emissions accumulate in the ambient environment to form a stock, according to:

$$S' = \varphi s - lS, \ \varphi > 0, l > 0, \ S(0) = S_0 > 0, S(t) \geq 0.$$

Carrying capacity depends on the emissions' stock. Thus the stock of emissions may change the carrying capacity according to:

$$K(t) = \omega(S(t)) = A - \varepsilon S(t), \ \varepsilon > 0, K(t) \geq 0 \tag{34}$$

$$K' = -\varepsilon S' = -\varepsilon(\varphi s - lS). \tag{35}$$

Assuming that ε is small, we are considering a situation in which the evolution of the pollutant's stock in the ambient environment induces a slow evolution of the carrying capacity K. In this case we have time-scale separation between the fast resource and pollution dynamics, and the slow dynamics of carrying capacity. If $\varepsilon = 0$, then carrying capacity is fixed and does not respond to changes in pollution stock. If ε is small but is ignored, that is we take $\varepsilon \to 0$, the carrying capacity is treated as fixed, while in reality it is slowly changing in response to changes in the pollution stock. This is the source of the time-scale externality.

Using (34) to solve for S and replacing in (35), we obtain

$$K' = -\varepsilon\left(\varphi s - l\left(\frac{A-K}{\varepsilon}\right)\right).$$

Defining γ so that $l = \gamma\varepsilon$, we obtain

$$K' = \varepsilon\left(\gamma(A-K) - \varphi\sum_{i=1}^{I} s_i\right).$$

In this case the dynamical system can written in slow time as:

$$\varepsilon x'(\tau) = \rho x(\tau)\left(1 - \frac{x(\tau)}{K(\tau)}\right) - \sum_{j=1}^{J} h_j(\tau) \tag{36}$$

$$K'(\tau) = \gamma(A - K(\tau)) - \varphi\sum_{i=1}^{I} s_i(\tau), \ \gamma = \frac{l}{\varepsilon} > 0, \tag{37}$$

$$K(0) = A - \varepsilon S(0) = K_0 > 0 \tag{38}$$

or in fast time as:

$$x'(t) = \rho x(t)\left(1-\frac{x(t)}{K(t)}\right)-\sum_{j=1}^{J}h_j(t) \tag{39}$$

$$K'(t) = \varepsilon\left(\gamma(A-K(t))-\varphi\sum_{i=1}^{I}s_i(t)\right). \tag{40}$$

Given the dynamics (36)-(37), we can define the regulator's problem in slow time[6] as the problem of choosing harvesting effort and emission paths to maximize discounted aggregate benefits from harvesting and emissions net of environmental damages associated with the ambient pollutant stock, or

$$\max_{\mathbf{E},\mathbf{s}}\int_0^{\infty}\mathrm{e}^{-\delta\tau}\left[\sum_{j=1}^{J}\pi_j(x,E_j)+\sum_{i=1}^{I}B_i(s_i)-D\left(\sum_{i=1}^{I}s_i\right)\right]d\tau \tag{41}$$
$$\textit{subject to}\ \ (36)-(37)$$

where $D(\cdot)$, $D'>0, D''\geq 0$ is a damage function related to emissions. The current value Hamiltonian for this problem is defined as:

$$\begin{aligned}
H &= \pi+\lambda_1 f_1+\lambda_2 f_2\\
\pi &= \pi(K,x,\mathbf{E},\mathbf{s})=\sum_{j=1}^{J}\pi_j(x,E_j)+\sum_{i=1}^{I}B_i(s_i)-D\left(\sum_{i=1}^{I}s_i\right)\\
f_1 &= f_1(K,x,\mathbf{E},\mathbf{s})=\rho x\left(1-\frac{x}{K}\right)-\sum_{j=1}^{J}h_j\\
f_2 &= f_2(K,x,\mathbf{E},\mathbf{s})=\gamma(A-K)-\varphi\sum_{i=1}^{I}s_i
\end{aligned}$$

where $\lambda_1(\tau),\lambda_2(\tau)$ are the associated costate variables. Using the results of Appendix 3 the optimality conditions resulting from the application of the maximum principle to problem (41) are:

$$H_{\mathbf{E}}=0, \mathbf{E}=(E_1,...,E_J),\ H_{\mathbf{s}}=0, \mathbf{s}=(s_{1,..,}s_I) \tag{42}$$
$$\begin{aligned}
\varepsilon\left(\lambda_1'-\delta\lambda_1\right) &= -H_x\\
\lambda_2'-\delta\lambda_2 &= -H_K\\
\varepsilon x' &= f_1(K,x,\mathbf{E},\mathbf{s})\\
K' &= f_2(K,x,\mathbf{E},\mathbf{s})
\end{aligned} \tag{43}$$

along with Benveniste -Scheinkman transversality conditions at infinity. Calculating the derivatives in (42)-(43) we obtain the following fast/slow system of equations characterizing the optimal solution:

[6] We denote with δ the discount rate in slow time, i.e., the ten year discount rate.

$$p\beta q x^{\alpha} {E_j}^{\beta-1} - w - \lambda_1 \beta q x^{\alpha} {E_j}^{\beta-1} = 0, j = 1.., J \tag{44}$$

$$B_i^{'}(s_i) - D'\left(\sum_{i=1}^{I} s_i\right) - \lambda_2 \varphi = 0, i = 1.., I \tag{45}$$

$$\varepsilon\left(\lambda_1' - \delta\lambda_1\right) + (p - \lambda_1)\alpha q x^{\alpha-1} \sum_{j=1}^{J} {E_j}^{\beta} + \lambda_1 \rho (1 - 2\frac{x}{K}) = 0$$

$$\lambda_2' - \delta\lambda_2 + \lambda_1 \rho \frac{x^2}{K^2} - \gamma\lambda_2 = 0$$

$$\varepsilon x' = \rho\left(1 - \frac{x}{K}\right) - \sum_{j=1}^{J} h_j$$

$$K' = \gamma\left(A - K\right) - \varphi \sum_{i=1}^{I} s_i.$$

4.0.3 Numerical simulations

In order to obtain tractable results we consider, without loss of generality that, $B_i(s_i) = \sqrt{s_i}$, $I = 2$ and that $D(\cdot) = (\cdot)^2$ is a quadratic damage function. Furthermore we assume that $\alpha = \beta = 1/2$ and $\varphi = 1$. System (44) consists of six equations. The first two of them are algebraic equations from which we can solve for the control variables of our problem. Thus we obtain:

$$E_j = (\frac{(p - \lambda_1)\beta q x^{\alpha}}{w})^{\frac{1}{1-\beta}}, E_j^{\beta} = \frac{(p - \lambda_1)\beta q x^{\alpha}}{w} \tag{46}$$

$$1/(2\sqrt{s_i}) - 2\sum_{i=1}^{I} s_i - \lambda_2 \varphi = 0.$$

The system of the remaining four equations is a system with fast and slowly evolving variables. In particular we obtain the following set of equations which characterizes the evolution along an optimal path of biomass and pollution stock and their corresponding shadow values in slow time τ.

$$\varepsilon\left(\lambda_1' - \delta\lambda_1\right) + (p-\lambda_1)\alpha q x^{\alpha-1}\sum_{j=1}^{J} E_j{}^{\beta} + \lambda_1\rho(1-2\frac{x}{K}) = 0 \quad (47)$$
$$\lambda_2' - \delta\lambda_2 + \lambda_1\rho\frac{x^2}{K^2} - \gamma\lambda_2 = 0$$
$$\varepsilon x' = \rho\left(1-\frac{x}{K}\right) - \sum_{j=1}^{J} h_j$$
$$K' = \gamma\left(A-K\right) - \varphi\sum_{i=1}^{I} s_i.$$

The above system is called "the slow system" while rescaling, as $\tau = \varepsilon t$, we obtain the so called "fast system"

$$\left(\lambda_1' - \varepsilon\delta\lambda_1\right) + (p-\lambda_1)\alpha q x^{\alpha-1}\sum_{j=1}^{J} E_j{}^{\beta} + \lambda_1\rho(1-2\frac{x}{K}) = 0 \quad (48)$$
$$\lambda_2' + \varepsilon\{-\delta\lambda_2 + \lambda_1\rho\frac{x^2}{K^2} - \gamma\lambda_2\} = 0$$
$$x' = \rho x\left(1-\frac{x}{K}\right) - \sum_{j=1}^{J} h_j$$
$$K' = \varepsilon\{\gamma\left(A-K\right) - \varphi\sum_{i=1}^{I} s_i\}$$

where in the case of the fast system the derivatives are evaluated with respect to fast evolving time t.

The above systems (47)-(48) can be rewritten in a matrix notation as:

$$\mathit{Fast}\ \begin{array}{c} d\mathbf{X}/dt = F(\mathbf{X},\mathbf{K},\varepsilon) \\ d\mathbf{K}/dt = \varepsilon G(\mathbf{X},\mathbf{K},\varepsilon) \end{array},\ \mathit{Slow}\ \begin{array}{c} \varepsilon d\mathbf{X}/d\tau = F(\mathbf{X},\mathbf{K},\varepsilon) \\ d\mathbf{K}/d\tau = G(\mathbf{X},\mathbf{K},\varepsilon) \end{array} \quad (49)$$

with $\mathbf{X} = (\lambda_1, x)^T, \mathbf{K} = (\lambda_2, K)^T$ the vectors of fast and slow variables respectively. Furthermore, let $F = (F_1, F_2)^T$ and $G = (G_1, G_2)^T$ with

$$\begin{aligned} F_1 &= \varepsilon\delta\lambda_1 - (p-\lambda_1)\alpha q x^{\alpha-1}\sum_{j=1}^{J} E_j{}^{\beta} - \lambda_1\rho(1-2\frac{x}{K}) \\ F_2 &= \rho x\left(1-\frac{x}{K}\right) - \sum_{j=1}^{J} h_j \\ G_1 &= \delta\lambda_2 - \lambda_1\rho\frac{x^2}{K^2} + \gamma\lambda_2 \\ G_2 &= \gamma\left(A-K\right) - \varphi\sum_{i=1}^{I} s_i, \end{aligned}$$

with E_j^{β} and s defined through (46) Setting $\varepsilon = 0,$ in the fast system we define the *layer problem,* in which the carrying capacity is fixed independent of the pollution stock. Setting $\varepsilon = 0$ in the slow system we define the *reduced problem*, in which the fast variable is treated as a variable which has relaxed to its steady-state value.

The steady-state value of the relaxed fast variable evolves slowly, as the slow variable moves towards its own steady-state value. This movement takes place along the so-called slow manifold. To approximate the slow manifolds which characterize the solution of our problem, Fenichel's invariant manifold theorem can be applied (Fenichel 1979). The application of this theorem requires three conditions. (i) The functions F, G should be continuous. (ii) The second condition is related to the reduced problem and requires the existence of functions of the form $\mathbf{X} = \mathbf{H}^o(\mathbf{K}) = [H_1^o(\mathbf{K}), H_2^o(\mathbf{K})]$ such that $F(\mathbf{H}^o(\mathbf{K}), \mathbf{K}, \varepsilon = 0) = 0,$ that is the fast evolving variables should be solved as functions of the slow variables. In particular taking into account relationship (46) which gives the effort rate and manipulating we obtain that:

$$x = K\frac{(p-\lambda_1)\alpha + \lambda_1}{(p-\lambda_1)\alpha + 2\lambda_1}$$

with λ_1 being the solution of

$$(p-\lambda_1)^2 q^2 \alpha\beta J w^{-1} + \lambda_1 \rho \frac{-(p-\lambda_1)\alpha}{(p-\lambda_1)\alpha + 2\lambda_1} = 0.$$

Adopting the specific parameter setting:

$$J = 2, p = 10, w = 5, q = 0.048, \rho = 0.45, \delta = 0.05,$$

we derive the following solutions for the fast variables as functions of the slow variables:[7]

$$\begin{aligned}
(x, \lambda_1)_1 &= (0.33318K, -14460.9) \\
(x, \lambda_1)_2 &= (\frac{K}{2}, 10) \\
(x, \lambda_1)_3 &= (0.999539K, 0.00230506).
\end{aligned}$$

These solutions are candidates for the slow manifold and indicate that the stock of the renewable resource depends on the slow varying carrying capacity that responds to changes in the stock of pollution. A regulator ignoring the slow dynamics of carrying capacity would have regarded x as being independent of S. (iii) Finally, accordingly to the third condition, the real parts of the eigenvalues of the Jacobian matrix $\mathbf{J} = \frac{\partial F}{\partial \mathbf{X}}(\mathbf{H}^o(\mathbf{K}), \mathbf{K}, \varepsilon = 0)$ should be nonzero. Negative real parts induce an attracting slow manifold, while if

[7] We used Mathematica for some of the calculations at the current section.

there is at least one positive real part, the manifold is repelling. In our case the matrix $\mathbf{J}$ is given by

$$
\begin{aligned}
\mathbf{J} &= \frac{\partial F}{\partial \mathbf{X}} = \begin{bmatrix} \frac{\partial F_1}{\partial \lambda_1} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial \lambda_1} & \frac{\partial F_2}{\partial x} \end{bmatrix} = \begin{bmatrix} -J_{11} & J_{12} \\ 0 & J_{22} \end{bmatrix} \\
J_{11} &= J_{22} = \rho(1 - 2\frac{(p-\lambda_1)\alpha + \lambda_1}{(p-\lambda_1)\alpha + 2\lambda_1}) - (p-\lambda_1) j q^2 \alpha\beta / w \\
J_{12} &= -\frac{1}{K}\frac{(p-\lambda_1)\alpha + 2\lambda_1}{w\{(p-\lambda_1)\alpha + \lambda_1\}} j\alpha\beta(\alpha-1)(p-\lambda_1)^2 q^2 + 2\lambda_1 \rho \frac{1}{K},
\end{aligned}
$$

while the eigenvalues of this matrix are given by

$$
eig_{1,2} = \pm\{(p-\lambda_1) j q^2 \alpha\beta / w - \rho(1 - 2\frac{(p-\lambda_1)\alpha + \lambda_1}{(p-\lambda_1)\alpha + 2\lambda_1})\}.
$$

For the solution $(x, \lambda_1)_2 = (\frac{K}{2}, 10)$ the eigenvalues are equal to zero so Fenichel's theorem has no application. From the two remaining cases the solution, $(x, \lambda_1)_1$ is rejected on economic grounds because it is associated with a negative shadow value for the renewable resource, while the solution $(x, \lambda_1)_3 = (0.999539K, 0.00230506)$ is acceptable with real eigenvalues: $(eigen_3 = \pm 9.99308)$.

Fenichel's theorem extends then the analysis to an arbitrary small parameter $\boldsymbol{\varepsilon}$ and provides the characterization of the optimal slow manifold M_ε. Let $M_\varepsilon = \{(\mathbf{X}, \mathbf{K}) \in \mathbb{R}^4 : \mathbf{X} = (\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}), \mathbf{K}, \boldsymbol{\varepsilon})\}$ such that :

$$
d\mathbf{K}/d\tau = G(\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}), \mathbf{K}, \boldsymbol{\varepsilon}), \tag{50}
$$

where the vector $\mathbf{H}^{\boldsymbol{\varepsilon}}(\mathbf{K}) = \mathbf{H}^{\mathbf{0}}(\mathbf{K}) + \varepsilon \mathbf{H}^{(\mathbf{1})}(\mathbf{K}) + ...$ as $\boldsymbol{\varepsilon} \to 0$, with

$$
\begin{aligned}
\mathbf{H}^0(\mathbf{K}) &= \mathbf{H}^0(\mathbf{K}) \qquad (51) \\
\mathbf{H}^{(1)}(\mathbf{K}) &= [\frac{\partial F}{\partial \mathbf{X}}]^{-1}[\frac{\partial \mathbf{H}^{\mathbf{0}}}{\partial \mathbf{K}} G - \frac{\partial F}{\partial \boldsymbol{\varepsilon}}] \\
\frac{\partial F}{\partial \boldsymbol{\varepsilon}} &= [\frac{\partial F_1}{\partial \boldsymbol{\varepsilon}}, \frac{\partial F_2}{\partial \boldsymbol{\varepsilon}}]^T = [\delta\lambda_1, 0]^T \\
\frac{\partial \mathbf{H}^{\mathbf{0}}}{\partial \mathbf{K}} &= \begin{bmatrix} \frac{\partial H_1^o(\mathbf{K})}{\partial \lambda_2} & \frac{\partial H_1^o(\mathbf{K})}{\partial K} \\ \frac{\partial H_2^o(\mathbf{K})}{\partial \lambda_2} & \frac{\partial H_2^o(\mathbf{K})}{\partial K} \end{bmatrix}.
\end{aligned}
$$

Then the dynamical system (50) characterizes optimal paths of the slow variable and its shadow value, when controls are chosen optimally and the interaction between the slow and the fast variables are accounted for on the slow manifold M_ε.

4.1 The optimal slow manifold and regulation

The slow manifold $(x, \lambda_1)_3 = (0.999539K, 0.00230506)$ implies that the stock of biomass is proportional to the carrying capacity, while carrying capacity evolves slowly in response to the evolution of the pollution stock, according to (50). To solve (50) we need to characterize, according to (51), the vector $\mathbf{H}^{\epsilon}(\mathbf{K})$ using our parametrization. This leads to the dynamical system:

$$\begin{bmatrix} d\lambda_2/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\lambda_2 - \lambda_1^{\epsilon}\rho(0.999539)^2 + \gamma\lambda_2 \\ \gamma\left(A-K\right) - \varphi\sum_{i=1}^{I} s_i, \end{bmatrix} \tag{52}$$

with steady state $(\lambda_2^*, K^*) = (0.00010291, 24.9)$ which is a saddle point.[8] To determine the stable manifold we assume an initial value $K(0) = 15$ and we apply a multiple shooting method, which for an initial state $(0.009, 15)$ the system converges to the steady state $(0.00010291, 24.9)$ for $\tau = 1.25$.

The economic interpretation of this result is the following. Let the initial value for the carrying capacity be $K(0) = 15$, then if the regulator imputes a shadow value in the carrying capacity with initial value 0.009 the optimal paths in slow time for the carrying capacity $K(\tau)$, its shadow cost, the emission flow $s(\tau)$ and the harvesting effort $E(\tau)$ are depicted in Figure 4, with the system being on the stable manifold for the slow variables. The trajectories can be transformed to fast time by appropriate rescaling.

In terms of policy design the fact that carrying capacity has a shadow value λ_2 implies that the emission tax should internalize two externalities, the straightforward pollution externality captured by the damage function $D(\cdot)$ and the most subtle time-scale externality captured by the shadow value if carrying capacity. From (45) the optimal emission tax can be defined as:

$$T^*(t) = D'\left(\sum_{i=1}^{I} s_i^*(t)\right) + \lambda_2^*(t)\,\varphi \tag{53}$$

where the term $D'\left(\sum_{i=1}^{I} s_i^*(t)\right)$ internalizes the emission externality, the term $\lambda_2^*(t)\,\varphi$ internalizes the time scale externality, and $s_i^*(t)$, $\lambda_2^*(t)$ are the optimal paths depicted in Figure 4. Alternatively the path $s_i^*(t)$ can be used as the basis for quantity regulation. The path $E^*(t)$ can be used as a basis for quantity regulation for the renewable resource.

Using the optimal slow manifold $(x, \lambda_1)_3 = (0.999539K, 0.00230506)$ and the optimal steady state $(\lambda_2^*, K^*) = (0.00010291, 24.9)$ for the slow variables, it follows that the optimal steady state for the biomass and its shadow value, when the carrying capacity has relaxed to its long-run optimal steady state, will be $(x^*, \lambda_1^*) = (0.999539 * 24.9, 0.00230506) = (24.888, 0.00230506)$.

5 Ignoring the time scale externality

An idea of the possible cost, in terms of efficient regulation, from ignoring the time scale separation can be obtained by solving again the regulator's

[8] Detailed calculations are presented in Appendix 4.

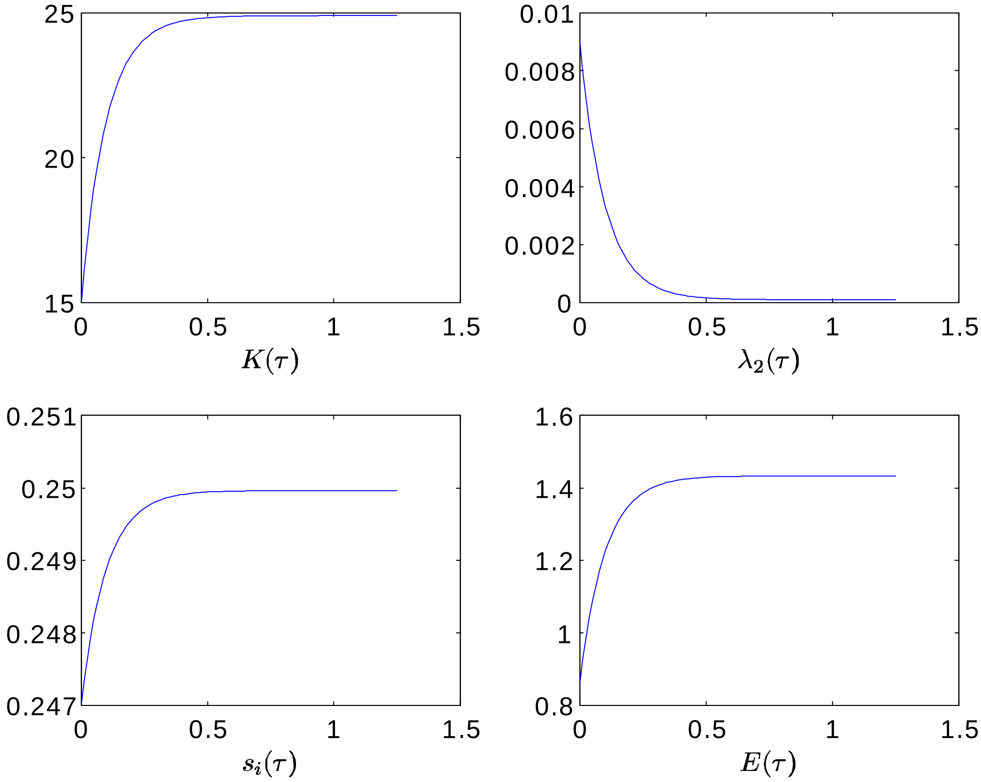

Figure 4: Evolution of the slow system

problem assuming a constant carrying capacity. In this case the regulators problem will be:

$$\max_{\mathbf{E}} \int_0^{\infty} \mathrm{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j (x, E_j) = \pi(x, \mathbf{E}) \right] dt \tag{54}$$

$$s.t., \qquad x'(t) = \rho x(t) \left(1 - \frac{x(t)}{\bar{K}} \right) - \sum_{j=1}^{J} h_j, x(0) = x_0.$$

We run our simulations again assuming that $\bar{K} = A = 25$, where A was defined in (34).[9] In this case we derive the following steady states for the renewable resource:

$$X_1 = (x, m)_1 = (24.7452, 0.0465056), X_2 = (x, m)_2 = (6.39496, -716.759)$$
$$X_3 = (x, m)_3 = (0, -1716.05), X_4 = (x, m)_4 = (0, -0.0582733)$$

[9] The full solution is presented in Appendix 5.

where, X_1 X_3, X_4, are saddle points, and X_2 is an unstable focus.[10] Disregarding the extinction solutions X_3, X_4 and the unstable focus X_2 we observe that for the acceptable solution X_1 the steady state for the biomass $x = 24.7452$ is close to the optimal steady state when the time scale externality is accounted for $x^* = 24.888$. Thus in this case, (i.e., where the fixed carrying capacity is in the neighborhood of the optimal steady state carrying capacity when the slowly evolving carrying capacity is taken into account) regulating by ignoring the time scale separation leads to steady states which are very close to regulation when time scale separation is accounted for. On the other hand if the fixed carrying capacity deviates a lot from the optimal steady state then the resulting path and steady state for the renewable resource will deviate from the corresponding optimal path.

If the regulator assumes a fixed carrying capacity $\bar{K} = 30$, then the acceptable steady state for the renewable resource will be $X_1 = (x, m)_1 = (29, 6942, 0.0465056)$, while a fixed carrying capacity of $\bar{K} = 15$ results in an acceptable steady state $X_1 = (x, m)_1 = (14.8471, 0.0465056)$.[11] Thus the fixed-carrying-capacity paths deviate from the optimal paths. This deviation can be clearly seen if we consider the impact from the industrial sector on the carrying capacity, which exists but is not taken into account when the regulator assumes a fixed K. Assume that the regulator chooses and implements optimal emissions $(\hat{s}_1, \hat{s}_2)$ by solving the static problem

$$\max_{s_i} \sum_{i=1}^{I} B_i(s_i) - D\left(\sum_{i=1}^{I} s_i\right). \tag{57}$$

Then pollution will accumulate according to $S'(t) = \varphi(\hat{s}_1 + \hat{s}_2) - lS(t)$, $S(0) = S_0$. Let $\hat{S}(t)$ be the resulting path of accumulated pollution. Then the carrying capacity will evolve slowly according to:

$$K(t) = A - \varepsilon \hat{S}(t), K(0) = A - \varepsilon S_0. \tag{58}$$

If, for example, we use our parametrization with $\varphi = 2, l = 0.4, \varepsilon = 0.04$, $S_0 = 1$, and $A = 30$, then $\hat{S}(t) = 2.5 - 1.5\mathrm{e}^{-0.4t}$ and $K(t) = 30 - 0.04\hat{S}(t)$. Thus carrying capacity is slowly reducing. Since our numerical

[10] It is likely that, a Skiba point exists for initial values of x in the neighborhood of X_2 with branches converging either to X_1 with positive biomass at the steady state, or to resource extinction. However this is beyond the scope of the present paper.

[11] The full solutions are

$$\begin{aligned} X_1 &= (x,m)_1 = (29, 6942, 0.0465056), X_2 = (x,m)_2 = (7.67395, -716.759) \\ X_3 &= (x,m)_3 = (0, -1716.05), X_4 = (x,m)_4 = (0, -0.0582733), \end{aligned} \tag{55}$$

$$\begin{aligned} X_1 &= (x,m)_1 = (14.8471, 0.0465056), X_2 = (x,m)_2 = (3.83697, -716.759) \\ X_3 &= (x,m)_3 = (0, -1716.05), X_4 = (x,m)_4 = (0, -0.0582733) \end{aligned} \tag{56}$$

X_2 is unstable focus and all the rest are saddle points.

results indicate that the optimal steady-state biomass is declining along with carrying capacity, treating the carrying capacity as fixed by ignoring the time scale externality, implies that the regulatory scheme will induce excess harvesting.

6 Concluding Remarks

In the present paper we identify a time scale externality related to the common characteristic of ecosystems to contain state variables which evolve in different time scales, fast or slow. If economic agents take decisions that affect these state variables by ignoring time scale separation then a time scale externality is introduced. To study the time scale externality we analyzed a renewable resource management problem when carrying capacity is evolving slowly, either in response to an exogenous forcing, or in response to pollution accumulating in the industrial sector of the economy. Using singular perturbation methods we analyze the problem of a regulator seeking to internalize the time scale externality and we derive the optimal paths for the fast and the slow variables along with the optimal regulatory scheme that includes the adjustment induced by time scale separation. We also show, mainly through numerical simulations, that ignoring the time scale externality could lead to inefficient regulation.

Areas for further research include the introduction of nonconvexities in ecosystems dynamics. In particular if pollution dynamics, which induce a slow variation of the carrying capacity, are characterized by non convexities then the slow manifold might contain more than one feasible branches. In this case an additional task of optimal regulation would be to identify the optimal slow branch and steer the fast system towards this branch.

Uncertainty is also another open issue, in particular the case where the evolution of the fast variable might be characterized by risk, or measurable uncertainty, and the evolution of the slow variable might be characterized by ambiguity. The application of robust control methods when the structure of uncertainty differs according to the time scale, could be an interesting area for further research.

7 Appendix

7.1 Appendix 1: Adiabatic system: Cooperative solution

System (10) has the following pairs of solutions, where m^* and x^* are determined as functions of $K(\tau)$.[12]

$$\begin{aligned} S_1 &= (m^*(K(\tau)), x^*(K(\tau))_1 = (-1716.05, 0) \\ S_2 &= (m^*(K(\tau)), x^*(K(\tau))_2 = (-0.0582733, 0) \\ S_3 &= (m^*(K(\tau)), x^*(K(\tau))_3 = (-716.759, 0.255798K) \\ S_4 &= (m^*(K(\tau)), x^*(K(\tau))_4 = (0.0465056, 0.989808K) \end{aligned} \quad (59)$$

with matrix A defined as:

$$\begin{aligned} A &= \begin{bmatrix} r - \rho\left(1 - \frac{2x}{K}\right) + J\frac{1}{2w}(p-m)q^2 & 2m\rho/K \\ J\frac{1}{2w}xq^2 & \rho\left(1 - \frac{2x}{K}\right) - J\frac{1}{2w}(p-m)q^2 \end{bmatrix}, \\ A &= \begin{bmatrix} 0,05 + T & 0,9\frac{m}{K} \\ 0.0004608x & -T \end{bmatrix}, \ T = 0.0004608(10-m) - 0,45\left(1 - \frac{2x}{K}\right). \end{aligned}$$

Note that solutions S_1 and S_2 lead to extinction, while solutions S_3 and S_4 lead to positive biomass along the equilibrium branch. The corresponding eigenvalues are shown below:

$$\begin{aligned} S_1 &: \{0.395364, -0.345364\}, \ S_2 : \{0.445365, -0.395365\} \\ S_3 &: \{0.025 + 0.237501i, 0.025 - 0.237501i\}, \ S_4 : \{0.495434, -0.445434\} \end{aligned} \quad (60)$$

We calculate the eigenvalues by using the values given in (9) while for x, K we use the four pairs given by (59). Thus S_3 is unstable while all others solutions are saddle points which is a result compatible with the optimal control structure of the problem.

Therefore all equilibrium branches $(m^*(K(\tau)), x^*(K(\tau)))$ of (11) admit an adiabatic solution $(\overline{m}(\tau), \overline{x}(\tau))$. With any of these solutions we can associate an adiabatic approximation of order one which is obtained as (Berglund 1998):

$$\begin{aligned} (\overline{m}(\tau), \overline{x}(\tau))^T &= (m^*(\tau), x^*(\tau))^T + \varepsilon u(\tau, \varepsilon) + \bigcirc(\varepsilon^2) \\ u(\tau, \varepsilon) &= -A_i^{-1} w(\tau, \varepsilon), \ w(\tau, \varepsilon) = -[((m^*(\tau))', (x^*(\tau))')^T], i = 1, .., 4. \end{aligned} \quad (61)$$

The adiabatic approximating solution (12) describes how the stock of biomass evolves under optimal regulation given the slow evolution of carrying

[12] We used the Mathematica package for the calculations in the current section.

capacity. Taking $\varepsilon = 0.04$, so that the fast time unit is 1/25 of the slow time unit, biomass dynamics at the regulators equilibrium satisfy:

$$\begin{aligned}
S_1 &: \quad \overline{x}(\tau) = 0 + \bigcirc(\varepsilon^2) \ , \ S_2 : \overline{x}(\tau) = 0 + \bigcirc(\varepsilon^2) && (62)\\
S_3 &: \quad \overline{x}(\tau) = 0.255798K(\tau) + 0.0296219K'(\tau) + \bigcirc(\varepsilon^2) && (63)\\
S_4 &: \quad \overline{x}(\tau) = 0.989808K(\tau) - 0.0888812K'(\tau) + \bigcirc(\varepsilon^2) && (64)\\
K(\tau) &= K_0\left(1 + \alpha_0\tau + \alpha_1\cos(\beta_0\tau)\right) && (65)\\
K'(\tau) &= K_0\left(1 + \alpha_0 - \beta_0\alpha_1\sin(\beta_0\tau)\right) && (66)
\end{aligned}$$

As described in the main text we study S_4.

7.2 Appendix 2: Adiabatic system: Open loop Nash Equilibrium

Using the parameter setting of the cooperative solution with two players

$$J = 2, p = 10, w = 5, q = 0,048, \rho = 0.45, r = 0,05,$$

we obtain the equilibrium branches $(m^*(K(\tau)), x^*(K(\tau)))$ by setting$(\varepsilon m'(K(\tau)), \varepsilon x'(K(\tau))) = (0,0)$:

$$\begin{aligned}
S_1 &= (m^*(K(\tau)), x^*(K(\tau))_1 = (-1721.08, 0)\\
S_2 &= (m^*(K(\tau)), x^*(K(\tau))_2 = (-0.0290515, 0)\\
S_3 &= (m^*(K(\tau)), x^*(K(\tau))_3 = (-715.07, 0.257529K)\\
S_4 &= (m^*(K(\tau)), x^*(K(\tau))_4 = (0.0233078, 0.989784K)
\end{aligned}$$

Matrix A in the OLNE case takes the form

$$\begin{aligned}
A &= \begin{bmatrix} T1 & 2m\rho/K \\ J\frac{1}{2w}xq^2 & \rho\left(1 - \frac{2x}{K}\right) - \frac{J}{2w}(p-m)q^2 \end{bmatrix},\\
T1 &= r - \rho\left(1 - \frac{2x}{K}\right) + \frac{1}{2w}(p-m)q^2 + (J-1)\frac{1}{4w}\ (p-2m)q^2
\end{aligned}$$

Note that as in the cooperative case there are two solutions, S_1 and S_2 which lead to extinction, while solutions S_3 and S_4 lead to positive biomass along the equilibrium branch. The associated eigenvalues are shown below

$$\begin{aligned}
S_1 &: \ \{0.39653, -0.347682\},\ S_2 : \{0.445379, -0.396531\}\\
S_3 &: \ \{0.024424 + 0.238083i, 0.024424 - 0.238083i\},\ S_4 : \{0.494261, -0.445413\}
\end{aligned}$$

Thus S_3 is unstable while all others solutions are saddle points which is a result compatible with the optimal control structure of the problem.

As in the cooperative case we can associate an adiabatic approximation based on:

$$\begin{aligned}(\overline{m}(\tau),\overline{x}(\tau))^T &= (m^*(\tau),x^*(\tau))^T+\varepsilon u(\tau,\varepsilon)+\bigcirc(\varepsilon^2), where \quad (67)\\ u(\tau,\varepsilon) &= -A_i^{-1}w(\tau,\varepsilon), w(\tau,\varepsilon)=-(m^*(\tau),x^*(\tau))^{T\prime}, i=1,..,4\end{aligned}$$

Taking $\varepsilon = 0.04$, biomass dynamics at the OLNE satisfy:

$$S_1 : \overline{x}(\tau)=0+\bigcirc(\varepsilon^2)\ ,\ S_2:\overline{x}(\tau)=0+\bigcirc(\varepsilon^2) \quad (68)$$

$$S_3 : \overline{x}(\tau)=0.257529K(\tau)+0.0296259K'(\tau)+\bigcirc(\varepsilon^2) \quad (69)$$

$$S_4 : \overline{x}(t)=0.989784K(\varepsilon t)--0.088885K'(\varepsilon t)+\bigcirc(\varepsilon^2) \quad (70)$$

$$K(\tau) = K_0(1+\alpha_0\tau+\alpha_1\cos(\beta_0\tau)) \quad (71)$$

$$K'(\tau) = K_0(1+\alpha_0-\beta_0\alpha_1\sin(\beta_0\tau)) \quad (72)$$

Focussing on S_4 which is characterized by saddle point stability and positive biomass we derive the corresponding adiabatic solutions for the optimal effort E. In fast time the optimal paths for biomass and effort are as shown in the main text.

7.3 Appendix 3: The maximum principle in fast/slow time

Consider the following problem in slow time τ.

$$\begin{aligned}&\max_u \int_0^\infty e^{-\delta\tau}U(x_1,x_2,u)\,d\tau \quad (73)\\ &s.t. \quad \varepsilon\dot{x}_1=f_1(x_1,x_2,u)\\ &\dot{x}_2=f_2(x_1,x_2,u)\end{aligned}$$

where x_1 is the fast variable, x_2 is the slow variable and the control $u = (u_1, u_2)$ affects both variables. [13]

Writing:

$$\int_0^\infty e^{-\delta\tau}U(x_1,x_2,u)\,d\tau=\int_0^\infty e^{-\delta\tau}obj\,d\tau \quad (74)$$

$$obj=U(x_1,x_2,u)+\lambda_1(\tau)[f_1(x_1,x_2,u)-\varepsilon\dot{x}_1]+\lambda_2(\tau)[f_2(x_1,x_2,u)-\dot{x}_2]$$

Integrate by parts the terms $e^{-\delta\tau}\lambda_1(\tau)\varepsilon\dot{x}_1$, $e^{-\delta\tau}\lambda_2(\tau)\dot{x}_2$, with $\lambda_1(\tau), \lambda_2(\tau)$ the associated Lagrange multipliers, to obtain these two terms:

$$\int_0^\infty e^{-\delta\tau}\lambda_1(\tau)\varepsilon\dot{x}_1d\tau=-\int_0^\infty e^{-\delta\tau}\varepsilon x_1\left(\dot{\lambda}_1-\delta\lambda_1\right)d\tau+e^{-\delta\tau}\varepsilon x_1(\tau)\lambda_1(\tau)\Big|_0^\infty$$

$$\int_0^\infty e^{-\delta\tau}\lambda_2(\tau)\dot{x}_2d\tau=-\int_0^\infty e^{-\delta\tau}x_2\left(\dot{\lambda}_2-\delta\lambda_2\right)d\tau+e^{-\delta\tau}x_2(\tau)\lambda_2(\tau)\Big|_0^\infty$$

[13] Note that δ is the discount rate in slow time.

Substitute the above two relationships into (74) we obtain:

$$\int_0^\infty e^{-\delta\tau} U(x_1, x_2, u)\, d\tau = \tag{75}$$
$$\int_0^\infty e^{-\delta\tau}\{U(x_1, x_2, u) + \lambda_1(\tau) f_1(x_1, x_2, u) + \lambda_2(\tau) f_2(x_1, x_{2,} u)\} d\tau +$$
$$\int_0^\infty e^{-\delta\tau} \varepsilon x_1 \left(\dot{\lambda}_1 - \delta\lambda_1\right) d\tau + \varepsilon x_1(0)\lambda_1(0) +$$
$$\int_0^\infty e^{-\delta\tau} x_2 \left(\dot{\lambda}_2 - \delta\lambda_2\right) d\tau + x_2(0)\lambda_2(0).$$

Taking the comparison control $u^*(\tau) + \alpha h(\tau) = (u_1^*(\tau) + \alpha h_1(\tau), u_2^*(\tau) + \alpha h_2(\tau))$ where $u^*(\tau)$ is the optimal control, and considering the corresponding state variable generated by the comparison control $(y_1(\tau,\alpha), y_2(\tau,\alpha)) = (x_1(\tau), x_2(\tau))$ the value of the above integral is a function of the parameter α so we can define:

$$\begin{aligned} J(\alpha) &= \int_0^\infty e^{-\delta\tau} U(y_1(\tau,\alpha), y_2(\tau,\alpha), u^*(\tau) + \alpha h(\tau))\, d\tau \\ &= \int_0^\infty e^{-\delta\tau}\{U(y_1(\tau,\alpha), y_2(\tau,\alpha), u^*(\tau) + \alpha h(\tau)) + \\ &\quad \lambda_1(\tau) f_1(y_1(\tau,\alpha), y_2(\tau,\alpha), u^*(\tau) + \alpha h(\tau)) + \\ &\quad \lambda_2(\tau) f_2(y_1(\tau,\alpha), y_2(\tau,\alpha), u^*(\tau) + \alpha h(\tau))\} d\tau = \\ &\quad \int_0^\infty e^{-\delta\tau} \varepsilon y_1(\tau,\alpha)\left(\dot{\lambda}_1 - \delta\lambda_1\right) d\tau + \varepsilon y_1(0,\alpha)\lambda_1(0) + \\ &\quad \int_0^\infty e^{-\delta\tau} y_2(\tau,\alpha)\left(\dot{\lambda}_2 - \delta\lambda_2\right) d\tau + y_2(0,\alpha)\lambda_2(0). \end{aligned} \tag{76}$$

Evaluate the derivative $J'(\alpha)$ and since $u^*(\tau)$ is the optimal control the following must be satisfied

$$\begin{aligned} J'(0) &= 0, where \\ J'(a) &= \int_0^\infty e^{-\delta\tau} U_a(y_1(\tau,\alpha), y_2(\tau,\alpha), u^*(\tau) + \alpha h(\tau))\, d\tau. \end{aligned} \tag{77}$$

Manipulating we obtain the maximum principle as:

$$\begin{aligned} U_{u_1} + \lambda_1 f_{1u_1} + \lambda_2 f_{2u_1} &= 0 \\ U_{u_2} + \lambda_1 f_{1u_2} + \lambda_2 f_{2u_2} &= 0 \\ \varepsilon\left(\dot{\lambda}_1 - \delta\lambda_1\right) + U_{x_1} + \lambda_1 f_{1x_1} + \lambda_2 f_{2x_1} &= 0 \\ \left(\dot{\lambda}_2 - \delta\lambda_2 + U_{x_2} + \lambda_1 f_{1x_2} + \lambda_2 f_{2x_2}\right) &= 0. \end{aligned} \tag{78}$$

Defining the Hamiltonian H as:

$$H = U(x_1, x_2, u) + \lambda_1(\tau) f_1(x_1, x_2, u) + \lambda_2(\tau) f_2(x_1, x_2, u)$$

the optimality conditions regarding our initial problem (74) become:

$$\begin{aligned} H_{u_1} &= 0 \\ H_{u_2} &= 0 \\ \varepsilon\left(\dot{\lambda}_1 - \delta\lambda_1\right) &= -H_{x_1} \\ \dot{\lambda}_2 - \delta\lambda_2 &= -H_{x_2} \end{aligned}$$

or our systems takes the following form:

$$\begin{gathered} U_{u_1} + \lambda_1 f_{1u_1} + \lambda_2 f_{2u_1} = 0 \\ U_{u_2} + \lambda_1 f_{1u_2} + \lambda_2 f_{2u_2} = 0 \\ \varepsilon\left(\dot{\lambda}_1 - \delta\lambda_1\right) + U_{x_1} + \lambda_1 f_{1x_1} + \lambda_2 f_{2x_1} = 0 \\ \dot{\lambda}_2 - \delta\lambda_2 + U_{x_2} + \lambda_1 f_{1x_2} + \lambda_2 f_{2x_2} = 0 \\ \varepsilon\dot{x}_1 = f_1(x_1, x_2, u) \\ \dot{x}_2 = f_2(x_1, x_2, u)\,. \end{gathered} \tag{79}$$

Setting $\varepsilon = 0$ and solving on the slow manifold, we obtain the system:

$$\begin{aligned} U_{u_1} + \lambda_1 f_{1u_1} + \lambda_2 f_{2u_1} &= 0 \\ U_{u_2} + \lambda_1 f_{1u_2} + \lambda_2 f_{2u_2} &= 0 \\ U_{x_1} + \lambda_1 f_{1x_1} + \lambda_2 f_{2x_1} &= 0 \\ \dot{\lambda}_2 - \delta\lambda_2 + U_{x_2} + \lambda_1 f_{1x_2} + \lambda_2 f_{2x_2} &= 0 \\ 0 &= f_1(x_1, x_2, u) \\ \dot{x}_2 &= f_2(x_1, x_2, u)\,. \end{aligned} \tag{80}$$

So we need to solve for u, x_1, λ_1 in terms of x_2, λ_2 and substitute into the differential equations for x_2, λ_2. We can draw phase diagrams. Then we can simulate the full model.

7.4 Appendix 4: The Slow Manifold

$$\begin{gathered} \mathbf{H}^0(\mathbf{K}) = \mathbf{H}^0(\mathbf{K}) \\ \mathbf{H}^{(1)}(\mathbf{K}) = [\frac{\partial F}{\partial \mathbf{X}}]^{-1}[\frac{\partial \mathbf{H^0}}{\partial \mathbf{K}} G - \frac{\partial F}{\partial \boldsymbol{\varepsilon}}] \\ \frac{\partial F}{\partial \boldsymbol{\varepsilon}} = [\delta\lambda_1, 0]^T \\ \frac{\partial \mathbf{H^0}}{\partial \mathbf{k}} = \begin{bmatrix} 0 & -0.999539 \\ 0 & 0 \end{bmatrix} \\ [\frac{\partial F}{\partial \mathbf{X}}]^{-1} = \frac{1}{D}\begin{bmatrix} -eig & -J_{12} \\ 0 & eig \end{bmatrix}, \\ D = -eig^2, eig = -\rho(1 - 2\frac{(p-\lambda_1)\alpha + \lambda_1}{(p-\lambda_1)\alpha + 2\lambda_1}) + (p-\lambda_1) j q^2 \alpha\beta/w. \end{gathered}$$

With $\varepsilon = 0.04$, the optimal trajectories of the fast variables $\mathbf{X}^{\varepsilon}$ as functions of the slow variables on the slow manifold are given by:

$$\mathbf{X}^{\varepsilon} = (\lambda_1^{\boldsymbol{\varepsilon}}, x^{\boldsymbol{\varepsilon}})^T = \begin{bmatrix} 0.00230506 \\ 0.999539K \end{bmatrix} + 0.04 \begin{bmatrix} 0.04(\frac{1}{eig} G_2 - \delta(0.00230506)) \\ 0 \end{bmatrix} \tag{81}$$
$$= \begin{bmatrix} 0.00400277G_2 + 0.00230045 \\ 0.999539K \end{bmatrix}$$

Then the dynamics of the slow variables on M_{ε} are given by:

$$\begin{bmatrix} d\lambda_2/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\lambda_2 - \lambda_1^{\boldsymbol{\varepsilon}} \rho \frac{(x^{\boldsymbol{\varepsilon}})^2}{K^2} + \gamma\lambda_2 \\ \gamma\left(A - K\right) - \varphi \sum_{i=1}^{I} s_i \end{bmatrix}. \tag{82}$$

Taking into account that $1/(2\sqrt{s_i}) - 2\sum_{i=1}^{I} s_i - \lambda_2\varphi = 0$ and adopting the following parameterization

$$D\left(\cdot\right) = (\cdot)^2, \delta = 0.05, \varphi = 2, A = 25, \varepsilon = 0.04, l = 0.4, \gamma = l/\varepsilon = 10, I = 2$$

we obtain the dynamical system characterizing the slow variables as:

$$\begin{bmatrix} d\lambda_2/d\tau \\ dK/d\tau \end{bmatrix} = \begin{bmatrix} \delta\lambda_2 - \lambda_1^{\boldsymbol{\varepsilon}} \rho(0.999539)^2 + \gamma\lambda_2 \\ \gamma\left(A - K\right) - \varphi \sum_{i=1}^{I} s_i, \end{bmatrix}. \tag{83}$$

The steady states of the above system give the full system equilibria and their derivatives, with respect to $\mathbf{K}$, determine stability properties when evaluated at a specific steady state. Manipulating and taking into account (81), (82) and the condition for s_i, we obtain for the steady state (λ_2, K):

$$\begin{bmatrix} 0 \\ K \end{bmatrix} = \begin{bmatrix} \delta\lambda_2 - 0.00230045 \quad \rho(0.999539)^2 + \gamma\lambda_2 \\ \frac{\varphi}{\gamma}\sum_{i=1}^{I} s_i - A \end{bmatrix}, or \tag{84}$$

$$\begin{bmatrix} \lambda_2 \\ K \end{bmatrix} = \begin{bmatrix} 0.00230045\rho(0.999539)^2/(\delta + \gamma) \\ \frac{\varphi}{\gamma}\sum_{i=1}^{I} s_i - A \end{bmatrix} = \begin{bmatrix} 0.00010291 \\ 24.9 \end{bmatrix}, \tag{85}$$

$$s_i = 0.249966. \tag{86}$$

The stability matrix for system (83) *is*

$$J = \begin{bmatrix} (\delta + \gamma) & 0 \\ (A - \frac{\varphi}{\gamma}\sum_{i=1}^{I} s_i)\lambda_2 & -\gamma \end{bmatrix}$$

with associated determinant equal to $-(\gamma + \delta)\gamma < 0$, and trace $\delta > 0$. Thus the steady state $(\lambda_2^*, K^*) = (0.00010291, 24.9)$ is a saddle point.

7.5 Appendix 5: Renewable resource management with fixed carrying capacity

Assuming that harvesting which is the variable of interest takes place in fast time it is natural to study the cooperative solution for the following regulator's problem.

$$\max_{\mathbf{E}} \int_0^\infty \mathrm{e}^{-rt} \left[\sum_{j=1}^{J} \pi_j (x, E_j) = \pi(x, \mathbf{E}) \right] dt \tag{87}$$

$$s.t., \qquad x'(t) = \rho x(t) \left(1 - \frac{x(t)}{\bar{K}}\right) - \sum_{j=1}^{J} h_j, x(0) = x_0$$

with functions and parameters being as in previous sections.

$$\begin{aligned} h_j &= qx^\alpha E_j^\beta \; , \alpha > 0, 0 < \beta < 1, j = 1, ..J. \\ \mathbf{E} &= (E_1, ..., E_J) . \end{aligned}$$

Defining the current value Hamiltonian H as:

$$H(x, E_j, m) = \pi(x, \mathbf{E}) + m(\rho x \left(1 - \frac{x}{\bar{K}}\right) - \sum_{j=1}^{J} h_j) \tag{88}$$

with $\pi_j (x, E_j) = pqx^\alpha E_j^\beta - wE_j, h_j = qx^\alpha E_j^\beta$ we obtain the following optimality conditions:

$$H_{E_j}(x, E_j, m) = pqx^\alpha \beta E_j^{\beta-1} - w - \beta mqx^\alpha E_j^{\beta-1} = 0, j = 1, ..J. \tag{89}$$

$$H_{E_j E_j} = (p - m)(\beta - 1)qx^\alpha \beta E_j^{\beta-2} < 0,$$

$$m' = rm - H_x(x, E_j, m) =$$

$$rm - \alpha pqx^{\alpha-1} \sum_{j=1}^{J} E_j^\beta - m\{\rho \left(1 - \frac{2x}{\bar{K}}\right) - \alpha qx^{\alpha-1} \sum_{j=1}^{J} E_j^\beta\}$$

$$x'(t) = \rho x(t) \left(1 - \frac{x(t)}{\bar{K}}\right) - \sum_{j=1}^{J} h_j.$$

Solving for E_j,we obtain

$$E_j = (\frac{p - m}{w} qx^\alpha \beta)^{\frac{1}{1-\beta}}, , j = 1, ..J$$

and thus the (89) becomes:

$$\begin{aligned} m' &= rm - m\rho \left(1 - \frac{2x}{\bar{K}}\right) - J\alpha(\frac{\beta}{w})^{\frac{\beta}{1-\beta}}((p - m)q)^{\frac{1}{1-\beta}} x^{\frac{\alpha+\beta-1}{1-\beta}} \\ x' &= \rho x \left(1 - \frac{x}{\bar{K}}\right) - J(\frac{\beta}{w})^{\frac{\beta}{1-\beta}}(p - m)^{\frac{\beta}{1-\beta}} q^{\frac{1}{1-\beta}} x^{\frac{\alpha}{1-\beta}} \end{aligned} \tag{90}$$

We study the nature of the steady states of our system by adopting the usual parameter values. In particular for $\beta = \alpha = 1/2$, with

$$J = 2, p = 10, w = 5, q = 0,048, \rho = 0.45, r = 0,05, \bar{K} = \{25, 30, 15\}$$

we obtain the results of section 5.

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